The Bounded Axiom A Forcing Axiom

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Abstract

We introduce the Bounded Axiom A Forcing Axiom(BAAFA). It turns out that it is equiconsistent with the existence of a regular Σ_2-correct cardinal and hence also equiconsistent with BPFA. Furthermore we show that, if consistent, it does not imply the Bounded Proper Forcing Axiom(BPFA).

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A plenitude of Forcing Axioms has been considered during the last thirty years. The class of forcing notions satisfying Axiom A has been somewhat neglected, though. We herewith want to draw attention to what we define to be the Bounded Axiom A Forcing Axiom.

Notation 1 *In this paper Σ_{cl} names the class of all countably closed notions of forcing, the class of forcing notions satisfying the countable chain condition is denoted $\mathcal{C}_{\mathrm{cc}}$. By $\mathcal{AA}$ we mean the class of all forcing notions satisfying Axiom A, by $\mathcal{P}_{\mathrm{rop}}$ the class of all proper notions of forcing and by $\mathcal{S}_{\mathrm{sp}}$ the class of forcing notions preserving the stationarity of subsets of $\aleph_1$.*

The class of all complete Boolean algebras by $\mathfrak{cba}$, $\mathrm{ro}(\mathbb{P})$ *denotes the regular open algebra of a forcing notion $\mathbb{P}$. $\overline{\overline{X}}$ is the cardinality and $\varphi^{\prime\prime}X$ the pointwise image of X under φ.* Func *refers to the class of all functions. For an ordinal α and a function f in a forcing extension by $\dot{f}(\alpha)$ we mean a name for $f(\alpha)$.*

For the definition and the properties of the regular open algebra of a forcing notion see [Ku], lemma II.3.3 and section VII.7.

Definition 2 *A forcing notion $\mathbb{P} = \langle P, \leqslant^0_{\mathbb{P}}\rangle$ satisfies* Axiom A *if and only if there is a sequence $\langle \leqslant^n_{\mathbb{P}} | n \in \omega \setminus 1\rangle$ of partial orders on P such that*

1. *$p \leqslant^{n+1}_{\mathbb{P}} q$ implies $p \leqslant^n_{\mathbb{P}} q$ for all $p, q \in P$ and all $n < \omega$.*
2. *If $\langle p_n | n < \omega\rangle$ is a sequence of conditions with the property $\forall n < \omega$: $p_{n+1} \leqslant^n_{\mathbb{P}} p_n$ then there exists a conditon p such that $p \leqslant^n_{\mathbb{P}} p_n$ for all $n < \omega$.*
3. *For every $n < \omega$, every $p \in \mathbb{P}$ and every antichain A there exists a $q \leqslant^n_{\mathbb{P}} p$ such that the antichain $\{r \in A | r \|^0_{\mathbb{P}} q\}$ is countable.*

Here $p\|^0_{\mathbb{P}}q$ means that p and q are compatible with respect to the partial ordering $\leqslant^0_{\mathbb{P}}$.

Both all countably closed and all ccc forcings satisfy Axiom A. All Axiom A forcings are proper. So the situation is as follows:

$$\Sigma_{\mathrm{cl}} \cup \mathcal{C}_{\mathrm{cc}} \subset \mathcal{AA} \subset \mathcal{P}_{\mathrm{rop}} \tag{1}$$

That countably closed notions of forcing satisfy Axiom A is seen by setting $\leqslant^n$ to be simply $\leqslant^0$ for all $n \in \omega \setminus 1$. That ccc forcings satisfy Axiom A follows

by setting $\leqslant_n$ to be the identity for all $n \in \omega \setminus 1$. If some notion of forcing satisfies Axiom A Player II has a winning strategy in the proper game below any condition of this forcing notion. Hence Axiom A forcings are proper. For a detailed proof see [Je], p. 604.

Definition 3 *Let $\mathcal{C}$ be a class of complete Boolean algebras and κ, λ be cardinals. The* Bounded Forcing Axiom for $\mathcal{C}$, κ and λ $\big($BFA$(\mathcal{C}, \kappa, \lambda)\big)$ *states the following:*

Whenever $\mathbb{P} \in \mathcal{C}$ and $\mathcal{A}$ is a family of less than κ maximal antichains of cardinality less than λ there exists an $\mathcal{A}$-generic filter.

Definition 4

- *Bounded Martin's Maximum*(BMM) is BFA$(\mathcal{S}_{\text{sp}} \cap \mathfrak{cba}, \aleph_2, \aleph_2)$.
- The *Bounded Proper Forcing Axiom*(BPFA) is BFA$(\mathcal{P}_{\text{rop}} \cap \mathfrak{cba}, \aleph_2, \aleph_2)$.

Whereas satisfying the countable chain condition, being proper and preserving the stationarity of subsets of $\aleph_1$ are properties of a forcing notion, meaning them holding for a poset or its Boolean completion is equivalent in each case, the same does not seem to be true for Axiom A. One direction, however can be proven.

Remark 5 *If a poset satisfies Axiom A, then its Boolean completion does so too.*

If $\mathbb{P}$ is a poset satisfying Axiom A, witnessed by the sequence of partial orderings $\langle \leqslant^n_{\mathbb{P}} \mid n < \omega \rangle$ and $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \text{ro}(\mathbb{P})$ is the canonical dense embedding into its Boolean completion then for $a, b \in \text{ro}(\mathbb{P})$ set $a \leqslant^0_{\text{ro}(\mathbb{P})} b$ iff $a \leqslant_{\text{ro}(\mathbb{P})} b$ and for every $n \in \omega \setminus 1$ set $a \leqslant^n_{\text{ro}(\mathbb{P})} b$ if either $a = b$ or $a <_{\text{ro}(\mathbb{P})} b$, $a \in \delta_{\mathbb{P}}{}^{\text{“}}\mathbb{P}$ and if $b \in \delta_{\mathbb{P}}{}^{\text{“}}\mathbb{P}$ then $\delta^{-1}_{\mathbb{P}}(a) \leqslant^n_{\mathbb{P}} \delta^{-1}_{\mathbb{P}}(b)$.

Definition 6

The *Bounded Axiom A Forcing Axiom*(BAAFA) is BFA$(\mathcal{AA} \cap \mathfrak{cba}, \aleph_2, \aleph_2)$. (2)

Note that by remark being a complete Boolean algebra satisfying Axiom A is equivalent to being a completion of an Axiom A poset!

Definition 7 *An ordinal α is called Σ_n-correct iff $V_\alpha \prec_{\Sigma_n} V$.*

For regular cardinals, being Σ_n-correct is a large cardinal notion. In fact the regular Σ_1-correct cardinals are precisely the strongly inacessible cardinals. On the one hand, $H_\kappa = V_\kappa$ holds for inacessible κ and any Σ_2-assertion believed by a H_λ for an uncountable cardinal λ is true, hence inacessibles are Σ_1-correct. On the other hand, since any infinite cardinal is a limit ordinal and for any $\lambda < \kappa$ it is $\Sigma_1\big(\{\mathcal{P}(\lambda)\}\big)$-expressible that 2^λ exists, Σ_1-correct cardinals have to be strong limits.

Our main theorem is the following:

Theorem 8 *If there is a regular Σ_2-correct cardinal κ then there is a generic extension by a notion of forcing satisfying both Axiom A and the κ-c.c. so that in the generic extension* ZFC $+ 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 +$ BAAFA *holds, yet* BPFA *fails.*

Note that regular Σ_2-correct cardinals are also known as Σ_2-reflecting!

In order to prove theorem 8 we employ an iterated forcing construction analogous to both the one of Kunen in [Ku] for the consistency proof of Martin's Axiom and the one of Shelah in [G-S] for the consistency proof of BPFA.

In order for this construction to be analogous to the two above we need the following theorem.

Theorem 9 (Koszmider, 1993) *Suppose $\big(\langle \mathbb{P}_\alpha, \leqslant_\alpha, \mathbb{1}_\alpha\rangle, \langle \pi_\alpha, \preccurlyeq_\alpha, \iota_\alpha\rangle\big)$ is an iteration of forcing notions with countable support of length κ such that $\mathbb{1}_\alpha \Vdash_{\mathbb{P}_\alpha}$ "π_α satisfies Axiom A." for all $\alpha < \kappa$. Then $\mathbb{P}_\kappa$ satisfies Axiom A.*

See [Ko] for a proof.

Lemma 10 *That a notion of forcing $\mathbb{P}$ satisfies Axiom A is a $\Sigma_2(\{\mathbb{P}\})$-property.*

Proof. Let $\mathbb{P} = \langle P, \leqslant^0\rangle$ be a notion of forcing. $\mathbb{P}$ satisfies Axiom A iff the following holds:

$$
\begin{aligned}
&\exists X, \langle \leqslant^n | n \in \omega \setminus 1\rangle, f\Big(\forall S \subset P : S \in X, \mathrm{dom}(f) = P \times X \times \omega \times \omega, \qquad (3)\\
&\forall n < \omega\big(\leqslant^n \text{ is a partial ordering of } P \text{ and } \forall p, q \in P(p \leqslant^{n+1} q \to p \leqslant^n q)\big),\\
&\forall \langle p_n | n < \omega\rangle\big((\forall n < \omega : p_{n+1} \leqslant^n q_n) \to \exists p \in P \forall n < \omega : p \leqslant^n p_n\big)\\
&\text{and } \forall p \in P \forall n < \omega \forall A \in X(A \text{ is an antichain}\\
&\to \exists r \in P(r \leqslant^n p \wedge \{s \in A | s\|^0 r\} \subset f''(\{p\} \times \{A\} \times \{n\} \times \omega))\Big)
\end{aligned}
$$

Here f plays the role of a family of Skolem-functions witnessing the countability of the $\{s \in A | s \|^0 r\}$'s. (3) is indeed a Σ_2-formula since all statements expressed in english and $\mathrm{dom}(f) = Q \times X \times \omega \times \omega$ are Σ_0-expressible. $\dashv$

Remark 11 *If $\alpha \in \mathrm{Lim}$, $\mathbb{P}_\alpha$ is an iterated forcing construction of length α with finite or countable support, G_α is $\mathbb{P}_\alpha$-generic, $\big(\beta < \mathrm{cf}(\alpha)\big)^{V[G_\alpha]}$ and $S \in \mathcal{P}(\beta) \cap V[G_\alpha]$ then already $S \in V[G_\gamma]$ for some $\gamma < \alpha$.*

Now we are going to prove theorem 8. So let κ be regular and Σ_2-correct. We use a standard bookkeeping device to force BAMAFA with an iterated forcing $\mathbb{Q}_\kappa = \Big\langle \big\langle (\mathbb{Q}_\gamma, \leqslant_\gamma, \mathbf{1}_\gamma), \langle \pi_\gamma, \preccurlyeq_\gamma, \iota_\gamma \rangle \big\rangle \big| \gamma < \kappa \Big\rangle$ of length κ with countable support. To this end, choose a surjection $f : \kappa \setminus \mathrm{Lim} \twoheadrightarrow \kappa^2$ such that $f(\gamma) < \kappa \cdot (\gamma + 1)$ for all $\gamma < \kappa$. Furthermore for $\gamma < \kappa$ let

$$e_\gamma : \kappa \twoheadrightarrow \{\langle \dot{\mathbb{S}}, \dot{\mathcal{A}} \rangle | \dot{\mathcal{A}}, \dot{\mathbb{S}} \in H_\kappa \text{ are } \mathbb{Q}_\gamma\text{-names and } \mathbf{1}_\gamma \Vdash_{\mathbb{Q}_\gamma} \text{“}\dot{\mathbb{S}} \in H_{\check{\kappa}} \text{ is a Boolean algebra and } \dot{\mathcal{A}} \text{ is a family of maximal antichains in } \mathbb{S}.\text{”}\} \tag{4}$$

be an enumeration. This is possible since κ is in particular inaccessible and hence $\overline{\overline{H_\kappa}} = \kappa$. Whenever $\eta \leqslant \gamma$ one canonically can embed $\mathbb{Q}_\eta$ into $\mathbb{Q}_\gamma$ so we always can conceive of $\mathbb{Q}_\eta$-names as $\mathbb{Q}_\gamma$-names in such a situation. So for the names $\dot{\mathbb{S}}$, $\dot{\mathcal{A}}$ from (4) let $\beta^\gamma_{\langle \dot{\mathbb{S}}, \dot{\mathcal{A}} \rangle}$ be a $\mathbb{Q}_\gamma$-name from H_κ such that

$$\mathbf{1}_\gamma \Vdash_{\mathbb{Q}_\gamma} \text{“If there is a complete Boolean algebra } \mathbb{B} \text{ satisfying Axiom A and extending } \dot{\mathbb{S}} \text{ such that every antichain of } \dot{\mathcal{A}} \text{ is still a maximal antichain in } \mathbb{B} \text{ then } \beta^\gamma_{\langle \dot{\mathbb{S}}, \dot{\mathcal{A}} \rangle} \text{ is such a } \mathbb{B}\text{, otherwise } \beta^\gamma_{\langle \dot{\mathbb{S}}, \dot{\mathcal{A}} \rangle} \text{ is the trivial Boolean algebra.”.} \tag{5}$$

Now we can use f for bookkeeping. Set $\pi_\gamma := \beta^\gamma_{e_\eta(\xi)}$ iff $f(\gamma) = \kappa \cdot \eta + \xi$ for $\gamma \in \kappa \setminus \mathrm{Lim}$. If $\gamma < \kappa$ is a limit ordinal then let π_γ be a name for the forcing notion which adds a surjection from $\aleph_1$ to γ with countable conditions, that is $\mathbb{R} := \{p | p \in \mathrm{Func} \wedge \mathrm{dom}(p) \in [\aleph_1]^{<\omega_1} \wedge \mathrm{ran}(p) \subset \gamma\}$ and $p \leqslant_{\mathbb{R}} q$ whenever $p \supset q$.

Note that since the π_γ are all chosen from H_κ all initial segments of our iterated forcing construction are in H_κ too. Beside this recognize that we have $\mathbf{1}_\gamma \Vdash_{\mathbb{Q}_\gamma}$ “π_γ satisfies Axiom A.” for all $\gamma < \kappa$.

- $\mathbb{Q}_\kappa$ has the κ-c.c.—Proof:
  Suppose towards a contradiction that $\{q_\alpha|\alpha<\kappa\}$ was an antichain in $\mathbb{Q}_\kappa$. Consider $\mathcal{B}:=\{\mathrm{supt}(q_\alpha)|\alpha<\kappa\}$. We distinguish two cases:

  - If $\overline{\overline{\mathcal{B}}}=\kappa$ apply [Ku], lemma II.1.6. This is just a generalized Δ-system-lemma. So there is a Δ-system $\mathcal{C}\subset\mathcal{B}$ with root r of size κ. Then define $\mathcal{D}:=\{q_\alpha|\alpha<\kappa\wedge\mathrm{supt}(q_\alpha)\in\mathcal{C}\}$.
  - If $\overline{\overline{\mathcal{B}}}<\kappa$ then by the pigeonhole principle there exists an $r\in[\kappa]^{<\omega_1}$ and an $s\in[\kappa]^\kappa$ such that $\forall\alpha\in s:\ \mathrm{supt}(q_\alpha)=r$. Define $\mathcal{D}:=\{q_\alpha|\alpha\in s\}$.

  Set $\gamma:=\sup(r)+1$. In both cases $\{q\restriction\gamma|q\in\mathcal{D}\}$ is an antichain of size κ in $\mathbb{Q}_\gamma$. But $\mathbb{Q}_\gamma\in H_\kappa$ so in particular $\overline{\overline{\mathbb{Q}_\gamma}}<\kappa\not\downarrow$ $\dashv$ ($\mathbb{Q}_\kappa$ has the κ-c.c..)

  This immediately implies that κ remains a cardinal in the generic extension.

- Next we want to see that $V[G_\kappa]\models$ "$2^{\aleph_1}\leqslant\kappa$"—Proof:
  From fact 11 we know that every subset of $\aleph_1$ in $V[G_\kappa]$ is already in some $V[G_\gamma]$ where $\gamma<\kappa$. Every set in $\mathcal{P}(\aleph_1)\cap V[G_\gamma]$ is represented by a nice $\mathbb{Q}_\gamma$-name. Let $\lambda_\gamma:=\overline{\overline{\mathbb{Q}_\gamma}}+\aleph_1$. There are at most $(2^{\lambda_\gamma})^{\aleph_1}=2^{\lambda_\gamma}<\kappa$ such names. So there are at most $\Sigma_{\gamma<\kappa}2^{\lambda_\gamma}\leqslant\kappa\cdot\kappa=\kappa$ subsets of $\aleph_1$ in $V[G_\kappa]$. Note that at this point we needed κ's regularity.

  $\dashv$ ($V[G_\kappa]\models$ "$2^{\aleph_1}\leqslant\kappa$")

  Since $2^{\aleph_1}>\aleph_1$ this also shows for the second time that κ is not collapsed and so remains a cardinal in the generic extension. Moreover it is now clear that $V[G_\kappa]\models$ "$\kappa=\aleph_2$". Note that if we succeed in showing $V[G_\kappa]\models$ BAAFA because of BAAFA $\Rightarrow$ $\mathrm{MA}_{\aleph_1}$ $\Rightarrow$ $2^{\aleph_0}\geqslant\aleph_2$ we will have proved everything apart from the failure of BPFA.

BAAFA holds by the following argument:

We work in $V[G_\kappa]$. Let $\mathbb{B}^*$ be a Boolean algebra satisfying Axiom A and $\mathcal{A}^*=\{A_\beta|\beta<\aleph_1\}$ be a family of at most $\aleph_1$ maximal antichains of cardinality at most $\aleph_1$. Let $\mathbb{S}^*$ be the subalgebra finitely generated by $\bigcup_{\beta<\aleph_1}A_\beta$. We can choose a Boolean algebra $\mathbb{B}$ isomorphic to $\mathbb{B}^*$ by an isomorphism $\psi:\mathbb{B}^*\simeq\mathbb{B}$ such that the subalgebra finitely generated by $\bigcup_{\beta<\aleph_1}\psi``A_\beta$ is in $H_{\aleph_2}$. Let $\mathbb{S}$ denote this subalgebra. $\mathbb{S}$ and $\mathcal{A}:=\{\psi``A_\beta|\beta<\aleph_1\}$ are already present in some $V[G_\gamma]$ for $\gamma<\kappa$. At some point later on $\mathcal{A}$ and $\mathbb{S}$ are considered, say at $\eta\in\kappa\setminus\gamma$.

Now we consider the following statement:

$$\text{There is a Boolean algebra } \mathbb{B}' \text{ satisfying Axiom A and extending } \mathbb{S} \text{ such that every antichain from } \mathcal{A} \text{ is still a maximal antichain in } \mathbb{B}'. \tag{6}$$

Note that this is a Σ_2-statement. Since it is true in the final $V[G_\kappa]$ as witnessed by $\mathbb{B}$ there is a condition forcing this. Since the iteration uses countable support every(such) condition is already in a $\mathbb{Q}_\zeta$ for a $\zeta \in \kappa \setminus \gamma$. In fact in every $V[G_\xi]$ for $\xi \in \kappa \setminus \gamma$ there is a $\mathbb{B}'$ witnessing the truth of (6). given by the following

Claim 12 *If $q \in \frac{\mathbb{Q}_\kappa}{G_\xi}$ is a condition of the tail-forcing and β is a $\frac{\mathbb{Q}_\kappa}{G_\xi}$-name such that $q \Vdash_{\frac{\mathbb{Q}_\kappa}{G_\xi}}$ "β is a Boolean superalgebra of $\mathbb{S}$ satisfying Axiom A such that every maximal antichain in $\check{\mathcal{A}}$ stays maximal in β" and if one sets $\mathbb{R} := \{r \in \frac{\mathbb{Q}_\kappa}{G_\xi} | r \leqslant_{\frac{\mathbb{Q}_\kappa}{G_\xi}} q\}$, then $\mathrm{ro}(\mathbb{R} \star \beta^+)$ is isomorphic to an algebra witnessing the truth of (6).*

Proof. First note that $\mathrm{ro}(\mathbb{R} \star \beta^+)$ can be seen as an extension of $\mathbb{S}$ via the following embedding:

$$\chi : \mathbb{S} \longrightarrow \mathrm{ro}(\mathbb{R} \star \beta^+) \tag{7}$$

$$s \longmapsto \begin{cases} \delta\big((q, \check{s})\big) & \text{iff } s \neq 0_{\mathbb{S}} \\ 0_{\mathrm{ro}(\mathbb{R} \star \beta^+)} & \text{otherwise.} \end{cases} \tag{8}$$

Let us choose an algebra $\mathbb{B}' \supset \mathbb{S}$ which is isomorphic to $\mathrm{ro}(\mathbb{R} \star \beta^+)$. Now suppose towards a contradiction that there is an $A \in \mathcal{A}$ and a $b' \in \mathbb{B}'$ which is incompatible with all elements from $\mathcal{A}$. By density of $\delta''(\frac{\mathbb{Q}_\kappa}{G_\xi} \star \beta^+)$ in $\mathbb{B}'$ there is a $(q, \tau) \in \frac{\mathbb{Q}_\kappa}{G_\xi} \star \beta^+$ such that $\delta((q, \tau)) \leqslant_{\mathbb{B}'} b'$. We have

$$\forall a \in A : \delta((q, \tau)) \perp_{\mathbb{B}'} \chi(a) \tag{9}$$

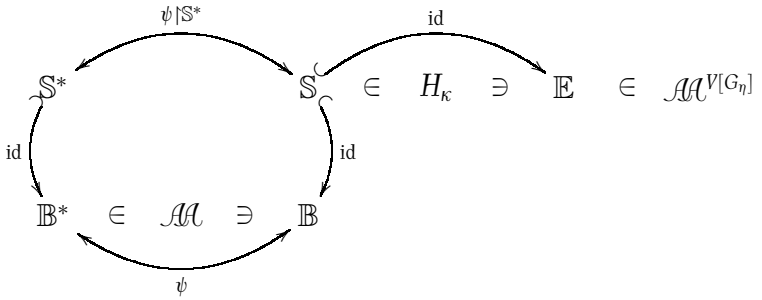
Since δ is a dense embedding it follows immediately that

$$\forall a \in A : (q, \tau) \perp_{\frac{\mathbb{Q}_\kappa}{G_\xi} \star \beta^+} (p, \check{a}). \tag{10}$$

But then

$$\forall a \in A : q \Vdash_{\frac{\mathbb{Q}_\kappa}{G_\xi}} \text{“}\tau \perp_{\beta^+} \check{a}\text{”}. \tag{11}$$

Since κ is still Σ_2-correct in $V[G_\eta]$ we can choose a witness $\mathbb{E} \in \mathcal{AA} \cap \mathfrak{cba} \cap H_\kappa$. Hence we force nontrivially in step η. So $V[G_{\eta+1}] \subset V[G_\kappa]$ contains an $\mathcal{A}$-generic filter over $\mathbb{E}$. But then it also contains an $\mathcal{A}^*$-generic filter over $\mathbb{B}^*$ which is what we wanted to prove.



We are going to show that BPFA fails in the generic extension. Indeed we are going to show more than that. We are going to show that $\neg$ BFA $\left(\{\mathrm{ro}(\mathbb{P}_{\mathrm{acfc}})\}, \aleph_2, \aleph_2\right)$ for some specific proper notion of forcing $\mathbb{P}_{\mathrm{acfc}}$ which is the following one:

Definition 13

$$\mathbb{P}_{acfc} := \{p | p \in [\aleph_1 \times \aleph_1]^{<\omega} \wedge \exists f \supset p : f : \aleph_1 \longrightarrow \aleph_1 \textit{ is a normal function.}\} \tag{12}$$

The conditions are ordered by extension, bigger conditions being stronger.

This forcing is due to Baumgartner, [Ba], and adds a club with finite conditions—hence the index. It is well-known that this is a proper notion of forcing failing to satisfy Axiom A. The latter is usually proved by employing a variant of the following game. Since up to now it seems to have no name, we refer to it as the *strong proper game*.

Definition 14 *Let $\mathbb{P}$ be a notion of forcing and $p \in \mathbb{P}$ a condition. The* strong proper game for $\mathbb{P}$ below p *is played as follows: In move $n < \omega$ Player I plays a maximal antichain A_n to which Player II responds by playing a countable $B_n \subset A_n$. After ω moves Player II wins iff there is a $q \leqslant p$ such that the antichain B_n is predense below q for every $n < \omega$.*

See for example [Je], p. 613.

We need the following fact:

Lemma 15 *If $\mathbb{P}$ is a notion of forcing which satisfies Axiom A then for every $p \in \mathbb{P}$ Player II has a winning strategy in the strong proper game for $\mathbb{P}$ below p.*

An important point is that $\mathbb{P}_{\text{acfc}}$ is absolute between transitive models of set theory with the same $\aleph_1$. In particular this means that it is absolute between our ground model and its generic extension by $\mathbb{Q}_\kappa$. These are consequences of the fact that transitve models sharing their $\aleph_1$ also have the same finite sets of countable ordinals together with the following lemma.

Lemma 16 *$p \in \mathbb{P}_{acfc}$ is a $\Delta_1(\{\aleph_1, p\})$-relation.*

Proof. The original definition yields the $\Sigma_1(\{\aleph_1, p\})$-definition. The following formula provides the $\Pi_1(\{\aleph_1, p\})$-definition.

$$p \in \text{Func} \wedge \text{dom}(p) \subset \aleph_1 \wedge \nexists g : \omega \hookrightarrow \text{dom}(p) \wedge \forall g, \langle g_\gamma | \gamma < \beta \rangle, \alpha \in \text{dom}(p) \tag{13}$$
$$\big((\alpha < \beta \wedge \beta \in \text{dom}(p) \wedge \nexists \gamma \in \text{dom}(p) : \alpha < \gamma \wedge \gamma < \beta) \rightarrow \big(p(\alpha) < p(\beta) \wedge$$
$$\beta \leqslant p(\beta) \wedge \nexists \gamma < \beta (g : p(\beta) \setminus p(\alpha) \longrightarrow \gamma \setminus \alpha \text{ is order-preserving.}) \wedge (\beta \in \text{Lim}$$
$$\rightarrow \forall \gamma < p(\beta) \exists \eta < \beta \nexists \zeta < \beta (g_\eta : p(\beta) \setminus \gamma \longrightarrow \zeta \setminus \eta \text{ is order-preserving.}))\big)\big)$$

Let us first check that this is necessary. So suppose in search for a contradiction that f were a normal function extending p yet there were counterexamples $g, \langle g_\gamma | \gamma < \beta \rangle, \alpha \in \text{dom}(p)$ to (13). Since (13) is supposed to fail $\alpha < \beta$ and $\beta \in \text{dom}(p)$ have to hold. We know that $p(\alpha) < p(\beta)$ and $\beta \leqslant p(\beta)$ because f is a normal function extending p. We distinguish two cases.

- β is a successor ordinal. (13) fails so $\exists \gamma < \beta (g : p(\beta) \setminus p(\alpha) \longrightarrow \gamma \setminus \alpha$ is order-preserving.). $f \supset p$ is in particular order-preserving hence $g \circ \big(f \restriction (\beta \setminus \alpha)\big) : \beta \setminus \alpha \longrightarrow \gamma \setminus \alpha$ is order-preserving which is absurd.
- β is a limit ordinal. Then $\exists \gamma < \beta (g : p(\beta) \setminus p(\alpha) \longrightarrow \gamma \setminus \alpha$ is order-preserving.) which yields a contradiction as we just saw or $\exists \gamma < p(\beta) \forall \eta < \beta \exists \zeta < \beta (g_\eta : p(\beta) \setminus \gamma \longrightarrow \zeta \setminus \eta$ is order-preserving.). Fix a $\gamma < p(\beta)$ such that $\forall \eta < \beta \exists \zeta < \beta (g_\eta : p(\beta) \setminus \gamma \longrightarrow \zeta \setminus \eta$ is order-preserving.). Define $\eta := \min\{\xi < \beta | f(\xi) \geqslant \gamma\}$ and choose a $\zeta < \beta$ such that $g_\eta : p(\beta) \setminus \gamma \longrightarrow \zeta \setminus \eta$ is order-preserving. So $\big(g_\eta \circ f \restriction (\beta \setminus \eta)\big) : \beta \setminus \eta \longrightarrow \zeta \setminus \eta$ is order-preserving. Contradiction!

Now we have to show that this is sufficient. For $x \in p$ let x_d, x_r be abbreviations for $\bigcup \mathrm{dom}(\{x\})$ and $\bigcup \mathrm{ran}(\{x\})$ respectively. Let $e : n \longleftrightarrow p$ be an order-preserving enumeration of p, that is $l < m$ implies $e(l)_d < e(m)_d$. We are going to inductively construct a normal function f_n extending p, provided that (13) is fulfilled. To begin with, let $f_1 := \{e(0)\}$. Suppose we succeeded in defining a normal function f_m extending $e``m$ for some $m < n$. We are now going to define $f_{m+1} \supset e`(m+1)$. In order to do this we distinguish two cases:

- $e(m)_d$ is a successor ordinal. Then the following is a normal function extending $e`(m+1)$:

$$f_{m+1} : e(m)_d + 1 \longrightarrow e(m)_r + 1 \tag{14}$$
$$\xi \longmapsto \begin{cases} f_m(\xi) & \text{iff } \xi \leqslant e(m-1)_d \\ e(m-1)_r + \big(- e(m-1)_d + \xi \big) & \text{iff } \xi \in e(m)_d \setminus e(m-1)_d \\ e(m)_r & \text{iff } \xi = e(m)_d \end{cases}$$

  This could only fail to be a normal function if $f_{m+1}(\gamma) = e(m)_r$ for some $\gamma < e(m)_d$. But then $f_{m+1}^{-1} \restriction \big(e(m)_r \setminus e(m-1)_r\big)$ would be an order-preserving function from $e(m)_r \setminus e(m-1)_r$ into $\gamma \setminus e(m-1)_d$ contradicting (13).

- $e(m)_d$ is a limit ordinal. Choose a sequence $\langle \chi_n | n < \omega \rangle$ which is cofinal in $e(m)_r$. Let $n(\xi)$ denote the smallest $n < \omega$ such that $\chi_n > \xi$. Then the following is a normal function extending $e`(m+1)$.

$$f_{m+1} : e(m)_d + 1 \longrightarrow e(m)_r + 1 \tag{15}$$
$$\xi \longmapsto \begin{cases} f_m(\xi) & \text{iff } \xi \leqslant e(m-1)_d \\ \sup_{\vartheta < \xi} f_{m+1}(\vartheta) & \text{iff } \xi \in \mathrm{Lim} \cap e(m)_r \setminus e(m-1)_d \\ \chi_{(n \circ f)(\vartheta)} & \text{iff (16)} \\ f_{m+1}(\vartheta) + 1 & \text{iff (17)} \\ e(m)_r & \text{iff } \xi = e(m)_d. \end{cases}$$

  Here (16) and (17) stand for the following cases:

$$\xi = \vartheta + 1 \text{ for some } \vartheta \in e(m)_d \setminus e(m-1)_d \text{ and} \tag{16}$$
$$\mathrm{otyp}\big(e(m)_d \setminus \vartheta\big) \leqslant \mathrm{otyp}\big(e(m)_r \setminus \chi_{(n \circ f_{m+1})(\vartheta)}\big).$$
$$\xi = \vartheta + 1 \text{ for some } \vartheta \in e(m)_d \setminus e(m-1)_d \text{ and} \tag{17}$$
$$\mathrm{otyp}\big(e(m)_d \setminus \vartheta\big) > \mathrm{otyp}\big(e(m)_r \setminus \chi_{(n \circ f_{m+1})(\vartheta)}\big).$$

The first two cases ensure that f_{m+1} is continuous on $e(m)_d$ so it could only fail to be a normal function by being discontinous at $e(m)_d$. But the discrimination between the cases (16) and (17) ensures both that $f_{m+1}{}^{\text{“}}e(m)_d \subset e(m)_r$ and that $f_{m+1}{}^{\text{“}}e(m)_d$ is cofinal in $e(m)_r$. In order to see that the latter holds suppose otherwise. We will reach a contradiction by constructing a sequence of functions $\langle g_\gamma | \gamma < \beta \rangle$ such that (13) fails. Let n be minimal such that $\chi_n \geqslant \sup\left(f_{m+1}{}^{\text{“}}e(m)_d\right)$. The choice of n implies that case (17) obtains for all successor ordinals ξ such that $f_{m+1}(\xi) > \chi_{n-1}$. But this means that there is a $\gamma < e(m)_d$ such that for all $\eta \in e(m)_d \setminus \gamma$ we have $\mathrm{otyp}\left(e(m)_d \setminus \eta\right) > \mathrm{otyp}\left(e(m)_r \setminus \chi_n\right)$. So for all $\eta \in e(m)_d \setminus \gamma$ we can choose a $\zeta_\eta < e(m)_d$ and an order-preserving. $g_\eta : e(m)_r \setminus \chi_n \longrightarrow \zeta_\eta \setminus \eta$. By additionally setting $g_\xi := g_\gamma$ for all $\xi < \gamma$ we define a family $\langle g_\xi | \xi < e(m)_d \rangle$ of functions.

But (13) implies that $\forall \gamma < e(m)_r \exists \eta < e(m)_d \nexists \zeta < e(m)_d (g_\eta : e(m)_r \setminus \gamma \longrightarrow \zeta \setminus \eta$ is order-preserving.).Contradiction!

$\dashv$

We need another fact, recall that an ordinal α is called *indecomposable* iff $\beta, \gamma < \alpha$ implies $\beta + \gamma < \alpha$ or equivalently iff $\alpha = \omega^\beta$ for some $\beta \leqslant \alpha$.

Lemma 17 *Let $\langle \alpha_n | n < \omega \rangle$ be a strictly increasing sequence of indecomposable countable ordinals and $\langle \beta_n | n < \omega \rangle$ be a sequence of ordinals such that $\beta_n < \alpha_{n+1}$ holds for all $n < \omega$. The following sets are dense in $\mathbb{P}_{acfc}$:*

$$D^{\langle \alpha_n | n<\omega \rangle}_{\langle \beta_n | n<\omega \rangle} := \{p \in \mathbb{P}_{acfc} | \exists n < \omega : \alpha_n \in \mathrm{dom}(p) \wedge p(\alpha_n) \geqslant \beta_n\}. \tag{18}$$

Proof. Choose a $p \in \mathbb{P}_{\text{acfc}}$ and a normal function $f \supset p$ witnessing this. Define $\alpha := \sup_{n<\omega} \alpha_n$, $\eta := \max\left(\mathrm{dom}(p) \cap \alpha\right)$ and $n := \min\{m < \omega | \eta < \alpha_m\}$. We distinguish two cases.

- $f(\alpha_n) \geqslant \beta_n$. Then let $q := p \cup \{\langle \alpha_n, f(\alpha_n) \rangle\}$. $q \in D^{\langle \alpha_n | n<\omega \rangle}_{\langle \beta_n | n<\omega \rangle}$ and f testifies that $q \in \mathbb{P}_{\text{acfc}}$ at all.
- $f(\alpha_n) < \beta_n$. In this case define $q := p \cup \{\langle \alpha_n, \beta_n + \alpha_n \rangle\}$. Again $q \in$

$D^{\langle\alpha_n|n<\omega\rangle}_{\langle\beta_n|n<\omega\rangle}$. The following normal function witnesses that $q \in \mathbb{P}_{\text{acfc}}$.

$$g : \aleph_1 \longrightarrow \aleph_1 \tag{19}$$
$$\delta \longmapsto \begin{cases} f(\delta) & \text{iff } \delta \in (\eta+1) \cup \left(\aleph_1 \setminus (\alpha_{n+1}+1)\right) \\ \beta_n + \delta & \text{iff } \delta \in (\alpha_{n+1}+1) \setminus (\eta+1) \end{cases}$$

g clearly is a normal function on $\eta+1$, $(\alpha_{n+1}+1)\setminus(\eta+1)$ and $\aleph_1\setminus(\alpha_{n+1}+1)$ because f and $\delta \mapsto \beta_n + \delta$ are normal functions. Moreover we have

$$g(\eta) = f(\eta) < f(\alpha_n) < \beta_n < \beta_n + \eta + 1 = g(\eta+1) \tag{20}$$
$$\text{and } g(\alpha_{n+1}) = \beta_n + \alpha_{n+1} = \alpha_{n+1} \tag{21}$$
$$< \alpha_{n+1} + 1 \leqslant f(\alpha_{n+1}+1) = g(\alpha_{n+1}+1).$$

In any of both cases $q \in \mathbb{P}_{\text{acfc}} \cap D^{\langle\alpha_n|n<\omega\rangle}_{\langle\beta_n|n<\omega\rangle}$ is an extension of p. Since p was arbitrarily chosen this shows the density of $D^{\langle\alpha_n|n<\omega\rangle}_{\langle\beta_n|n<\omega\rangle}$. ⊣

Now we show $\mathbf{1}_{\mathbb{Q}_\kappa} \Vdash_{\mathbb{Q}_\kappa}$ "$\neg$ BFA $\left(\{\text{ro}(\mathbb{P}_{\text{acfc}})\}, \aleph_2, \aleph_2\right)$". So assume towards a contradiction that there is a $q \in \mathbb{Q}_\kappa$ such that $q \Vdash_{\mathbb{Q}_\kappa}$ 'BFA $\left(\{\text{ro}(\mathbb{P}_{\text{acfc}})\}, \aleph_2, \aleph_2\right)$". Define $\mathcal{D} := \{D \subset \mathbb{P}_{\text{acfc}} | D \text{ is dense.}\}$. Since κ is rendered $\aleph_2$ in the generic extension we have $\mathbf{1}_{\mathbb{Q}_\kappa} \Vdash_{\mathbb{Q}_\kappa}$ "$\overline{\overline{\mathcal{D}}} < \aleph_2$" and $\mathbf{1}_{\mathbb{Q}_\kappa} \Vdash_{\mathbb{Q}_\kappa}$ "$\forall D \in \mathcal{D} : \overline{\overline{D}} < \aleph_2$". Let $G \ni q$ be a $\mathbb{Q}_\kappa$-generic filter. We work in $V[G]$. Let $\mathbb{B} := \text{ro}(\mathbb{P}_{\text{acfc}})$ and let $\delta : \mathbb{P}_{\text{acfc}} \longrightarrow \mathbb{B}$ be the canonical dense embedding. Let $\mathcal{D}_{\mathbb{B}} := \{\delta``D | D \in \mathcal{D}\}$. Since $q \in G$ our assumption implies the existence of a $\mathcal{D}_{\mathbb{B}}$-generic filter H in $V[G]$. Since the sets $D_\alpha = \{p \in \mathbb{P}_{\text{acfc}} | \alpha \in \text{dom}(p)\}$ are both in V and dense in $\mathbb{P}_{\text{acfc}}$, one can define a normal function as follows:

$$f : \aleph_1 \longrightarrow \aleph_1 \tag{22}$$
$$\alpha \longmapsto \text{"the unique } \beta \text{ such that } \delta(\{\langle\alpha,\beta\rangle\}) \in H."$$

Back to V by lemma 15 Player II has a winning strategy in the strong proper game for $\mathbb{Q}_\kappa$ below q. Let $\dot{f}$ be a $\mathbb{Q}_\kappa$-name for f.

In the role of Player I in the strong proper game for $\mathbb{Q}_\kappa$ below q in move 0 we set $\alpha_0 = 0$ and play a maximal antichain A_0 which decides $\dot{f}(0)$. In general our opponent in move n plays a countable $B_n \subset A_n$ according to her strategy. We set $\beta_n := \sup\left(\{\beta < \aleph_1 | \exists q \in B_n : q \Vdash_{\mathbb{Q}_\kappa} \text{"}\dot{f}(\alpha_n) = \beta\text{"}\}\right) + 1$ and choose an

indecomposable countable ordinal $\alpha_{n+1} \in \aleph_1 \setminus \beta_n$. Finally we play a maximal antichain A_{n+1} which decides $\dot{f}(\alpha_{n+1})$.

The play of this game yields a sequence of countable indecomposable ordinals $\langle \alpha_n | n < \omega \rangle$ and a sequence of ordinals $\langle \beta_n | n < \omega \rangle$ such that $\beta_n < \alpha_{n+1}$ for all $n < \omega$. So we can apply lemma 17. We choose a witness $r \leqslant_{\mathbb{Q}_\kappa} q$ for II's strategy to be winning. Choose an $s \leqslant_{\mathbb{Q}_\kappa} r$ and a $p \in D^{\langle \alpha_n | n<\omega \rangle}_{\langle \beta_n | n<\omega \rangle}$ such that $s \Vdash_{\mathbb{Q}_\kappa}$ "$\delta(p) \in H$". But then by definition of $D^{\langle \alpha_n | n<\omega \rangle}_{\langle \beta_n | n<\omega \rangle}$, $s \Vdash_{\mathbb{Q}_\kappa}$ "$\dot{f}(\alpha_n) \geqslant \beta_n$ for some $n < \omega$. So B_n cannot be maximal below r. $\lightning$

The following theorem shows that theorem 8 is optimal in terms of consistency strength.

Theorem 18 (Goldstern-Shelah-Todorčević)

$$\mathrm{BFA}\left(\mathrm{ro}^{"}(\Sigma_{\mathrm{cl}} \star \mathcal{C}_{\mathrm{cc}}), \aleph_2, \aleph_2)\right) \Longrightarrow \text{"}\aleph_2 \textit{ is regular and } \Sigma_2\textit{-correct in } L\text{."} \tag{23}$$

Here $\Sigma_{\mathrm{cl}} \star \mathcal{C}_{\mathrm{cc}}$ refers to the class of forcing notions which can be conceived of as a two-step-iteration of a countably closed notion of forcing with a forcing notion satisfying the countable chain condition. Since $\mathrm{ro}^{"}(\Sigma_{\mathrm{cl}} \star \mathcal{C}_{\mathrm{cc}}) \subset \mathcal{M}$ we get as a corollary

Corollary 19

$$\mathrm{BAFA} \Longrightarrow \text{"}\aleph_2 \textit{ is regular and } \Sigma_2\textit{-correct in } L\text{."} \tag{24}$$

A comment seems to be in order here. In [G-S] Shelah already proved that $\aleph_2$ is regular and Σ_2-correct—or in his terminology 'reflecting"—if a Bounded Forcing Axiom holds for notions of forcing which are attained by finitely iterating countably closed forcings with forcings satisfying the ccc. In fact the corollary above already follows from this proof. Todorčević's simpler proof which only uses a two-step-iteration can be read in [As], p. 134.

In [Mo] Moore proved the following theorem.

Theorem 20 $\mathrm{BPFA} \Rightarrow 2^{\aleph_0} = \aleph_2$

This leads to the following problem.

Problem 21 *Does* BAFA *decide the size of the continuum?*

In fact Bagaria asked a similar question in [Bag], p.16.

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