MATH6280 A - Measure Theory
Midterm Examination
November 3, 2016

(1) Let $(X, \mathcal{M}, \mu)$ be a measure space and $A_1, A_2, \dots$ be a countable collection of sets in $\mathcal{M}$ with the property that for every distinct $i, j \in \mathbb{N}^+$ we have $\mu(A_i \cap A_j) = 0$. Show that $\sum_{i=1}^{\infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i)$.
Solution. The inequality $\sum_{i=1}^{\infty} \mu(A_i) \geq \mu(\bigcup_{i=1}^{\infty} A_i)$ is true for any countable collection of sets from $\mathcal{M}$, so it is enough to prove $\sum_{i=1}^{\infty} \mu(A_i) \leq \mu(\bigcup_{i=1}^{\infty} A_i)$. Let $N = \bigcup_{i \neq j} A_i \cap A_j$. Then $\mu(N) \leq \sum_{i \neq j} \mu(A_i \cap A_j) = 0$. Now note that the sets $A_i' = A_i \setminus N$ are pairwise disjoint. Clearly, $\mu(A_i') \leq \mu(A_i)$ and $\mu(A_i') = \mu(A_i') + \mu(N) \geq \mu(A_i)$. Thus, $\mu(A_i) = \mu(A_i')$ for every $i \in \mathbb{N}$. So by the additivity of μ we get

$$\mu(\bigcup_{i=1}^{\infty} A_i) \geq \mu(\bigcup_{i=1}^{\infty} A_i') = \sum_{i=1}^{\infty} \mu(A_i') = \sum_{i=1}^{\infty} \mu(A_i).$$

(2) (a) Define the notion of an outer measure!
Solution. If X is a set, an outer measure on X is a function $\rho : \mathcal{P}(X) \to [0, \infty]$ with the following properties:
(i) $\rho(\emptyset) = 0$,
(ii) for every $A, B \subset X$ with $A \subset B$, we have $\rho(A) \leq \rho(B)$
(iii) for every $A_1, A_2, \dots \subset X$ we have $\rho(\bigcup_{i=1}^{\infty} A_i)) \leq \sum_{i=1}^{\infty} \rho(A_i)$.

(b) Let λ^* be the outer measure associated to the Lebesgure measure, i. e., for $A \subset \mathbb{R}$ define $\lambda^*(A) = \inf\{\sum_{i=1}^{\infty} \lambda(A_i) : A_i \text{ are open intervals}, A \subset \bigcup_{i=1}^{\infty} A_i\}$. Prove that *for every* $A \subset \mathbb{R}$ there exists a G_δ set B such that $A \subset B$ and $\lambda^*(A) = \lambda^*(B)$.
Solution. By the definition of λ^* for every $n \in \mathbb{N}^+$ there exists a countable collection of open intervals $A_1^n, A_2^n, \dots$ such that $A \subset \bigcup_{i=1}^{\infty} A_i^n$ and $\lambda^*(A) + \frac{1}{n} \geq \sum_{i=1}^{\infty} \lambda(A_i^n)$. Let $B_n = \bigcup_{i=1}^{\infty} A_i^n$ and define $B = \bigcap_{n=1}^{\infty} B_n$.
Clearly, $A \subset B_n$ for each n, so $A \subset B$ as well. Thus, by the monotonicity of an outer measure we get $\lambda^*(A) \leq \lambda^*(B)$.
On the other hand $B \subset B_n$ where the latter set is the union of the intervals $(A_i^n)_{i=1}^{\infty}$. So $\lambda^*(B) \leq \sum_{i=1}^{\infty} \lambda(A_i^n)$ for every n, hence $\lambda^*(B) \leq \lambda^*(A) + \frac{1}{n}$ holds for every n, which shows that $\lambda^*(A) = \lambda^*(B)$.

(3) Let $f : \mathbb{R} \to [0, \infty)$ be a Borel measurable function and consider the set $S = \{(x, y) : 0 \leq y \leq f(x)\}$. Show that
(a) S is $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ measurable,
(b) $(\lambda \times \lambda)(S) = \int_{\mathbb{R}} f d\lambda$.

Solution.
(a) Since all rectangles with Borel sides are in the σ-algebra $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ and $S \subset \{(x, y) : y \geq 0\}$, it is enough to prove that $T = \{(x, y) : y \geq 0\} \setminus S$ can be expressed as a countable union of such rectangles. We will show that $T = \bigcup_{q \in \mathbb{Q}} f^{-1}([0, q]) \times (q, \infty)$, note that as f is Borel measurable the sets $f^{-1}([0, q])$ are in $B_{\mathbb{R}}$ so the sets $f^{-1}([0, q]) \times (q, \infty)$ are in $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

Now for every (x, y) with $y \geq 0$ we have

$$(x, y) \in T \iff (x, y) \notin S \iff f(x) < y \iff$$

there exists a rational number q with $f(x) < q < y \iff$

$$(x, y) \in \bigcup_{q \in \mathbb{Q}} f^{-1}([0, q]) \times (q, \infty),$$

which shows the desired equality.

(b) By Fubini's theorem and the measurability of S we get

$$(\lambda \times \lambda)(S) = \int_{\mathbb{R}} \lambda(S_x)\, d\lambda(x)$$

but $S_x = [0, f(x)]$ and $\lambda([0, f(x)]) = f(x)$, so

$$(\lambda \times \lambda)(S) = \int_{\mathbb{R}} \lambda(S_x)\, d\lambda(x) = \int_{\mathbb{R}} f(x)\, d\lambda(x).$$

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space, $f : X \to [0, \infty)$ be measurable with $\int_X f d\mu < \infty$ and $\varepsilon > 0$. Prove that there exists a $\delta > 0$ such that for every $B \in \mathcal{M}$ with $\mu(B) < \delta$ we have $\int_B f d\mu < \varepsilon$.

Solution. By the definition of $\int_X f\, d\mu = \sup\{\int_X \phi\, d\mu : 0 \leq \phi \leq f, \phi \text{ is a simple function}\}$. Since $\int_X f$ is finite, there exists a simple function ϕ with $0 \leq \phi \leq f$ such that $\int f\, d\mu < \int \phi\, d\mu + \frac{\varepsilon}{2}$. Note that so for every measurable set B we have

$$(*) \qquad \int_B f\, d\mu - \int_B \phi\, d\mu = \int_B f - \phi\, d\mu \leq \int_X f - \phi\, d\mu < \frac{\varepsilon}{2},$$

where the first inequality holds because $f - \phi \geq 0$.

ϕ is simple and nonnegative, so it can be expressed in the form $\sum_{i=1}^n a_i \chi_{E_i}$ for some measurable sets $E_1, \ldots, E_n$ and nonnegative reals $a_1, \ldots, a_n$. Let $K = 1 + \max\{a_i : 1 \leq i \leq n\}$ and $\delta < \frac{\varepsilon}{2K}$. If B is an arbitrary measurable set with $\mu(B) < \delta$ then $\int_B \phi\, d\mu \leq K\mu(B) < K\frac{\varepsilon}{2K} = \frac{\varepsilon}{2}$. Thus, using this and (*) yields

$$\int_B f\, d\mu < \int_B \phi\, d\mu + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$