Measure theory notes

0.1. **Vitali's theorem.**

Definition 0.1. We say that a function μ is a *natural total measure on* $\mathbb{R}$ if $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ and

(1) for every collection $A_1, A_2, \dots \subset \mathbb{R}$ of pairwise disjoint sets we have
$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n),$$
(2) for every $x \in \mathbb{R}$ and every $A \subset \mathbb{R}$ we have $\mu(x + A) = \mu(A)$,
(3) for every $x, y \in \mathbb{R}$ with $x < y$
$$\mu([x, y]) = y - x.$$

Theorem 0.2. *There is no natural total measure on* $\mathbb{R}$.

Proof. Suppose towards contradiction that μ is such a measure. Notice first that Property 1 implies monotonicity, that is, if $A \subset B$ then $\mu(A) \leq \mu(B)$: indeed,
$$\mu(B) = \mu(B \setminus A) + \mu(A) \geq \mu(A).$$

Now we define a relation $\sim$ on $[0, 1]$ as follows: let $x \sim y$ if and only if $x - y \in \mathbb{Q}$. It is easy to check that $\sim$ is an equivalence relation:

- $x \sim x$, since $x - x \in \mathbb{Q}$,
- $x \sim y$ implies $y \sim x$, since $y - x = -(x - y)$,
- if $x \sim y$ and $y \sim z$ then $x \sim z$, since $x - z = (x - y) + (y - z)$ and the sum of two rational numbers is rational.

Let $V \subset [0, 1]$ be a set that intersects each equivalence class at exactly one point. Notice that if r_1, r_2 are distinct rationals then $V + r_1 \cap V + r_2 = \emptyset$: otherwise, $v_1 + r_1 = v_2 + r_2$ was true for some distinct $v_1, v_2 \in V$, so $v_1 - v_2 \in \mathbb{Q}$, i. e., $v_1 \sim v_2$ contradicting the choice of V.

Note that if $x \in [0, 1]$ then there exists a $v \in V$ with $x \sim v$, in other words, $x - v \in \mathbb{Q}$ (note also that $x, v \in [0, 1]$ implies $x - v \in [-1, 1]$), therefore each $x \in [0, 1]$ is covered by some of the sets $(V + r)_{r \in \mathbb{Q} \cap [-1,1]}$. Thus,
$$[0, 1] \subset \bigcup_{r \in \mathbb{Q} \cap [-1,1]} V + r \subset [-1, 2],$$
where the last containment follows for $V \subset [0, 1]$.

So, using the fact that the sets $(V + r)_{r \in \mathbb{Q}}$ are disjoint translates of V and the monotonicity of μ, we obtain
$$1 = \mu([0, 1]) \leq \mu(\bigcup_{r \in \mathbb{Q} \cap [-1,1]} V + r) \leq \mu([-1, 2]) = 3$$
and by the countability of $\mathbb{Q}$ we get
$$1 \leq \sum_{r \in \mathbb{Q} \cap [-1,1]} \mu(V + r) \leq 3.$$
Hence, $1 \leq \sum_{i=1}^{\infty} \mu(V) \leq 3$, thus on the one hand $\sum_{i=1}^{\infty} \mu(V)$ is bounded by 3, so $\mu(V) = 0$, on the other hand the sum is non-zero, a contradiction.

□

0.2. **Product of σ-algebras.** It is not hard to see that if X is a set, $\mathcal{E} \subset \mathcal{P}(X)$, that is, $\mathcal{E}$ is a family of subsets of X, then there exists a smallest σ-algebra containing $\mathcal{E}$. We call this set the σ-algebra generated by $\mathcal{E}$ and denote it by $\mathcal{M}(\mathcal{E})$.

Definition 0.3. Suppose that $\mathcal{M}_i \subset \mathcal{P}(X_i)$ is a collection of σ-algebras on the sets X_i for $i \in I$. We define the *product* of the σ-algebras $\mathcal{M}_i$ as the σ-algebra of subsets of $\prod_{i \in I} X_i$ generated by the sets of the form $\{\pi_i^{-1}(A_i) : A_i \in \mathcal{M}_i\}$, where π_i denotes the projection map "onto the ith coordinate" from $\prod_{i \in I} X_i$ to X_i. We denote this σ-algebra by $\otimes_{i \in I} \mathcal{M}_i$.

In particular, if $I = \{1, 2\}$ then the σ-algebra $\mathcal{M}_1 \otimes \mathcal{M}_2$ is generated by the sets of the form $A_1 \times X_2$ and $X_1 \times A_2$, where $A_1 \in \mathcal{M}_1$ and $A_2 \in \mathcal{M}_2$.

Theorem 0.4. *Suppose that the families of sets $\mathcal{E}_i \subset \mathcal{M}_i$ generate the σ-algebras $\mathcal{M}_i$ for every $i \in I$. Then the σ-algebra generated by $\mathcal{F}_1 = \{\pi_i^{-1}(A_i) : A_i \in \mathcal{E}_i, i \in I\}$ is exactly $\otimes_{i \in I} \mathcal{M}_i$. If moreover for every $i \in I$ we have $X_i \in \mathcal{E}_i$ and I is countable then the σ-algebra generated by the family $\mathcal{F}_2 = \{\prod_{i \in I} A_i : A_i \in \mathcal{E}_i\}$ is $\otimes_{i \in I} \mathcal{M}_i$.*

Proof. We prove the first statement first. Clearly, by $\mathcal{E}_i \subset \mathcal{M}_i$ and the definition of $\otimes_{i \in I} \mathcal{M}_i$ we have $\mathcal{F}_1 \subset \otimes_{i \in I} \mathcal{M}_i$, therefore $\mathcal{M}(\mathcal{F}_1) \subset \otimes_{i \in I} \mathcal{M}_i$.

In order to prove $\mathcal{M}(\mathcal{F}_1) \supset \otimes_{i \in I} \mathcal{M}_i$ it is enough to prove that $\mathcal{M}(\mathcal{F}_1)$ contains a generating set of the σ-algebra $\otimes_{i \in I} \mathcal{M}_i$. So we prove that

$$(*) \qquad \mathcal{M}(\mathcal{F}_1) \supset \{\pi_i^{-1}(A_i) : A_i \in \mathcal{M}_i, i \in I\},$$

the latter is by definition a generating set of $\otimes_{i \in I} \mathcal{M}_i$. Define for every $i \in I$ the set

$$\mathcal{A}_i = \{A \in \mathcal{M}_i : \pi_i^{-1}(A) \in \mathcal{M}(\mathcal{F}_1)\}.$$

Obviously, $\mathcal{A}_i \subset \mathcal{M}_i$ and by the definition of $\mathcal{F}_1$ we have $\mathcal{E}_i \subset \mathcal{A}_i$ for every $i \in I$.

We claim that $\mathcal{A}_i$ is a σ-algebra, note that by $\mathcal{E}_i \subset \mathcal{A}_i$ this will imply $\mathcal{M}_i = \mathcal{A}_i$. Let $B, B_1, B_2, \dots \in \mathcal{A}_i$ for some $i \in I$. Then, since $\pi_i^{-1}(B) \in \mathcal{M}(\mathcal{F}_1)$ by definition of $\mathcal{A}_i$ and $\mathcal{M}(\mathcal{F}_1)$ being a σ-algebra we get that $\pi_i^{-1}(X_i \setminus B) = (\prod_{i \in I} X_i) \setminus \pi_i^{-1}(B) \in \mathcal{M}(\mathcal{F}_1)$, so $X_i \setminus B \in \mathcal{A}_i$ as well, so $\mathcal{A}_i$ is closed under taking complements. Similarly, using $\pi_i^{-1}(\bigcup_n B_n) = \bigcup_n \pi_i^{-1}(B_n)$ we get that $\mathcal{A}_i$ is closed under countable unions.

Hence $\mathcal{A}_i = \mathcal{M}_i$, but this means that for every $A_i \in \mathcal{M}_i$ we have $\pi_i^{-1}(A_i) \in \mathcal{M}(\mathcal{F}_1)$, in other words, (*) holds and we are done.

We prove now that $\mathcal{M}(\mathcal{F}_2) = \otimes_{i \in I} \mathcal{M}_i$. Since $X_i \in \mathcal{E}_i$ we have that if $A_i \in \mathcal{E}_i$ the sets of the form $\pi_i^{-1}(A_i)$ are elements of $\mathcal{F}_2$, that is, $\mathcal{F}_1 \subset \mathcal{F}_2$. Thus, from the first part we get $\mathcal{M}(\mathcal{F}_2) \supset \mathcal{M}(\mathcal{F}_1) = \otimes_{i \in I} \mathcal{M}_i$. For the reverse just notice that for every set of the form $\prod_{i \in I} A_i$ where $A_i \in \mathcal{E}_i$ we have

$$\prod_{i \in I} A_i = \bigcap_{i \in I} \pi_i^{-1}(A_i).$$

If I is countable, then the above intersection is a countable intersection of elements of the σ-algebra $\otimes_{i \in I} \mathcal{M}_i$, thus, $\prod_{i \in I} A_i \in \otimes_{i \in I} \mathcal{M}_i$, so $\mathcal{F}_2 \subset \otimes_{i \in I} \mathcal{M}_i$ which finishes the proof.

□

Corollary 0.5. *If I is countable, then $\otimes_{i \in I} \mathcal{M}_i$ is generated by the sets of the form $\prod_{i \in I} A_i$ where $A_i \in \mathcal{M}_i$.*

Proof. Let $\mathcal{E}_i = \mathcal{M}_i$ in Theorem 0.4. □

0.3. **Elementary families.**

Definition 0.6. Let X be a set. A collection $\mathcal{E}$ of subsets of X is called an *elementary family* if

(1) for every $A, B \in \mathcal{E}$ we have $A \cap B \in \mathcal{E}$,
(2) for every $A \in \mathcal{E}$ there exist pairwise disjoint sets $A_1, \dots, A_k \in \mathcal{E}$ with $A^c = A_1 \cup \dots \cup A_n$.

Proposition 0.7. *Suppose that $\mathcal{E}$ is an elementary family. Let $\mathcal{A} = \{A_1 \cup \dots \cup A_n : A_1, \dots, A_n$ are disjoint, $A_i \in \mathcal{E}\}$, that is, the collection of finite disjoint unions of sets from $\mathcal{E}$.*

Proof. First we prove by induction on n that whenever $A_1, \dots, A_n \in \mathcal{E}$ then $\cup_{i=1}^n A_i \in \mathcal{A}$. For $n = 1$ this is obvious, so suppose that we have proved the statement for every natural $\leq n$.

Let $A_1, \dots, A_n, A_{n+1} \in \mathcal{E}$ be arbitrary. By the inductive hypothesis

$$A_1 \cup \dots \cup A_n \cup A_{n+1} = B_1 \cup \dots \cup B_k \cup A_{n+1},$$

for some $B_1, \dots, B_k \in \mathcal{E}$ are disjoint. But then

$$B_1 \cup \dots \cup B_k \cup A_{n+1} = \\ = \bigcup_{i=1}^k (B_i \cap A_{n+1}^c) \cup A_{n+1},$$

and the sets $B_i \cap A_{n+1}^c$ (for $1 \leq i \leq k$) and A_{n+1} are pairwise disjoint and by Property 2 we have $B_i \cap A_{n+1}^c = \bigcup_{l=1}^m B_i \cap E_l$, where E_l are pairwise disjoint sets from $\mathcal{E}$. Therefore, $A_1 \cup \dots \cup A_n \cup A_{n+1} \in \mathcal{A}$.

Thus, since every finite union of elements of $\mathcal{A}$ is a finite union of elements $\mathcal{E}$, we get that$\mathcal{A}$ is closed under finite unions.

To see that it is closed under taking complements let $A_1, \dots, A_n \in \mathcal{E}$ pairwise disjoint, then

$$(A_1 \cup \dots \cup A_n)^c = \bigcap_{i=1}^n A_i^c.$$

By Property 2 for each $i \leq n$ we get sets $(E_j^i)_{1 \leq j \leq n_i} \in \mathcal{E}$ with $A_i = \cup_{j=1}^{n_i} E_j^i$. Thus,

$$\bigcap_{i=1}^n A_i^c = \bigcap_{i=1}^n (\bigcup_{j=1}^{n_i} E_j^i) = \bigcup \{E_{j_1}^1 \cap E_{j_2}^2 \cap \dots \cap E_{l_n}^n : 1 \leq j_i \leq n_i, 1 \leq i \leq n\},$$

which is the finite union of sets in $\mathcal{E}$, consequently, by the first part of the proof, an element of $\mathcal{A}$. $\square$

0.4. **Regularity properties of measures.** As we have seen, for a measure μ defined on a σ-algebra we can define the appropriate outer measure μ^*, which will be a measure on the collection of (Caratheodory-) measurable sets. Moreover, on this collection μ^* is the unique measure extending μ, provided that μ was σ-finite. Hence, we do not lose anything if we identify μ with this extension to the measurable sets. In particular, we will call a set μ-measurable, if it is measurable with respect to the outer measure μ^*, and let $\mu(A) = \mu^*(A)$ if A is measurable etc.

Theorem 0.8. *Let μ be a finite measure on the σ-algebra of $\mathcal{B}_X$ for some metric space X. Then for every μ measurable set E we have*

(1) $\mu(E) = \inf\{\mu(U) : U \supset E, U$ *is open*$\}$,
(2) $\mu(E) = \sup\{\mu(F) : F \subset E, F$ *is closed*$\}$.

Proof. Let $\mathcal{A}$ be the collection of those μ measurable sets for which the conditions 1-2 hold.

$\mathcal{A}$ contains every closed set: indeed, if F is closed, then 2 is obvious. To see that 1 holds, use that F is G_δ (which can be proved very similarly to the case $X = \mathbb{R}$), so $F = \bigcap_{n=1}^\infty U_n$ where U_n is open. Letting $V_k = \bigcap_{n=1}^k U_n$ the sets V_k are open and form a decreasing sequence, with $F = \bigcap_{n=1}^\infty V_n$, by the finiteness and hence the continuity of μ we have $\mu(V_n) \to \mu(F)$ as $n \to \infty$.

$\mathcal{A}$ is a σ-algebra. HW

Thus, we get that $\mathcal{A} \supset \mathcal{B}_X$. Now, let A be an arbitrary μ measurable set. First we prove that we can find a $B \supset A$ with $B \in \mathcal{B}_{\mathbb{R}^d}$ and $\mu(B) = \mu(A)$: By the definition μ^* for every $m \in \mathbb{N}$ there exists sets $A_n^m \in \mathcal{B}_{\mathbb{R}^d}$ such that $A \subset \bigcup_{n=1}^\infty A_n^m$ and $\mu(A) > \mu(\bigcup_{n=1}^\infty A_n^m) - \frac{1}{m+1}$. By the finiteness of μ we get that

$$\mu((\bigcup_{n=1}^\infty A_n^m) \setminus A) = \mu(\bigcup_{n=1}^\infty A_n^m) - \mu(A) < \frac{1}{m+1}.$$

Let $B = \bigcap_{m=1}^\infty \bigcup_{n=1}^\infty A_n^m$. Then $B \in \mathcal{A}$ and $B \supset A$ and for every $m \in \mathbb{N}$ we have

$$\mu(B) - \mu(A) \leq \mu(\bigcup_{n=1}^\infty A_n^m) - \mu(A) \leq \frac{1}{m+1},$$

so $\mu(B) = \mu(A)$.

Thus, we can find $B \in \mathcal{B}_{\mathbb{R}^d}$ such that $B \supset A$ and $\mu(A) = \mu(B)$. Let $\varepsilon > 0$. Using that $B, B^c \in \mathcal{A}$ we get an open set $U \supset B \supset A$ and an open set $V \supset B^c$ such that

$$\mu(U) < \mu(B) + \varepsilon = \mu(A) + \varepsilon, \tag{*}$$

and

$$\mu(V) < \mu(B^c) + \varepsilon.$$

Using that μ is finite and letting $F = X \setminus V$ we get

$$\mu(F) = \mu(X \setminus V) = \mu(X) - \mu(V) \geq \mu(X) - (\mu(B^c) + \varepsilon) =$$
$$\mu(X) - (\mu(X) - \mu(B) + \varepsilon) \geq \mu(B) - \varepsilon = \mu(A) - \varepsilon.$$

The fact that ε was arbitrary and the last equation show that 2 holds for A, while (*) shows that 1 is also true. □

Corollary 0.9. *Suppose that μ is a measure on $\mathcal{B}_\mathbb{R}$ which is finite on every bounded set. Then μ is Borel regular. In particular, every Lebesgue-Stieltjes measure is Borel regular.*

Proof. Let $E \subset \mathbb{R}$ be a μ measurable set and fix $\varepsilon > 0$. For each $n \in \mathbb{Z}$ the interval $[n, n+1]$ is a metric space (with the usual metric) and the measure $\mu|_{[n,n+1]}$ is a finite Borel measure (note that for a set $B \subset [n, n+1]$ we have in $B \in \mathcal{B}_{[n,n+1]}$ iff $B \in \mathcal{B}_\mathbb{R}$ and also B is $\mu|_{[n,n+1]}$-measurable iff B is μ-measurable). Thus, by Theorem 0.8 it is regular, so there exist closed sets (in $[n, n+1]$ as a metric space) $F_n \subset E \cap [n, n+1)$ such that

$$\mu(E \cap [n, n+1)) < \mu(F_n) + \varepsilon \cdot 2^{-|n|-2}.$$

Note that (using that the intervals $[n, n+1]$ are closed in $\mathbb{R}$) the sets F_n and hence the set $F = \bigcup_{n\in\mathbb{Z}} F_n$ are closed in $\mathbb{R}$, $F \subset E$ and

$$\mu(E) = \sum_{n\in\mathbb{Z}} \mu(E \cap [n, n+1)) < \sum_{n\in\mathbb{Z}} (\mu(F_n) + \varepsilon \cdot 2^{-|n|-2}) < \mu(F) + \varepsilon.$$

This shows that 2 holds.

The proof of outer regularity is similar: find a U_n open subset in each $(n-1, n+1)$ such that $U_n \supset E \cap [n, n+1)$ and

$$\mu(U_n) < \mu(E \cap [n, n+1)) + \varepsilon \cdot 2^{-|n|-2}.$$

Then (again, by the fact that the intervals $(n-1, n+1)$ are open) the set $U = \bigcup_{n\in\mathbb{Z}} U_n$ is open in $\mathbb{R}$, $U \supset E$ and

$$\mu(U) \leq \sum_{n\in\mathbb{Z}} \mu(U_n) < \sum_{n\in\mathbb{Z}} (\mu(E \cap [n, n+1)) + \varepsilon \cdot 2^{-|n|-2}) < \mu(E) + \varepsilon,$$

so we are done. □

0.5. **Riemann integral and Lebesgure integral.** For an interval I we denote by $|I|$ the length of I.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. A sequence $a = x_0 < x_1 < \cdots < x_n = b$ is called a *partition* of the interval $[a, b]$. The *upper and lower sums* corresponding to the partition $P = (x_i)_{i=0}^n$ are defined as follows:

$$U(P, f) = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}),$$

and

$$L(P, f) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}).$$

Let

$$\overline{\int_a^b} f(x)dx = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\},$$

and

$$\underline{\int_a^b} f(x)dx = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}.$$

We say that f is Riemann integrable if $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$.

A sequence of partitions $P_k = (x_i^k)_{i=0}^n$ is called *infinitely refining*, if $\max_{1\leq i\leq n}(x_i^k - x_{i-1}^k) \to 0$ as $k \to \infty$. It is not hard to show that f is Riemann integrable if and only if for every infinitely refining partition sequence $P_k = (x_i^k)_{i=0}^n$ we have that

$$\lim_{k\to\infty} U(P_k, f) = \lim_{k\to\infty} L(P_k, f).$$

Theorem 0.10. *(Lebesgue's criterion of integrability) Let $f : [a, b] \to \mathbb{R}$ be a function. f is Riemann integrable if and only if f is bounded and the set $\{x \in [a, b] : f$ is not continuous at $x\}$ has Lebesgue measure zero.*

Before we begin the proof of the theorem we need a definition.

Definition 0.11. The *oscillation* of f on the interval I, $\omega_f(I)$, is defined by

$$\omega_f(I) = \sup_{x\in I} f(x) - \inf_{x\in I} f(x).$$

Whereas the *oscillation of f at a point x_0*, $\omega_f(x_0)$ is defined by

$$\omega_f(x_0) = \lim_{\varepsilon\to 0} \omega_f((x_0-\varepsilon, x_0+\varepsilon)).$$

Lemma 0.12. *(1) f is continuous at a point x_0 if and only if $\omega_f(x_0) = 0$,*
(2) if $c \in \mathbb{R}$, the set $\{x : \omega_f(x) \geq c\}$ is closed in $[a,b]$.

Proof. HW □

Proof. Thus, using our new notations, we have to proof that f is Riemann integrable if and only if f is bounded and the set $\{x : \omega_f(x) > 0\}$ has measure zero.
($\Rightarrow$) Suppose first that f is Riemann integrable. Then by the definition of the existence of the integral f is bounded, hence it is enough to prove that for every $m \in \mathbb{N}^+$ the set $D_m = \{x : \omega_f(x) > \frac{1}{m}\}$ has λ-measure zero. Thus, let $m \in \mathbb{N}^+$ and $\varepsilon > 0$ be given. By the definition of integrability there exists a partition $P = (x_i)_{i=0}^n$ such that $U(P,f) - L(P,f) < \frac{\varepsilon}{m}$. By the definition of U and L we get:

$$(*) \quad \frac{\varepsilon}{m} > U(P,f) - L(P,f) = \sum_{i=1}^n (\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x))(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^n \omega_f([x_{i-1},x_i])(x_i - x_{i-1}) \geq \sum_{[x_{i-1},x_i]\cap D_m \neq\emptyset} \frac{1}{m}(x_i - x_{i-1})$$

where the last inequality follows from the fact that whenever in an interval $[x_{i-1}, x_i]$ there exists a point p from D_m, the oscillation $\omega_f([x_{i-1},x_i]) \geq \omega_f(p) \geq \frac{1}{m}$. Obviously, D_m is covered by the collection of intervals $[x_{i-1}, x_i]$ for which $[x_{i-1},x_i] \cap D_m \neq \emptyset$. But multiplying the two sides of (*) by m we get that the total length of these intervals is less then ε, thus $\lambda(D_m) < \varepsilon$. Since ε was arbitrary, this finishes the proof of this direction.
($\Leftarrow$) Suppose now that the points of discontinuity form a measure zero set and f is bounded. Fix an $\varepsilon > 0$ and let K be such that $|f(x)| < K$ for each $x \in [a,b]$. We will construct a partition of P with $U(P,f) - L(P,f) < \varepsilon$. By Lemma 0.12 the set $\{x : \omega_f(x) \geq \frac{\varepsilon}{2(b-a)}\}$ is closed and, as the subset of $[a,b]$ it is compact, while by our assumption it is measure zero. Thus, there exists a finite collection of open intervals $I_1, \dots, I_k$ such that $\sum_{j=1}^k |I_j| < \frac{\varepsilon}{4K}$ and $\{x : \omega_f(x) \geq \frac{\varepsilon}{2(b-a)}\} \subset \bigcup_{j=1}^k I_j$.

Now the set $K = [a,b] \setminus \bigcup_{j=1}^k I_j$ is also closed and bounded, hence compact, and for every $x \in K$ we have $\omega_f(x) < \frac{\varepsilon}{2(b-a)}$. But then, using the definition of $\omega_f(x)$ for each $x \in K$ there exists an open interval U around x such that $\omega_f(U) < \frac{\varepsilon}{2(b-a)}$. Again, we can find a cover of K by finitely many such open intervals. Intersecting these intervals with K and $I_1, \dots, I_k$ and taking the endpoints we get a partition $a = x_0 < x_1 < \dots < x_n = b$ with the following property: for each i we have that $[x_{i-1}, x_i] \subset I_j$ for some j, or $\omega_f([x_{i-1},x_i]) < \frac{\varepsilon}{2(b-a)}$. Thus,

$$U(P,f) - L(P,f) = \sum_{i=1}^n (\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x))(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} \omega_f([x_{i-1}, x_i])(x_i - x_{i-1}) \leq$$

$$\leq \sum_{\substack{[x_{i-1},x_i]\subset I_j \\ \text{for some } j}} \omega_f([x_{i-1}, x_i])(x_i - x_{i-1}) + \sum_{\omega_f([x_{i-1},x_i])<\frac{\varepsilon}{2(b-a)}} \omega_f([x_{i-1}, x_i])(x_i - x_{i-1}) \leq$$

$$\leq 2K \sum_{j=1}^{k} |I_j| + \frac{\varepsilon}{2(b-a)}(b-a) \leq 2K\frac{\varepsilon}{4K} + \frac{\varepsilon}{2} = \varepsilon,$$

which finishes the proof of the theorem.

□