MINIMAL DEFINABLE GRAPHS WITH NO DEFINABLE TWO-COLORINGS

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Abstract. We sketch our results about the structure of Borel graphs with Borel chromatic number at least three ordered by the relation of injective Borel homomorphism.

In [CMSV19] we have shown that a Borel graph has no Borel two-coloring if and only if it contains a continuous homomorphic copy of a graph called $\mathbb{L}_0$. Now, we would like to consider the case when the homomorphism is required to be injective.

For all $n \in \mathbb{N}$, let L_n denote the graph on $\{(0), \ldots, (n)\}$ with respect to which (i) and (j) are neighbors if and only if $|i-j| = 1$. Given $b \in \mathbb{N}^\mathbb{N}$ and $\bar{s} = (s_{i,n})_{(i,n) \in 2 \times \mathbb{N}}$ such that $s_{i,n} \in \bigcup_{m \leq n} \{0, \ldots, b(m)\} \times 2^{n-m}$ for all $(i, n) \in 2 \times \mathbb{N}$, define graphs $H_{b,\bar{s},n}$ on $\bigcup_{m \leq n} \{0, \ldots, b(m)\} \times 2^{n-m}$ by setting $H_{b,\bar{s},0} = L_{b(0)}$ and letting $H_{b,\bar{s},n+1}$ be the acyclic connected graph containing $\{(s_i \frown (j))_{i<2} \mid j < 2 \text{ and } (s_i)_{i<2} \in H_{b,\bar{s},n}\}$ and $L_{b(n+1)}$ in which $(s_{0,n}, 0)$ is a neighbor of (0), and $(b(n+1))$ is a neighbor of $(s_{1,n}, 1)$. Set $X_b = \{(c, k, n) \in 2^\mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid k \leq b(n)\}$, define $\pi_{b,n} \colon X_b \cap (2^\mathbb{N} \times \mathbb{N} \times \{0, \ldots, n\}) \to \bigcup_{m \leq n} \{0, \ldots, b(m)\} \times 2^{n-m}$ by $\pi_{b,n}(c, k, m) = (k) \frown c \restriction (n - m)$ for all $n \in \mathbb{N}$, and let $\mathbb{H}_{b,\bar{s}}$ be the digraph on X_b consisting of all pairs of the form $((c_i, k_i, n_i))_{i<2}$ such that $(\pi_{b,n}(c_i, k_i, n_i))_{i<2} \in H_{b,\bar{s},n}$ and $\forall m \geq n \; c_0(m) = c_1(m)$, where $n = \max(n_0, n_1)$.

A *tower over the canonical undirectable forest of lines* is a graph of the form $\mathbb{L}_b = \mathbb{H}_{b,\bar{s}}$, where $b \in \{0\} \times \mathbb{N}^\mathbb{N}$ and $\bar{s} = (s_{i,n})_{(i,n) \in 2 \times \mathbb{N}}$ is given by $s_{i,0} = (0, i)$ and $s_{i,n} = (0)^n \frown (1) \frown (i)$ for all $i < 2$ and $n > 0$. A straightforward Baire category argument shows that if $b \in \{0\} \times (2\mathbb{N} + 1)^\mathbb{N}$, then $\mathbb{L}_b$ does not have a Borel two-coloring. Note that $\mathbb{L}_0$ is the graph $\mathbb{L}_b$, where $b(n) = 2n - 1$, for $n > 0$.

In order to give basis results for quasi-orders substantially stronger than homomorphism, we must introduce two more types of graphs.

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The *symmetrization of the graph of a tower over the odometer* is a graph of the form $\mathbb{\Sigma}_b = \mathbb{H}_{b,\bar{s}}$, where $b \in \{0\} \times \mathbb{N}^\mathbb{N}$ and $\bar{s} = (s_{i,n})_{(i,n) \in 2 \times \mathbb{N}}$ is given by $s_{0,n} = (0) \frown (1)^n \frown (0)$ and $s_{1,n} = (0) \frown (0)^n \frown (1)$. A straightforward Baire category argument shows that if b has alternating parity, then $\mathbb{\Sigma}_b$ does not have a Borel two-coloring.

The *symmetrization of the graph of a tower over the shift on* $[\mathbb{N}]^{\aleph_0}$ is a graph of the form $\mathbb{S}_{b,c} = \mathbb{H}_{b,\bar{s}}$, where $b \in \{0\} \times \mathbb{N}^\mathbb{N}$, $c \in \prod_{n \in \mathbb{N}} \{0, \ldots, b(n)\}$, and $\bar{s} = (s_{i,n})_{(i,n) \in 2 \times \mathbb{N}}$ is given by $s_{i,0} = (0, i)$ and $s_{i,n+1} = (c(n), i)$ for all $i < 2$ and $n \in \mathbb{N}$. A straightforward Baire category argument shows again that if $b \in \{0\} \times (2\mathbb{N} + 1)^\mathbb{N}$, then $\mathbb{\Sigma}_b$ does not have a Borel two-coloring.

Theorem 1. *Suppose that X is a Hausdorff space and G is an analytic graph on X with no Borel two-coloring. Then at least one of the following holds:*

1. *The graph G contains an odd cycle.*
2. *There exists $b \in \{0\} \times \mathbb{N}^\mathbb{N}$ of alternating parity for which there is an injective continuous homomorphism from $\mathbb{\Sigma}_b$ into G.*
3. *There exists $b \in \{0\} \times (2\mathbb{N} + 1)^\mathbb{N}$ for which there is an injective continuous homomorphism from $\mathbb{L}_b$ into G.*
4. *There exist $b \in \{0\} \times (2\mathbb{N} + 1)^\mathbb{N}$ and $c \in \prod_{n \in \mathbb{N}} \{0, \ldots, b(n)\}$ for which there is an injective continuous homomorphism from $\mathbb{S}_{b,c}$ into G.*

Moreover, if G is acyclic, then injective continuous homomorphism can be strengthened to continuous embedding.

Proof (Sketch). Using category- and measure-theoretic arguments, find a locally countable acyclic Borel subgraph of G with Borel chromatic number at least three for which there is a Borel way of selecting one or two ends from each connected component (see e.g. [Mil09]). A case-by-case analysis of the subgraph then yields the theorem. □

This result is sharp, in the sense that there are continuum-sized families of graphs of each of the three types (i.e., $\mathbb{\Sigma}_b$, $\mathbb{L}_b$, and $\mathbb{S}_{b,c}$) such that none has a Borel two-coloring, but every analytic graph on a Hausdorff space that admits an injective Borel homomorphism into at least two of them has a Borel two-coloring.

References

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