CHOICELESS RAMSEY THEORY OF LINEAR ORDERS

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Abstract. Motivated by work of Erdős, Milner and Rado, we investigate symmetric and asymmetric partition relations for linear orders without the axiom of choice. The relations state the existence of a subset in one of finitely many given order types that is homogeneous for a given colouring of the finite subsets of a fixed size of a linear order. We mainly study the linear orders $\langle {}^{\alpha}2, <_{lex}\rangle$, where α is an infinite ordinal and $<_{lex}$ is the lexicographical order. We first obtain the consistency of several partition relations that are incompatible with the axiom of choice. For instance we derive partition relations for $\langle {}^{\omega}2, <_{lex}\rangle$ from the property of Baire for all subsets of ${}^{\omega}2$ and show that the relation $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^2_2$ is consistent for uncountable regular cardinals κ with $\kappa^{<\kappa} = \kappa$. We then prove a series of negative partition relations with finite exponents for the linear orders $\langle {}^{\alpha}2, <_{lex}\rangle$. We combine the positive and negative results to completely classify which of the partition relations $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\bigvee_{\nu<\lambda} K_\nu, \bigvee_{\nu<\mu} M_\nu)^m$ for linear orders K_ν, M_ν and $m \leqslant 4$ and $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (K, M)^n$ for linear orders K, M and natural numbers n are consistent.

Contents

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1. Introduction

In this paper, we study the Ramsey theory of linear orders without the axiom of choice in the theory ZF. We work in this theory throughout the paper.

1.1. **Some Ramsey theory.** We begin with some definitions and facts from Ramsey theory. Structures with finitely many relations (usually linear orders) are denoted as K, L, M and a structure is identified with its underlying set. We use greek letters to denote ordinals, i.e. a cardinal ν is always assumed to be an ordinal.

Recall that for any order type τ we denote its reverse by τ^*. η denotes the order type of the rational numbers (the countable dense linear order without endpoints), λ the order type of the real numbers and—of course—ω is the order type of the natural numbers. For any order types σ and τ the order type $\sigma + \tau$ is the order type of a copy of σ to the left of a copy of τ. The order type $\sigma \cdot \tau$ (which is usually going to be written as $\sigma\tau$) consists of a copy of τ in which every point is replaced by a copy of σ. These conventions go back at least as far as to [914Ha, Chapter 4, §6] (cf. e.g. [962Ha] for an English version).

Definition 1.1.1. *Suppose that L, M are structures in the same signature and ν is a cardinal.*

(i) $[L]^M$ *denotes the set of substructures of L which are isomorphic to M.*
(ii) *Suppose that $f\colon [L]^M \to \nu$ is a colouring and $i < \nu$. A set $H \subseteq L$ is (f,i)-homogeneous if $f(x) = i$ for all $x \in [H]^M$.*
(iii) *Suppose that $f\colon [L]^M \to \nu$ is a colouring and $i < \nu$. A set $H \subseteq L$ is f-homogeneous if it is (f,i)-homogeneous for some $i < \nu$.*

We will consider the following partition relations.

Definition 1.1.2. *Suppose that K, L, M are structures and ν is a cardinal.*

(i) $L \longrightarrow (M)^K_\nu$ *states that for every colouring $f\colon [L]^K \to \nu$, there is some f-homogeneous $H \in [L]^M$.*
(ii) $L \longrightarrow [M]^K_\nu$ *states that for every $f\colon [L]^K \to \nu$, there is some $H \in [L]^M$ with* $\mathrm{ran}(f \restriction [H]^K) \neq \nu$.
(iii) $L \longrightarrow (M_0, \ldots, M_{n-1})^K$ *states that for every $f\colon [L]^K \to n$, there are $i < n$ and $H \in [L]^{M_i}$ such that H is (f,i)-homogeneous.*
(iv) $L \longrightarrow (M_{0,0} \vee \cdots \vee M_{0,k_0}, \ldots, M_{n-1,0} \vee \cdots \vee M_{n-1,k_{n-1}})^K$ *states that for every $f\colon [L]^K \to n$, there are $i < n$, $k \leqslant k_i$, and $H \in [L]^{M_{i,k}}$ such that H is (f,i)-homogeneous.*

If L is a linear order and each $M_{i,j}$ is an ordinal $\alpha_{i,j}$, then Definition 1.1.2(iv) is equivalent to $L \longrightarrow (\alpha_0, \ldots, \alpha_{n-1})^K$ where $\alpha_i \overset{df}{=} \min_{j \leqslant k_i} \alpha_{i,j}$ for every $i < n$.

We consider partition relations with exponent at least 2, and Proposition 1.1.3 below motivates the focus on linear orders. Let us first mention the case of exponent 1.

A structure L is *indivisible* if it satisfies $L \longrightarrow (L)^1_2$. If L is an indivisible structure with only one unary relation, then the relation is trivial, i.e. either full or empty. If L is any non-scattered countable linear order, i.e. L contains a copy of η, then L is indivisible. There are many interesting indivisible structures, for instance some countable metric spaces, cf. [007D].

If on the other hand L is a structure with a single binary relation, $L \longrightarrow (L)^2_2$ holds, and the domain of L can be linearly ordered (by a linear order which may be unrelated to L), then L is necessarily a linear order or trivial, by the following result. We will identify a relation with its restriction to the set of tuples with pairwise different coordinates.

Proposition 1.1.3. *Suppose that L is an infinite structure with a single binary relation and $L \longrightarrow (L)^2_2$.*

(1) *If the domain of L can be linearly ordered (by a linear order which may be unrelated to L), then L is a linear order or trivial, i.e. either full or empty.*
(2) *If the domain of L can be well-ordered, then L is a well-order with order type ω or a weakly compact cardinal.*

Proof. Note that $L \longrightarrow (L)^2_2$ implies $L \longrightarrow (L)^2_n$ for all $n \in \omega$. For the first claim, suppose that R_L is the binary relation of L and R is a linear order on the domain of L. Let

$$f_0(x,y) = 0 \text{ if } [(x,y) \in R \Rightarrow (x,y) \in R_L] \text{ and } [(y,x) \in R \Rightarrow (y,x) \in R_L]$$

$$f_1(x,y) = 0 \text{ if } [(x,y) \in R \Rightarrow (y,x) \in R_L] \text{ and } [(y,x) \in R \Rightarrow (x,y) \in R_L]$$

and choose the value 1 otherwise in each case. Let $f(x,y) = 2f_0(x,y) + f_1(x,y)$. The remaining claims follow. □

This generalises to dimensions $n \geqslant 3$ as follows. Let $P(X)$ denote the power set of a set X.

Definition 1.1.4. *Let $P(S_n)$ denote the power set of the symmetric group S_n.*

(i) *If L is a structure whose only relation is a linear order $<_L$ and $t \in P(S_n)$, let $L^{(t)}$ denote the structure whose only relation is the set of tuples $(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)})$ with $x_{\sigma(0)} <_L x_{\sigma(1)} <_L \cdots <_L x_{\sigma(n-1)}$ and $\sigma \in t$.*
(ii) *If M is a structure whose only relation is n-ary, then M is* induced by a linear order *if there is a linear order L with the same domain as M and some $t \subseteq P(S_n)$ with $M = L^{(t)}$.*

Proposition 1.1.5. (1) *If $\mathbb{N}$ is the structure of the natural numbers with the standard order and $t \in P(S_n)$, then $\mathbb{N}^{(t)} \longrightarrow (\mathbb{N}^{(t)})^m_k$ for all $k, m \in \omega$.*
(2) *Suppose that L is a structure whose only relation is n-ary relation for some $n \geqslant 2$, $L \longrightarrow (L)^n_2$, and the domain of L can be linearly ordered. Then L is induced by a linear order.*

Proof. The first claim follows from Ramsey's theorem. The second claim is proved as in Proposition 1.1.3. □

In this paper, we consider the following problem.

Problem 1.1.6. *Suppose that $n \geqslant 1$. For which pairs (L, M) of linear orders is there a linear order K with $K \longrightarrow (L, M)^n$?*

Since the answer depends on whether the axiom of choice holds, we consider Problem 1.1.6 in the following contexts.

(1) For arbitrary linear orders, assuming the the axiom of choice.
(2) For linear orders on ${}^{\kappa}\kappa$, the set of functions $f\colon \kappa \to \kappa$, assuming that $\kappa^{<\kappa} = \kappa$, so in particular ${}^{\mu}\kappa$ is well-ordered for all $\mu < \kappa$, but assuming that ${}^{\kappa}\kappa$ is not well-ordered.
(3) For arbitrary linear orders without the axiom of choice, and more specifically for linear orders on ${}^{\kappa}\kappa$ assuming that ${}^{\mu}2$ is not well-ordered for some $\mu < \kappa$.

For instance, the situation in (2) occurs in the model $L(P(\kappa))$ after forcing with $Col(\kappa, <\lambda)$, where $\lambda > \kappa$ is inaccessible, and (3) is fulfilled for linear orders of size at least $\aleph_1$ in models of the axiom of determinacy.

The *lexicographical order* $\langle {}^{\kappa}\kappa, <_{lex} \rangle$ is defined by $x <_{lex} y$ if $x \neq y$ and $x(\alpha) < y(\alpha)$ for the least $\alpha < \kappa$ with $x(\alpha) \neq y(\alpha)$.

Section 2 is concerned with partition relations for $\langle {}^{\gamma}2, <_{lex}\rangle$. Sections 3 and 4 is concerned with asymmetric negative partition relations without choice. The combined results of Chapter 2 and Chapters 3 and 4 determine which partition relations of the form $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (L, M)^n$ with $n \geqslant 2$ are consistent without choice.

1.2. **Partition relations assuming the axiom of choice.** We recall some known results on partition relations with choice. Partition relations for linear orders, in contrast to well-orders, were studied in [956ER, 962EH, 965Kr, 971E, 972EM, 974La].

Lemma 1.2.1. *Suppose that* ZFC *holds. Then* $L \not\longrightarrow (\omega^*, \omega)^2$ *for all linear orders* L.

Proof. The proof is similar to the proof of $\omega_1 \not\longrightarrow (\omega_1)^2_2$ in [933Si]. We consider a well-order on the domain of L and colour a pair depending on whether the well-order agrees with the natural order on this pair. □

This strongly limits the possibilities for positive partition relations under the axiom of choice. In particular, in any partition relation of the form $K \longrightarrow (L, M)^2$, we can assume that L, M are well-ordered, or that M is finite. Even for well-orders K, L, M, there are many difficult open questions for these relations (cf. [010HL, 979No, 993B, 999Ko, 008Jo, 010Sc, 014We]). Instead of considering these relations, we focus on linear orders L such that L, L^* are not well-ordered.

For partition relations with exponent at least 3, similar ideas as in Lemma 1.2.1 led to the following results.

Theorem 1.2.2 ([965Kr, Theorem 8] and [971E, Theorem 5])**.** *Suppose that* ZFC *holds. For any linear order* L

(1) $L \not\longrightarrow (4,\ \omega^* + \omega)^3$ *and*
(2) $L \not\longrightarrow (4,\ \omega + \omega^*)^3$.

The linear orders on the right side of the arrows are optimal, since $\omega \longrightarrow (\omega)^m_n$ and $\omega^* \longrightarrow (\omega^*)^m_n$ hold by Ramsey's theorem.

A further problem is to determine the valid partition relations which allow finitely many order types linked by a disjunction, instead of a single order type. For example, in the context of choice, the occurence of $\omega^* \vee \omega$ in a partition relation for a linear order states that there is an infinite homogeneous set with arbitrary order type. The occurence of $\omega^* + \omega \vee \omega + \omega^*$ in a partition relation for a linear order states that there is an infinite homogeneous set L such that L and L^* are not well-ordered.

Theorem 1.2.3 ([971E, Theorem 5])**.** *Suppose that* ZFC *holds. Then* $L \not\longrightarrow (5,\ \omega^* + \omega \vee \omega + \omega^*)^3$ *for all linear orders* L.

Question 1.2.4 ([971E, Remark on page 202])**.** *Suppose that* ZFC *holds. Is there a linear order* L *with* $L \longrightarrow (\omega^* + \omega \vee \omega + \omega^*, 4)^3$*?*

Let us mention two negative relations for $\langle {}^{\kappa}2, <_{lex}\rangle$ with choice. The topology on ${}^{\kappa}\kappa$ is given by the basic open sets $N_t = \{x \in {}^{\kappa}\kappa \mid t \subseteq x\}$ for $t \in {}^{<\kappa}\kappa$. A perfect subset of ${}^{\kappa}\kappa$ is a set of the form $[T] = \{x \in {}^{\kappa}\kappa \mid \forall \alpha < \kappa (x \upharpoonright \alpha \in T)\}$, where $T \subseteq {}^{<\kappa}\kappa$ is a perfect tree, i.e. a $<\kappa$-closed tree whose splitting nodes are cofinal in T.

Theorem 1.2.5 ([908Be])**.** *Suppose that* ZFC *holds and* $\kappa^{<\kappa} = \kappa$*. Then* $\langle {}^{\kappa}2, <_{lex}\rangle \not\longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^1_2$.

Proof. The counterexample is a κ-Bernstein set, i.e. a set $A \subseteq {}^{\kappa}\kappa$ such that A and its complement do not have perfect subsets. The set is constructed by diagonalization along an enumeration of the perfect subsets of ${}^{\kappa}\kappa$. □

A *meagre* subset of ${}^{\kappa}\kappa$ is a union of κ nowhere dense subsets of ${}^{\kappa}\kappa$, and a *comeagre* set is such that its complement is meagre.

Theorem 1.2.6. *Suppose that* ZFC *holds and* $\kappa^{<\kappa}=\kappa$. *Then* $\langle {}^\kappa 2, <_{lex}\rangle \not\longrightarrow (\langle {}^\kappa 2, <_{lex}\rangle, 3)^2$.

Proof. Suppose that $\langle C_\alpha \mid \alpha < 2^\kappa\rangle$ enumerates all perfect subsets of ${}^\kappa 2$. We choose an injective sequence $\langle x_\alpha, y_\alpha\rangle_{\alpha<2^\kappa}$ as follows. In step α, we find distinct $x_\alpha, y_\alpha \in C_\alpha$ with $x_\alpha \neq x_\beta$, $x_\alpha \neq y_\beta$, $y_\alpha \neq x_\beta$, and $y_\alpha \neq x_\beta$ for all $\beta < \alpha$. Let

$$\Gamma = \{\langle x_\alpha, y_\alpha\rangle \mid \alpha < 2^\kappa\} \cup \{\langle y_\alpha, x_\alpha\rangle \mid \alpha < 2^\kappa\}.$$

Let $f\colon [{}^\kappa 2]^2 \to 2$ denote the characteristic function of Γ, i.e. $f(x,y)=1$ if $(x,y)\in\Gamma$ and $f(x,y)=0$ otherwise. Note that every order preserving injection $f\colon \langle {}^\kappa 2, <_{lex}\rangle \hookrightarrow \langle {}^\kappa 2, <_{lex}\rangle$ is discontinuous in at most κ points for the following reason. Every point in which f is discontinuous defines a nontrivial interval in $\langle {}^\kappa\kappa, <_{lex}\rangle$, and the intervals from two distinct such points are are disjoint. It follows that f is continuous on a perfect set. This implies that for every isomorphism $f\colon \langle {}^\kappa 2, <_{lex}\rangle \to \langle {}^\kappa 2, <_{lex}\rangle$, there is a perfect set C such that $f \restriction C$ is a homeomorphism (cf. e.g. [014L, Corollary 5.3]). Hence $\langle {}^\kappa 2, \Gamma\rangle$ contains no independent set isomorphic to $\langle {}^\kappa 2, <_{lex}\rangle$ and no complete subgraph of size 3. □

1.3. **Partition relations assuming $\kappa^{<\kappa}=\kappa$.** We consider the lexicographical order $\langle {}^\kappa 2, <_{lex}\rangle$ for cardinals κ such that $\kappa^{<\kappa}=\kappa$, but ${}^\kappa\kappa$ is not necessarily well-ordered. The topology on ${}^\kappa\kappa$ is given by the basic open sets $N_t = \{x \in {}^\kappa\kappa \mid t \subseteq x\}$ for $t \in {}^{<\kappa}\kappa$. The following is proved in Theorem 2.3.2 below.

Theorem 1.3.1. *Suppose that V is a model of* ZFC *and κ is regular. There is a symmetric extension of V by a $<\kappa$-closed κ^+-c.c. forcing in which $\langle {}^\kappa 2, <_{lex}\rangle \longrightarrow (\langle {}^\kappa 2, <_{lex}\rangle)^2_2$ holds.*

Proof. See the proof of Theorem 2.3.2. □

It follows from Theorem 3.0.1 and Theorem 3.1.2 that Theorem 1.3.1 cannot be extended to exponent 3. For instance, the colouring which maps a triple to its splitting type does not have a large homogeneous set. The splitting type is defined as follows.

Definition 1.3.2. *Suppose that* $\gamma \in \mathrm{Ord}$.

(1) *For $x, y \in {}^\gamma\gamma$ let*
$$\delta_{x,y} = \delta(x,y) = \min\{\alpha < \gamma \mid x(\alpha) \neq y(\alpha)\}.$$
(2) *For $x, y \in {}^\gamma\gamma$ let*
$$\Delta_{x,y} = \Delta(x,y) = x \restriction \delta(x,y).$$
(3) *For $2 \leqslant n < \omega$ and $\vec{x} = \langle x_0,\ dots, x_{n-1}\rangle \in ({}^\gamma\gamma)^n$, let*
$$\delta(\vec{x}) = \langle \delta(x_0,x_1), \dots, \delta(x_{n-2},x_{n-1})\rangle.$$
(4) *Suppose that $\vec{a} = \langle a_0, a_1, \dots, a_{n-1}\rangle$, $\vec{b} = \langle b_0, \dots, b_{n-1}\rangle$ are lexicographically increasing tuples from ${}^\gamma 2$. Then $\vec{a}$, $\vec{b}$ are $<_{lex}$-isomorphic, in symbols $\vec{a} \approx \vec{b}$, if the unique order preserving function $\pi\colon \{a_i \mid i<n\} \to \{b_i \mid i<n\}$ satisfies $\pi(a_i) = b_i$ for all $i<n$.*
(5) *The* branching type *or* splitting type *of a $<_{lex}$-increasing tuple $\vec{a} = \langle a_0, a_1, \dots, a_{n-1}\rangle$ is $\tau(\vec{a})$, the representative of the isomorphism type of $\delta(\vec{a})$ obtained via the order-preserving mapping $\pi\colon \{\delta(a_0,a_1), \dots, \delta(a_{n-2},a_{n-1})\} \to |\{\delta(a_0,a_1), \dots, \delta(a_{n-2},a_{n-1})\}|$.*
(6) *(cf.* [981Bl]*) If $\vec{a} = (a_0, a_1, \dots, a_{n-1})$ is $<_{lex}$-increasing in ${}^\gamma\gamma$ and $\delta(\vec{a})$ is injective, then $\vec{a}$ is skew and the (length-order) pattern of $\vec{a}$ is the unique permutation $\rho = \rho(\vec{a})\colon (n-1) \to (n-1)$ such that $\delta(a_{\rho(0)}, a_{\rho(0)+1}) < \delta(a_{\rho(1)}, a_{\rho(1)+1}) < \dots < \delta(a_{\rho(n-2)}, a_{\rho(n-2)+1})$.*

The pattern, and similarly the branching type, describe in what order the paths a_i split apart as we proceed along the tree, cf. [981Bl]. Figure 2.2 below illustrates the branching types of quadruples and their clustering in symmetric pairs.

Remark 1.3.3. (1) *Note that if $\vec{a}$ is a $<_{lex}$-increasing n-tuple which is a skew subset, then the branching type $\tau(\vec{a})$ of $\vec{a}$ is a permutation of $n-1$ and every permutation of $n-1$ is realized as the branching type of a skew $<_{lex}$-increasing n-tuple.*

(2) *For every skew lexicographically increasing n-tuple* $\vec{a}$*, the branching type* $\tau(\vec{a})$ *is the inverse* $\tau(\vec{a}) = (\rho(\vec{a}))^{-1}$ *of the (length-order) pattern of* $\vec{a}$*.*

For any ordinal γ, a branch z through the tree ${}^{<\gamma}2$ may be regarded as the characteristic function of the set $x = \{\alpha < \gamma \mid z(\alpha) = 1\}$. Let $\bar{z}\colon \gamma \to 2$ be defined as $\bar{z}(\alpha) = 1 - z(\alpha)$. Then $\bar{z}$ is the characteristic function of $y = \gamma \setminus x$. Thus the map $\iota\colon {}^{\gamma}2 \to {}^{\gamma}2$, $\iota(z) = \bar{z}$ is an order reversing involution, i.e. $u <_{lex} v$ if and only if $\bar{v} <_{lex} \bar{u}$. Consequently, for any $<_{lex}$-increasing n-tuple $\vec{a}$ with $\tau(\vec{a}) = \sigma$, we have $\tau(\vec{b}) = \sigma^* = \sigma \circ \langle n-1, \ldots, 0\rangle$, the reverse of σ. This involution can be used to make symmetry arguments and we write $\bar{X}$ for $\{x \mid \bar{x} \in X\}$.

A *perfect subtree* of ${}^{<\gamma}\gamma$ is a subtree that is closed under increasing sequences such that the splitting nodes are cofinal. For any $n > 2$, any permutation of $n - 1 = \{0, 1, \ldots, n-2\}$, and any skew perfect tree T, one can show there is a $<_{lex}$-increasing tuple $\vec{x} \in [T]^{n-1}$ with $\rho(x) = \sigma$. So all skew types occur in every skew perfect tree.

Every perfect subtree of ${}^{\omega}\omega$ has a perfect skew subtree [981Bl, page 273, Lemma].

Lemma 1.3.4. *Suppose that* AC *holds and that* $\gamma > \omega$ *is a limit ordinal. Then every perfect subtree of* ${}^{<\gamma}\gamma$ *has a perfect skew subtree if and only if* $2^{\leq \operatorname{cof}(\gamma)} = \operatorname{cof}(\gamma)$*.*

Proof. Suppose that $2^{<\operatorname{cof}(\gamma)} = \operatorname{cof}(\gamma)$ and that T is a perfect subtree of ${}^{<\gamma}2$. First suppose that γ is regular. Then $2^{<\gamma} = \gamma$. Let $\langle t_\alpha \mid \alpha < \gamma\rangle$ enumerate ${}^{<\gamma}2$. It is straightforward to construct an order preserving embedding $f\colon {}^{<\gamma}2 \to T$ into a skew subtree of T by defining $f(t_\alpha)$ inductively for $\alpha < \gamma$. For arbitrary limit ordinals γ, the tree T can be thinned out to a perfect subtree that is isomorphic to ${}^{<\operatorname{cof}(\gamma)}2$. The previous case for $\operatorname{cof}(\gamma)$ implies the claim.

If γ is regular and $2^{<\gamma} > \gamma$, then there is no perfect skew tree of height γ for cardinality reasons. Suppose that γ is a limit ordinal with $2^{<\operatorname{cof}(\gamma)} > \operatorname{cof}(\gamma)$. Suppose that A is cofinal in γ with order type $\operatorname{cof}(\gamma)$. Let $T = \{s \in {}^{<\gamma}\gamma \mid \alpha \in A \Rightarrow s(\alpha) = 0\}$. There is no perfect skew subtree of T by the previous case. □

To prove the existence of homogeneous sets below, we need the following results on faithful embeddings.

Lemma 1.3.5 (Jean Larson)**.** *For every perfect subset* A *of* ${}^{\omega}2$*, there is an embedding* $e\colon {}^{\omega}2 \to A$ *such that* e *preserves order types and splitting types of finite subsets of skew subsets.*

Proof. Let ℓ denote the length of an element of ${}^{<\omega}2$. We define the *length-lexicographical* ordering on ${}^{<\omega}2$ by $s < t$ is and only if $(\ell(s) < \ell(t))$ or $\ell(s) = \ell(t)$ and $s <_{lex} t$. Note that this is a well-ordering.

Suppose that T is a perfect tree with $[T] \subseteq A$. We can inductively define an embedding $f\colon {}^{<\omega}2 \to T$ so that lexicographical order is preserved, extension $\sqsubseteq$ is preserved and $s < t$ implies that $\ell(f(s)) < \ell(f(t))$. We define $e\colon {}^{\omega}2 \to A$ by $f(t) = \bigcup_{k \in \omega} f(t \upharpoonright k)$. This is a faithful embedding, i.e. f preserves order types and splitting types of finite subsets of skew subsets. □

There are $(n-1)!$ branching types for lexicographically ordered n-tuples. Since a colouring of n-tuples can depend on the branching type, we consider sets which are separately homogeneous in each branching type.

Definition 1.3.6. $\langle {}^{\gamma}2, <_{lex}\rangle \longrightarrow_t (\langle {}^{\gamma}2, <_{lex}\rangle)^m_n$ *holds if for every colouring* $f\colon [{}^{\gamma}2]^m \to n$*, there is a set isomorphic to* $\langle {}^{\gamma}2, <_{lex}\rangle$ *which is separately homogeneous for* f *in each branching type.*

Partition relations $\longrightarrow_t$ for the linear order $\langle {}^{\omega}2, <_{lex}\rangle$ were considered by Blass [981Bl].

Lemma 1.3.7. *Suppose that* κ *is a regular cardinal.*

(1) *The linear orders* $\langle {}^{\kappa}2, <_{lex}\rangle$ *and* $\langle {}^{\kappa}\kappa, <_{lex}\rangle$ *are bi-embeddable.*
(2) *The linear orders* $\langle {}^{\omega}2, <_{lex}\rangle$*,* $\langle {}^{\omega}\omega, <_{lex}\rangle$*, and* λ *are bi-embeddable.*

Proof. The linear order $\langle {}^{\kappa}\kappa, <_{lex}\rangle$ is embeddable into $\langle {}^{\kappa}2, <_{lex}\rangle$ by the map $f\colon {}^{\kappa}\kappa \to {}^{\kappa}2$, where $f(\langle \alpha_i\rangle_{i<\kappa})$ is the concatenation of $1^{(\alpha_i)}{}^\frown 0$ for all $i<\kappa$. The linear order λ is isomorphic to $\langle {}^{\omega}\omega, <_{lex}\rangle \cdot \langle \mathbb{Z}, <\rangle$. □

Since these linear orders are bi-embeddable, they satisfy the same partition relations, as long as this does not refer to the splitting types.

Theorem 1.3.8. [981Bl] *$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow_t \langle {}^{\omega}2, <_{lex}\rangle^m_n$ holds for continuous colourings for all m,n.*

Proof. This is proved in [981Bl] using the Halpern-Läuchli theorem, cf. [966HL, Theorem 1]. To see that this holds without choice, suppose that a real x codes the continuous colouring. We apply Blass' theorem in $L[x]$ and obtain a closed set coded by a tree T. The statement that $[T]$ is homogeneous up to the branching type for the colouring coded by x is a Π^1_1 statement in x and T, and hence this holds in V. □

For uncountable cardinals κ, the analogue of Blass' theorem is connected with large cardinal properties of κ.

Theorem 1.3.9. *If $\kappa>\omega$ and $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow_t (\kappa^* \vee \kappa)^3_2$, then κ is weakly compact.*

Proof. If $f\colon [\kappa]^2 \to 2$ is a colouring, we define $g_f\colon [{}^{\kappa}2]^3 \to 2$ as follows. If $x,y,z \in {}^{\kappa}2$ are distinct and $A=\{x,y,z\}$, let $B=\{\delta(x,y),\delta(y,z),\delta(z,x)\}$ and $g_f(A) \overset{df}{=} f(B)$. Suppose that $H \subseteq {}^{\kappa}2$ is homogeneous for g_f up to the branching type and that H is isomorphic to κ^* or to κ. Then $I \overset{df}{=} \{\delta(x,y) \mid x,y\in H\}$ has order type κ and is homogeneous for f. □

Note that $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow_t (\kappa^* \vee \kappa)^2_2$ does not imply that κ is weakly compact, by Theorem 1.3.1.

Question 1.3.10. (1) *Is it consistent that $\kappa=\kappa^{<\kappa}>\omega$ and $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow_t (\langle {}^{\kappa}2, <_{lex}\rangle)^m_n$ holds for all m,n?*
(2) *If $\kappa=\kappa^{<\kappa}>\omega$ and $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow_t (\langle {}^{\kappa}2, <_{lex}\rangle)^3_2$, is κ measurable?*

1.4. **Partition relations in models of determinacy.** Partition relations for cardinals in models of determinacy have been intensively studied. Let us recall some results.

Definition 1.4.1. (1) *The* strong partition property *holds for a cardinal κ if $\kappa \longrightarrow (\kappa)^\kappa_\mu$ for all $\mu<\kappa$.*
(2) *Following* [970Ma]*, let θ denote the supremum of the ordinals α such that there is a surjection $f\colon P(\omega) \twoheadrightarrow \alpha$.*

Note that the strong partition property for ω is equivalent to the statement that all subsets of $[\omega]^\omega$ are Ramsey.

Theorem 1.4.2. (1) [976Pr] *The axiom of determinacy of games of reals $\mathsf{AD}_{\mathbb{R}}$ implies that ω has the strong partition property.*
(2) *Martin* [003Ka, Theorem 18.12], [004JM, 990Ja, 981K] *The axiom of determinacy AD implies that ω_1 has the strong partition property.*
(3) [008KW, 983KW] *Suppose that $V=L(\mathbb{R})$. Then AD holds if and only if there are unboundedly many strong partition cardinals below θ.*

It is both open whether the strong partition property for ω follows from AD, cf. [003Ka, Question 27.18] and what its consistency strength is, cf. [003Ka, Question 11.16]. By the next result the strong partition property for ω_1 surpasses its analogue for ω in consistency strength.

Theorem 1.4.3. (1) [977Ma, 5.1 Metatheorem] *It is consistent from an inaccessible cardinal that ω has the strong partition property.*
(2) [970Kl, 2.1 Theorem] *Every uncountable cardinal with the strong partition property is measurable.*

We ask which partition relations for linear orders hold if AD holds and $V = L(\mathbb{R})$. Note that the strong partition property for κ implies that $\langle {}^{\omega_1}2, <_{lex}\rangle$ is indivisible.

Theorem 1.4.4. *Suppose that κ has the strong partition property. Then $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^1_2$ holds.*

Proof. The claim follows from the strong partition property by identifying elements of $[\kappa]^\kappa$ with their characteristic functions in 2^κ. □

We ask whether this generalises to exponent 2.

Question 1.4.5. *Suppose that the axiom of determinacy holds in $V = L(\mathbb{R})$. Does this imply $\langle {}^{\omega_1}2, <_{lex}\rangle \longrightarrow (\langle {}^{\omega_1}2, <_{lex}\rangle)^2_2$?*

1.5. **Embedding linear orders into $\langle {}^{\kappa}2, <_{lex}\rangle$.** Every linear order of size κ embeds into $\langle {}^{\kappa}2, <_{lex}\rangle$ by a result of Hausdorff (cf. [949Ha, Chapter 6, Section 8]). If $\langle \kappa, <_L\rangle$ is a linear order, we map each $\gamma < \kappa$ to the characteristic function in ${}^{\kappa}2$ of the set of predecessors of γ in $<_L$ with $\alpha < \gamma$.

The negative partition results for suborders of $\langle {}^{\kappa}2, <_{lex}\rangle$ in the following sections suggest the question whether every linear order embeds into $\langle {}^{\kappa}2, <_{lex}\rangle$ for some cardinal κ. In models such that every linear order embeds into $\langle {}^{\kappa}2, <_{lex}\rangle$ for some cardinal κ, Theorem 4.3.2 and Theorem 4.6.1 hold for all linear orders.

Let $\mathbb{P}$ denote the forcing $P(\omega)$ ordered by inclusion up to finite error. We asked whether in a $\mathbb{P}$-generic extension of $L(\mathbb{R})$, there is a linear order which does not embed into $\langle {}^{\kappa}2, <_{lex}\rangle$ for any cardinal κ, if $L(\mathbb{R})$ is a model of determinacy. This was solved by Paul Larson in unpublished work (cf. Theorem 1.5.2 below).

The following is stated in [011CK, Section 1.1] without a proof.

Lemma 1.5.1. *Suppose that there is a measurable cardinal above ω Woodin cardinals. Let $(x, y) \in E_0$ if $x(n) = y(n)$ for all but finitely many n, for $x, y \in {}^{\omega}\omega$. Then there is no linear order in $L(\mathbb{R})$ of the equivalence classes of E_0.*

Proof. Suppose that in $L(\mathbb{R})$, $\phi(x, y, z, \alpha)$ defines a linear order on the equivalence classes of E_0, where $z \in {}^{\omega}2$ and $\alpha \in \mathrm{Ord}$. Let $\mathbb{Q}$ denote Cohen forcing. Suppose that (x, y) is $\mathbb{Q}^2$-generic over $L(\mathbb{R})$.

There is an elementary embedding $L(\mathbb{R}) \hookrightarrow L(\mathbb{R})^{V[x,y]}$ which fixes the ordinals by [001NZ, Theorem 1] . Therefore in $L(\mathbb{R})[x, y]$, ϕ defines a linear order on the equivalence classes of E_0 from α. Suppose that $(p, q) \Vdash^V_{\mathbb{Q}^2} \phi^{L(\mathbb{R})}(x, y, z, \alpha)$. Suppose that $(\bar{x}, x) \in E_0$, $(\bar{y}, y) \in E_0$, $p \subseteq \bar{y}$, and $q \subseteq \bar{x}$. Then $(p, q) \Vdash^V_{\mathbb{Q}^2} \phi^{L(\mathbb{R})}(\bar{y}, \bar{x}, z, \alpha)$. Since the definition of the linear order from α is invariant under E_0, this implies $(p, q) \Vdash^V_{\mathbb{Q}^2} \phi^{L(\mathbb{R})}(y, x, z, \alpha)$, contradicting the assumption. □

If U is an ultrafilter on ω, let $\langle \omega_U, <_U\rangle$ denote the ultrapower of the linear order $\langle \omega, <\rangle$ with U.

Theorem 1.5.2 (Paul Larson)**.** *Suppose that there is a measurable cardinal above ω Woodin cardinals and that U is $\mathbb{P}$-generic over $L(\mathbb{R})$. Then in $L(\mathbb{R})[U]$, the linear order $\langle \omega_U, <_U\rangle$ does not embed into $\langle {}^{\kappa}2, <_{lex}\rangle$ for any cardinal κ.*

Proof. Forcing with $\mathbb{P}$ preserves measurable cardinals by the Levy-Solovay theorem [010Cu, Theorem 9.6] and Woodin cardinals by [000HW, Corollary]. Therefore $M_\omega^\#$ is absolute between V and $V[G]$, where G is generic over V for a forcing in V_δ, where δ is the least Woodin cardinal. Then the supremum of the Woodin cardinals of M_ω is countable. Therefore M_ω satisfies the assumption A_κ in [001NZ, Theorem 1], where κ is below the least Woodin cardinal. Hence forcing with $\mathbb{P}$ does not add new sequences of ordinals, and in particular $\langle {}^{\kappa}2, <_{lex}\rangle = \langle {}^{\kappa}2, <_{lex}\rangle^{V[G]}$ for any $\mathbb{P}$-generic filter G over V.

The theories of $L(\mathbb{R})$ and $L(\mathbb{R})^{V[H]}$ are both determined by M_ω by [010St, Theorem 7.19] and hence equal, where H is $Col(\omega, <\kappa)$-generic over V and κ is the least inaccessible cardinal. Therefore we can apply [003DT, Corollary 7.4] to any colouring in $L(\mathbb{R})$.

Suppose that $p \in \mathbb{P}$ forces that $\dot{f}$ is such an embedding. Let $\mathbb{P}/p = \{q \in \mathbb{P} \mid q \leqslant p\}$. Let $g\colon [\omega]^\omega \times (\mathbb{P}/p) \to 2$, $g(x,q) = 0$ if q decides $\dot{f}(x)$, and $g(x,q) = 1$ otherwise.

There is an infinite set $A \subseteq \omega$ and a sequence $(c_i)_{i\in\omega}$ of subsets of ω of size 2 such that g is constant on $[A]^\omega \times \prod_i c_i$ by [003DT]. It follows from the definition of g that the value is 0. Therefore in $L(\mathbb{R})$, there is a linear order on the equivalence classes of E_0, contradicting Lemma 1.5.1. □

2. Partition relations for $\langle {}^\kappa\kappa, <_{lex}\rangle$

We consider the linear orders $\langle {}^\kappa\kappa, <_{lex}\rangle$ and $\langle {}^\kappa 2, <_{lex}\rangle$ for cardinals κ with $\kappa^{<\kappa} = \kappa > \omega$. These two linear orders are bi-embeddable and hence satisfy the same partition relations. To prove partition relations for linear orders, we will work with perfect sets.

Definition 2.0.1. (1) *A* perfect subtree of ${}^{<\kappa}\kappa$ *is a $<\kappa$-closed subtree of ${}^{<\kappa}\kappa$ whose branching nodes are cofinal.*
(2) *A* perfect subset of ${}^\kappa\kappa$ *is a set of the form $[T]$, where T is a perfect subtree of ${}^{<\kappa}\kappa$.*

We identify $[{}^\kappa 2]^n$ with the set of injective n-tuples $\langle x_0, \ldots, x_{n-1}\rangle$ in ${}^\kappa 2$ with $x_0 <_{lex} \cdots <_{lex} x_{n-1}$.

2.1. Partition relations for $\langle {}^\omega 2, <_{lex}\rangle$.

We first consider the linear order $\langle {}^\omega 2, <_{lex}\rangle$. The following is a variant of a theorem of Mycielski and Taylor.

The bounded topology on ${}^\kappa\kappa$ is given by the basic open sets $N_t = \{x \in {}^\kappa\kappa \mid t \subseteq x\}$ for $t \in {}^{<\kappa}\kappa$. We identify each set s in $[C]^n$ with the strictly $<_{lex}$-increasing n-tuple $\vec{s} = \langle s_0, \ldots, s_{n-1}\rangle$ with $s = \{s_0, \ldots, s_{n-1}\}$. Therefore the bounded topology induces a topology on $[C]^n$.

Lemma 2.1.1. *If $f\colon [{}^\omega 2]^m \to {}^\omega 2$ is Baire measurable, then there is a perfect set $C \subseteq {}^\omega 2$ such that $f \restriction [C]^m$ is continuous.*

Proof. Suppose that $\langle U_n \mid n \in \omega\rangle$ is a sequence of open dense subsets of $({}^\omega 2)^n$ such that f is continuous on their intersection. We construct a family $\langle t_s \mid s \in 2^n,\ n \in \omega\rangle$ by induction on n such that
(1) $t_s \subseteq t_u$ if $s \subseteq u$ and
(2) $N_{t_{s_i}} \times \cdots \times N_{t_{s_{m-1}}} \subseteq U_n$ if $s_0, \ldots, s_{m-1} \in 2^n$ and $s_i \neq s_j$ for all $i < j < m$.

Suppose that these properties hold for n. We first split each t_s for $s \in 2^n$ into $r_{s^\frown 0} = t_s^\frown 0$ and $r_{s^\frown 1} = t_s^\frown 1$. We enumerate the tuples $\vec{s} = \langle s_0, \ldots, s_{m-1}\rangle$ with $s_0, \ldots, s_{m-1} \in 2^{n+1}$ and $s_i \neq s_j$ for all $i < j < m$. Successively for each tuple $\vec{s}$, we extend r_{s_i} to t_{s_i} to fulfil (2) for this tuple. This implies the required properties. Let T denote the downwards closure of the set of t_s for $s \in 2^{<\omega}$. Then f is continuous on the set of m-tuples of distinct elements of $C = [T]$, and thus on $[C]^m$. □

Theorem 2.1.2. *Suppose that all sets of reals have the property of Baire.*
Then $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\langle {}^\omega 2, <_{lex}\rangle)^2_n$ for all n.

Proof. Note that $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\langle {}^\omega 2, <_{lex}\rangle)^2_2$ implies $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\langle {}^\omega 2, <_{lex}\rangle)^2_n$ for all $n \in \omega$. Suppose that $f\colon ({}^\omega 2)^2 \to 2$ is Baire measurable. There is a perfect set C such that $f \restriction [C]^m$ is continuous by Lemma 2.1.1. Since C is order isomorphic with $\langle {}^\omega 2, <_{lex}\rangle$, we can assume that $C = {}^\omega 2$. We can assume that there is no interval N_t such that f is constant on $[N_t]^2$ in colour 0. Using this assumption, we construct a family $(t_s)_{s\in 2^n,\ n\in\omega}$ by induction on n such that
(1) $t_s \subseteq t_u$ if $s \subseteq u$,
(2) $t_s <_{lex} t_u$ if $s <_{lex} u$,
(3) $f[N_{t_{s^\frown 0}} \times N_{t_{s^\frown 1}}] = \{1\}$ for all $s \in 2^n$.

This is possible since f is continuous. Let T denote that downwards closure of the set of t_s for $s \in 2^{<\omega}$. Then $f \restriction [T]^2$ is constant with value 1. □

Note that the assumption in Theorem 2.1.2 is consistent relative to ZF by [984Sh, 7.16 Theorem]. The consistency also follows as a special case of the result for cardinals κ with $\kappa^{<\kappa} = \kappa$ below.

The following result is used together with the negative partition relations in Chapter 3 to determine the consistent partition relations for $\langle {}^{\omega}2, <_{lex}\rangle$ with exponent 3.

Theorem 2.1.3. *Suppose that all sets of reals have the property of Baire. Then* $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\langle {}^{\omega}2, <_{lex}\rangle, 1+\omega^* \vee \omega+1)^3$.

Proof. Suppose that $f : [\langle {}^{\omega}2, <_{lex}\rangle]^3 \to 2$ is a colouring. We can assume that f is continuous by Lemma 1.3.5 and Lemma 2.1.1. Moreover we can assume that the colour $f(\vec{x})$ of a triple $\vec{x}$ depends only on the splitting type of $\vec{x}$ by Theorem 1.3.8. Let X_i for $i = 0, 1$ denote the set of $x \in {}^{\omega}2$ such that $x(n) = i$ for at most one n. Then X_0 and X_1 have order types $1 + \omega^*$ and $\omega + 1$, respectively and are homogenous.

If the colour of the splitting types for triples in X_0 and in X_1 is 0, then there is a homogeneous set of order type $\langle {}^{\omega}2, <_{lex}\rangle$ in colour 0. Otherwise one of the splitting types has colour 1. In this case, there is a homogeneous set in colour 1 of order type $1 + \omega^*$ or $\omega + 1$. □

Corollary 2.1.4. $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\langle {}^{\omega}2, <_{lex}\rangle, n)^3$ *for all natural numbers n.*

The following results are used in Chapter 4 to determine the consistent partition relations for $\langle {}^{\omega}2, <_{lex}\rangle$ with exponent 4.

Theorem 2.1.5. *Suppose that all sets of reals have the property of Baire. Then* $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\omega+1)^m_n$ *for all natural numbers m and n.*

Proof. Suppose that there is a colouring of $[{}^{\omega}2]^m$ in n colours. We can assume that the colour of skew tuples only depends on the splitting type by Lemma 1.3.5, Lemma 2.1.1 and Theorem 1.3.8. The set

$$S \overset{df}{=} \{x \in {}^{\omega}2 \mid |\{n < \omega \mid x(n) = 0\}| \leqslant 1\}.$$

has order type $\omega + 1$ and $[S]^m$ contains only the splitting type $\langle 0, 1, \ldots, m-1\rangle$. □

Note that the above is also a theorem in ZFC, cf. [970Ga, 986MP]. As stated before, a further problem is to determine the relations which allow finitely many order types linked by a disjunction, instead of a single order type. For example, assuming a fragment of choice, the occurence of $\omega^* \vee \omega$ in a partition relation for a linear order states that there is an infinite homogeneous set with arbitrary order type. The occurence of $\omega^* + \omega \vee \omega + \omega^*$ in a partition relation for a linear order states that there is an infinite homogeneous set such that L and L^* are not well-ordered.

2.2. **Terminology.** To run our arguments we are considering different kinds of quadruples and quintuples. We call $\vec{x} = \{x_0, x_1, x_2, x_3\}$ a *bouquet* if $\max(\delta(x_0, x_1), \delta(x_2, x_3)) < \delta(x_1, x_2)$ and we call it a *candelabrum* if $\delta(x_1, x_2) < \min(\delta(x_0, x_1), \delta(x_2, x_3))$. The remaining quadruples are called *combs* (cf. Figure 2.2).

If n is a natural number, $i \in \{0, 1\}$ and $\vec{x}$ is a $(2n+i+1)$-tuple we call $\vec{x}$ *dextral* if $\delta(x_0, x_n) < \delta(x_{n+i}, x_{2n+i})$ and *sinistral* otherwise. The attribute of being either dextral or sinistral is being referred to as *chirality.*

Furthermore we distinguish seven different kinds of quintuples. We may define them by recurring to the kinds of quadruples mentioned above. Suppose we are given a quintuple $\vec{p}$. Let $s_p \overset{df}{=} \Delta(p_0, p_4)$. We say that s *divides* $\vec{p}$ into $\{b \in \vec{p} \mid b \sqsupset s^\frown\langle 0\rangle\}$ and $\{b \in \vec{p} \mid b \sqsupset s^\frown\langle 1\rangle\}$. Using this terminology we may continue our definition as follows:

(a) $\vec{p}$ is a *cactus* if and only if s_p divides $\vec{p}$ in a comb of the same chirality and a branch.
(b) $\vec{p}$ is a *grape* if and only if s_p divides $\vec{p}$ in a comb of the opposite chirality and a branch.
(c) $\vec{p}$ is an *olivillo* if and only if s_p divides $\vec{p}$ in a bouquet of the same chirality and a branch.
(d) $\vec{p}$ is a *rose* if and only if s_p divides $\vec{p}$ in a bouquet of the opposite chirality and a branch.
(e) $\vec{p}$ is a *mistletoe* if and only if s_p divides $\vec{p}$ in a candelabrum and a branch.
(f) $\vec{p}$ is a *lilac* if and only if s_p divides $\vec{p}$ in a triple of the same chirality and a pair.

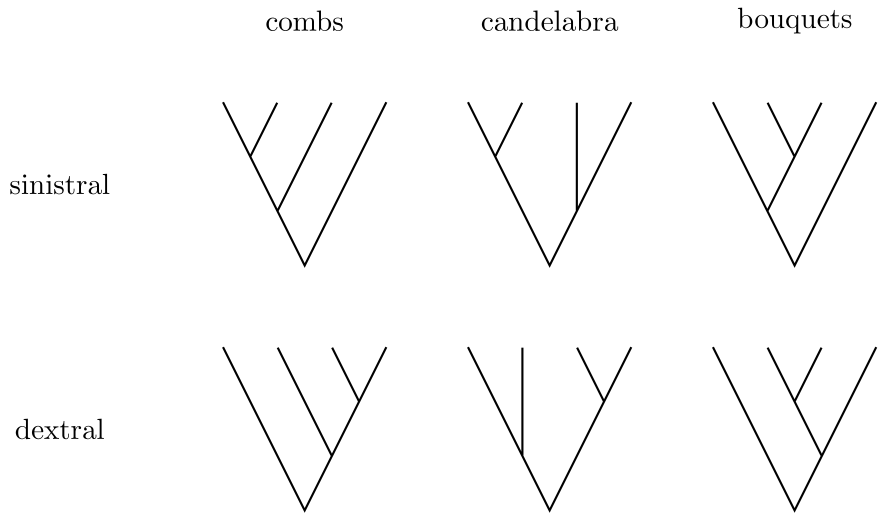


Figure 1. Bouquets, Candelabra and Combs

(g) $\vec{p}$ is a *guinea flower* if and only if s_p divides $\vec{p}$ in a a triple of the opposite chirality and a pair.

Finally there is one type of sextuples we are considering in our arguments and which we therefore want to name. So call $\vec{s} \in [{}^{\alpha}2]^6$ an *antler* if $\Delta(s_2, s_3) \sqsubseteq \Delta(s_1, s_2), \Delta(s_3, s_4)$ and $\Delta(h_{2i+1}, h_{2i+2}) \sqsubseteq \Delta(h_{4i}, h_{4i+1})$ for both $i \in \{0, 1\}$.

These are not the splitting types as used in [981Bl]. While our definition of chirality distinguishes between dextral and sinistral candelabra, this distinction is often irrelevant, in fact, most of the time the arguments used only concern the mutual relationship of splitting nodes along a single branch.

While there are 24 different splitting types of quintuples in the sense of [981Bl] it is, in this setting, more appropriate to only discern 14 types. Being a cactus, rose, olivillo or grape of a specified chirality amounts to a splitting type in the sense of [981Bl] but there are two different splitting types corresponding to being a mistletoe of a given chirality and three for being a lilac of a given chirality or a guinea flower of a given chirality.

As was shown before the number of different splitting types of n-tuples in this reduced sense is given by the n-th Catalan number, see e.g. [991HP], [996CG, page 101 et seqq.], [998LW], [999St], [005AO, page 119 et seqq.] or [015St].

Theorem 2.2.1. *Suppose that all sets of reals have the property of Baire. Then* $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (6,\ 1+\omega^* + \omega + 1 \vee m + \omega^* \vee \omega + n)^4$ *for all natural numbers m and n.*

Proof. Suppose without loss of generality that $m = n$ and that $f\colon [{}^{\omega}2]^4 \to 2$ is a colouring. We can assume that the value of f for skew tuples only depends on the splitting type by Lemma 2.1.1, Lemma 1.3.5 and Theorem 1.3.8. We define the following sets.

$$\begin{aligned}
X &\overset{df}{=} \{x \in {}^{\omega}2 \mid x(i) = x(0) \text{ for all but at most one natural numbers } i.\},\\
X_m &\overset{df}{=} \{0^m{}^\frown 1^k{}^\frown 0^\omega \mid k < \omega\} \cup \{0^k{}^\frown 1^\omega \mid k < m\},\\
Y &\overset{df}{=} \{0^\omega, 0^3{}^\frown\langle 1\rangle^\frown 0^\omega, 0^3{}^\frown 1^\omega, \langle 1\rangle^\frown 0^\omega, \langle 1, 0\rangle^\frown 1^\omega, 1^\omega\}.
\end{aligned}$$

The set X has order type $1 + \omega^* + \omega + 1$ and contains only combs and candelabra. The set X_m has order type $\omega + m$ and contains only combs and sinistral bouquets. $\bar{X}_m$ has order type $m + \omega^*$ and contains only combs and dextral bouquets. (Recall the involution defined on page 6.)

If at least one of the comb-types has colour 1, then there is an infinite homogeneous set in colour 1. Hence we can assume that these types have colour 0. Now if both candelabra-types have colour 0, then X is homogeneous in colour 0. If the type of sinistral bouquets has colour 0, then X_m is homogeneous in colour 0. If the type of dextral bouquets has colour 0, then $\bar{X}_m$ is homogeneous in

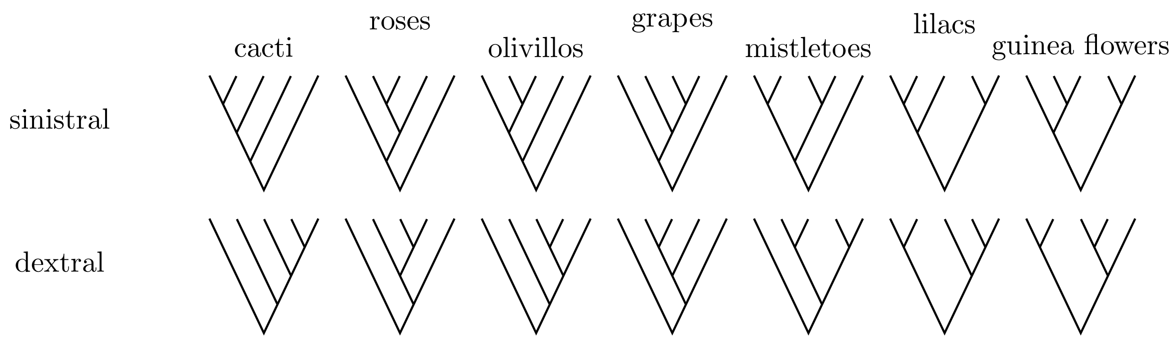


FIGURE 2. Seven Pentapetalae, cf. [009B$^{\&}$, 010M$^{\&}$]

colour 0. So we may assume that all both bouquet-types and one candelabrum-type get colour 1. If the latter is dextral then Y is homogeneous in colour 1, otherwise $\bar{Y}$ is. $\square$

Note that in contrast to Theorem 2.1.5 the previous theorem fails in ZFC by Theorem 4.4.1.

Theorem 2.2.2. *Suppose that all sets of reals have the property of Baire. Then* $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (5,\ \omega+1+\omega^* \vee 1+\omega^*+\omega+1)^4$ *holds.*

Proof. Suppose that $f\colon [{}^{\omega}2]^4 \to 2$ is a colouring. We can assume that the colour only depends on the splitting type by Lemma 2.1.1, Lemma 1.3.5 and Theorem 1.3.8. Suppose that there are no homogeneous sets with order types $\omega+1+\omega^*$ or $1+\omega^*+\omega+1$ in colour 0, and no homogeneous sets of size 5 in colour 1.

Let z denote the characteristic function of the odd numbers $n \in \omega$. We define the sets

$$\begin{aligned}
X &= \{x \in {}^{\omega}2 \mid x(i) = z(i) \text{ except in at most one place}\},\\
Y &= \{x \in {}^{\omega}2 \mid x(i) = x(0) \text{ except in at most one place}\},\\
F &= \{0^{\omega}, 0^\frown 1^{\omega}, 1^{2\frown} 0^{\omega}, \langle 1,1,0\rangle^\frown 1^{\omega}, 1^{\omega}\},\\
G &= \{0^{\omega}, 0^{3\frown} 1^{\omega}, 1^\frown 0^{\omega}, \langle 1,0\rangle^\frown 1^{\omega}, 1^{\omega}\}.
\end{aligned}$$

Then z is an element of X, X has order type $\omega+1+\omega^*$, and its quadruples are combs or bouquets. The set Y has order type $1+\omega^*+\omega+1$ and its quadruples are combs or candelabra. Both F and G are dextral guinea flowers which immediately implies that both $\bar{F}$ and $\bar{G}$ are sinistral guinea flowers.

If any comb-type would get colour 1 then there would be an infinite set homogeneous in colour 1. Hence as in the proof of Theorem 2.2.1 we may assume that all combs are of colour 0. Let us assume that X fails to be homogeneous for colour 0. Then one of the candelabra-types has to get colour 1. Similarly, from assuming that Y fails to be homogeneous in colour 0 we may infer that one of the bouquet-types has to get colour 1. The guinea flower F only contains dextral candelabra and dextral bouquets, $\bar{F}$ only contains sinistral candelabra and sinistral bouquets, G only contains sinistral candelabra and dextral bouquets and $\bar{G}$ only contains dextral candelabra and sinistral bouquets. Hence we inevitably end up with a quintuple homogeneous in colour 1. $\square$

The Theorem above is not provable in ZFC by Theorem 4.4.1 or Theorem 1.2.3. The following is analogous to Lemma 2.1.1 for Lebesgue measurable colourings.

Lemma 2.2.3. *Suppose that the Axiom of Dependent Choices* DC *holds. Suppose that* $f\colon [{}^{\omega}2]^m \to {}^{\omega}2$ *is a colouring such that* $f \restriction A$ *is Lebesgue measurable for all closed sets* $A \subseteq [{}^{\omega}2]^n$*. Then there is a perfect set* $C \subseteq {}^{\omega}2$ *such that* $f \restriction [C]^m$ *is continuous.*

Proof. If T is a subtree of ${}^{<\omega}\omega$, we denote its root by $\mathrm{root}(T)$ and its n-th splitting level by $\mathrm{split}_n(T)$. If $s \in T$, let $T/s = \{t \in T \mid s \subseteq t \text{ or } t \subseteq s\}$.

Since we work with $<_{lex}$ in the following construction, note that $<_{lex}$ is not total on $({}^{<\omega}2)^2$. We construct a family $\langle T_s \mid s \in 2^n,\ n \in \omega\rangle$ of perfect subtrees of ${}^{<\omega}2$ by induction on n such that

(1) $T_u \subsetneq T_s$ and $\mathrm{root}(T_s) \subsetneq \mathrm{root}(T_u)$ if $s \subsetneq u$,
(2) $\mathrm{root}(T_s) <_{lex} \mathrm{root}(T_u)$ if $s <_{lex} u$,
(3) if $s_0, \dots, s_{m-1} \in 2^n$ and $s_i \neq s_j$ for all $i < j < m$, then there is some $v \in 2^n$ such that $f[[T_{s_0}] \times \cdots \times [T_{s_{m-1}}]] \subseteq N_v$.

In the inductive construction, we will use the following result of Mycielski.

Claim. *If $A \subseteq \prod_{i<n} B_i$ has positive measure where each B_i is a closed subset of ${}^\omega 2$, then there are perfect sets $C_0, \dots, C_{n-1}$ with $\prod_{i<n} C_i \subseteq A$.*

Proof. See [967My, Theorem 1]. □

Suppose that T_u is constructed for all $u \in 2^n$. We choose two incompatible extensions t^0_u, t^1_u of $\mathrm{root}(T_u)$ for each $u \in 2^n$.

Let $\vec{r} = \langle r_i \mid i < k \rangle$ enumerate the sequences of the form $t^{j_i}_{u_i}$ for some $u_i \in 2^n$ and some $j_i < 2$. Let $S_i = T_{u_i}/r_i$ for all $i < k$.

We fix an enumeration of length ν of the strictly $<_{lex}$-increasing tuples $\vec{s} = \langle s_0, \dots, s_{m-1} \rangle$ of elements of $\vec{r}$. We will successively shrink $S_i = S^0_i$ to S^j_i for all $j < \nu$ and all $i < k$ as follows. Suppose that $j+1 < \nu$ and that S^j_i is defined for all $i < k$. Suppose that the tuple $\vec{s} = \langle r_{i_0}, \dots, r_{i_{m-1}} \rangle$ appears in this step of the enumeration. Let $B_{i_\zeta} = [S^j_{i_\zeta}]$ for $\zeta < m$. We choose some $v \in 2^n$ such that $A = (\prod_{\zeta<m} B_{i_\zeta}) \cap f^{-1}[N_v]$ has positive measure. We shrink $S^j_{i_\zeta}$ to $S^{j+1}_{i_\zeta}$ for all $\zeta < m$ by applying the previous claim to $A \subseteq \prod_{\zeta<m} B_{i_\zeta}$. Moreover, let $S^{j+1}_i = S^j_i$ for all $i < k$ such that $i \neq i_\zeta$ for all $\zeta < m$. Let $T_{u_i^\frown \langle j_i \rangle} = S^{\nu-1}_i$, where $r_i = t^{j_i}_{u_i}$ as defined above.

The trees T_u in this construction fusion to a perfect tree $T = \bigcup_{u \in 2^n} \mathrm{split}_{\leqslant n}(T_u)$ by conditions (1) and (2). Let $C = [T]$. It follows from condition (3) that $f \upharpoonright [C]^n$ is continuous. □

Theorem 2.2.4. *Suppose that the Axiom of Dependent Choices* DC *holds and that all sets of reals are Lebesgue measurable. Then the conclusions of Theorem 2.1.2, Theorem 2.1.3, and Theorem 2.2.2 hold.*

Proof. The proofs are identical to those for Baire measurable colourings, using Lemma 2.2.3 instead of Lemma 2.1.1. □

2.3. **Partition relations for $\langle {}^\kappa 2, <_{lex} \rangle$.** We now consider the analogous questions for $\langle {}^\kappa 2, <_{lex} \rangle$.

Lemma 2.3.1. *Suppose that κ is regular and V is a model of* ZFC.

(1) *Suppose that G is $Add(\kappa, 1)$-generic over V. Then in $V[G]$, for every function $f \colon [{}^\kappa 2]^n \to {}^\kappa 2$ definable from ordinals, there is a perfect set C such that $f \upharpoonright [C]^n$ is continuous.*
(2) *Suppose that H is $Add(\kappa, \lambda)$-generic over V and $\lambda \geqslant \kappa^+$. Then in $V[H]$, for every function $f \colon [{}^\kappa 2]^n \to {}^\kappa 2$ definable from ordinals and subsets of κ, there is a perfect set C such that $f \upharpoonright [C]^n$ is continuous.*

Proof. For the first claim, note that there is a perfect set C of $Add(\kappa, 1)$-generics in $V[G]$ such that the quotient forcing in $V[G]$ of each n-tuple $\vec{x} = (x_0, \dots, x_{n-1})$ of distinct elements of C is equivalent to $Add(\kappa, 1)$ by [016Sc]. Suppose that $\phi(\vec{x}, \alpha, t)$ holds in $V[G]$ if and only if $f(\vec{x}) \upharpoonright \alpha = t$, where ϕ is a formula with an ordinal parameter, which we omit. Then $V[G] \vDash \phi(\vec{x}, \alpha, t) \Leftrightarrow 1 \Vdash^{V[\vec{x}]}_{Add(\kappa,1)} \phi(\vec{x}, \alpha, t)$ for all $\vec{x} \in [C]^n$. Therefore $f(\vec{x}) \in V[\vec{x}]$ for all $\vec{x} \in [C]^n$.

Let $\psi(\vec{x}, \alpha, t)$ denote the formula $1 \Vdash^{V[\vec{x}]}_{Add(\kappa,1)} \phi(\vec{x}, \alpha, t)$. Let σ denote an $Add(\kappa, 1)^n$-name for the n-tuple of $Add(\kappa, 1)$-generic reals, so that $\sigma^{\vec{x}} = \vec{x}$ for all $\vec{x} \in [C]^n$.

Claim. *$f \upharpoonright [C]^n$ is continuous.*

Proof. If $\vec{x} \in [C]^n$ and $\alpha < \kappa$, then there is a condition $p \in Add(\kappa, 1)^n$ with $p \subseteq \vec{x}$ and $p \Vdash^V_{Add(\kappa,1)^n} \psi(\sigma, \alpha, f(\vec{x}) \upharpoonright \alpha)$. So $f(\vec{x}) \upharpoonright \alpha = f(\vec{y}) \upharpoonright \alpha$ for all $\vec{y} \in C$ with $p \subseteq \vec{y}$. This proves that $f \upharpoonright [C]^n$ is continuous. □

The proof of the second claim is analogous. We force with $Add(\kappa,1)^n$ over an intermediate model which contains the parameters and whose quotient forcing is equivalent to $Add(\kappa,\lambda)$. $\square$

We denote the power set of a set X by $P(X)$.

Theorem 2.3.2. *Suppose that κ is regular and V is a model of* ZFC*.*

(1) *Suppose that G is $Add(\kappa,1)$-generic over V. Then in $V[G]$*
$$\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^2_n$$
holds for all n and for all colourings $f\colon [{}^{\kappa}2]^2 \to 2$ definable from ordinals.

(2) *Suppose that H is $Add(\kappa,\lambda)$-generic over V and $\lambda \geqslant \kappa^+$. Then in $V[G]$*
$$\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^2_n$$
holds in $HOD_{P(\kappa)}$ and therefore in $L(P(\kappa))$ for all n.

Proof. It is sufficient to prove $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^2_2$. Suppose that $f\colon [{}^{\kappa}2]^2 \to 2$ is a colouring definable from ordinals in $V[G]$. There is a perfect set C such that $f \restriction [C]^2$ is continuous by Lemma 2.3.1. Since $\langle C, <_{lex}\rangle$ is order isomorphic to $\langle {}^{\kappa}2, <_{lex}\rangle$, we can assume that f is continuous.

We can assume that no interval in $\langle {}^{\kappa}2, <_{lex}\rangle$ is homogeneous for f in colour 0. Using this assumption, we construct a family $(t_s)_{s\in 2^\alpha,\ \alpha<\kappa}$ by induction on α such that

(1) $t_s \subseteq t_u$ if $s \subseteq u$ and
(2) $f[N_{t_{s^\frown 0}} \times N_{t_{s^\frown 1}}] = \{1\}$ for all $s \in 2^\alpha$.

The successor step is straightforward, since f is continuous. If $u \in 2^\beta$ and $\beta < \kappa$ is a limit, let $t_u = \bigcup_{s \subsetneq u} t_s$. Let T denote the downwards closure of the set of t_s for $s \in 2^{<\kappa}$. Then $f \restriction [T]^2$ is constant with value 1.

The proof of the second claim is analogous from the second claim in Lemma 2.3.1. $\square$

The size of 2^κ is measured by the ordinal θ_κ in contexts without choice.

Definition 2.3.3. *Let θ_κ denote the supremum of the ordinals α such that there is a surjection $f\colon P(\kappa) \twoheadrightarrow \alpha$.*

The following result shows that the partition relation $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^2_n$ is not linked to the size of θ_κ.

Corollary 2.3.4. *Suppose that κ is regular and V is a model of* ZFC*.*

(1) *There is a $<\kappa$-closed forcing $\mathbb{P}$ such that for any $\mathbb{P}$-generic filter G over V, $HOD^{V[G]}_{P(\kappa)}$ and $L(P(\kappa))^{V[G]}$ satisfy*
   (a) $\kappa = \kappa^{<\kappa}$,
   (b) $\theta_\kappa = \kappa^+$, *and*
   (c) $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^2_n$.
(2) *For any cardinal λ, there is a $<\kappa$-closed forcing $\mathbb{Q}$ such that for any $\mathbb{Q}$-generic filter H over V, $HOD^{V[H]}_{P(\kappa)}$ and $L(P(\kappa))^{V[G]}$ satisfy*
   (a) $\kappa = \kappa^{<\kappa}$,
   (b) $\theta_\kappa \geqslant \lambda$, *and*
   (c) $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^2_n$.

Moreover $HOD^{V[G]}_{P(\kappa)}$ and $L(P(\kappa))^{V[G]}$ satisfy dependent choice DC_κ for sequences of length κ.

Proof. For the first claim, we force GCH at κ with $Add(\kappa^+,1)$ and then apply Theorem 2.3.2 for $\lambda = \kappa^+$.

For the second claim, we force $\theta_\kappa \geqslant \lambda$ with the forcing $\mathbb{P}$ given by [012Lü, Theorem 1.5] and again apply Theorem 2.3.2 for $\lambda = \kappa^+$. Forcing with $\mathbb{P}$ followed by $<\kappa$-closed forcing does not decrease θ_κ.

The model $HOD_{P(\kappa)}^{V[G]}$ in Theorem 2.3.2 is closed under κ-sequences in $V[G]$ and therefore satisfies DC_κ. Every element of $L(P(\kappa))^{V[G]}$ is definable in $L(P(\kappa))^{V[G]}$ from an ordinal and a subset of κ. To prove DC_κ in $L(P(\kappa))^{V[G]}$ for a given relation, we construct a witnessing sequence in $V[G]$ with the ordinals in the definitions chosen as minimal. This sequence is an element of $L(P(\kappa))^{V[G]}$. □

3. Negative partition relations for triples

In the next two sections, we will prove negative partition properties for linear orders of the form $\langle {}^{\alpha}2, <_{lex}\rangle$. We limit ourselves to the case of two colours. In this chapter we are concerned with triples, whereas Chapter 4 deals with quadruples.

Theorem 2.1.2 cannot be improved to exponent 3 for asymmetric partition relations.

Theorem 3.0.1. *$\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (\omega^*, \omega)^3$ for all ordinals α.*

Proof. Suppose that $x, y, z \in {}^{\alpha}2$ with $x <_{lex} y <_{lex} z$. Let $f(x, y, z) = 0$ if $\Delta_{x,y} < \Delta_{y,z}$ and let $f(x, y, z) = 1$ otherwise. Suppose that H is homogeneous in colour 0 with order type ω^* and that $\langle x_i \mid i < \omega\rangle$ is the decreasing enumeration of H. Let $\alpha_i = \Delta_{x_i,x_{i+1}}$. Then $\langle \alpha_i\rangle_{i\in\omega}$ is decreasing. The argument for colour 1 is symmetric. □

Theorem 2.1.3 shows that Theorem 3.0.1 is optimal. The relation $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\langle {}^{\omega}2, <_{lex}\rangle)^2_n$ holds for all n if all sets of reals have the property of Baire by Theorem 2.1.2. This cannot be improved to exponent 3 in symmetric partition relations (cf. Theorem 3.1.2 below).

In the following proof, N^{α}_s denotes the set of branches in $\langle {}^{\alpha}2, <_{lex}\rangle$ extending a node $s \in {}^{<\alpha}2$. Note that we do not use the axiom of choice, as almost always in this paper.

Lemma 3.0.2. *Suppose that α, ν are infinite ordinals such that ν embeds into $\langle {}^{\alpha}2, <_{lex}\rangle$. Then $|\nu| \leqslant \alpha$.*

Proof. We fix an infinite ordinal α and an injective function $\psi\colon \alpha \times \alpha \to \alpha$ which exists by Hessenberg's Theorem, cf. [906He, page 108 et seqq.]. We may assume without loss of generality that α is indecomposable. Let A_ν denote the set of order-preserving embeddings $f\colon \nu \to {}^{\alpha}2$ and B_ν the set of injective functions $g\colon \nu \to \alpha$ for all ordinals $\nu \geqslant \alpha$. We will define $F_\nu : A_\nu \longrightarrow B_\nu$ by induction for all $\nu \geqslant \alpha$. This implies the claim.

The definition of $F_{\nu+1}$ from F_ν is straightforward. Suppose that $\nu \geqslant \alpha$ is a limit ordinal and $f \in A_\nu$. We aim to define $F_\nu(f)$. To this end, we define a sequence $\langle t_\zeta \mid \zeta < \rho\rangle$ for some $\rho \leqslant \kappa$ by induction.

Let t_0 be the unique splitting node of minimal height between elements in $\operatorname{ran}(f)$. If t_ξ is defined for some $\xi < \kappa$, let $t_{\xi+1}$ be the unique splitting node of minimal height between elements in $\operatorname{ran}(f)$ extending $t_\xi^\frown\langle 1\rangle$, if there is such a splitting node. For limit ordinals ξ, let t_ξ denote the unique splitting node between elements of $\operatorname{ran}(f)$ extending $\lim_{\zeta<\xi} t_\zeta$. Let $\rho \leqslant \alpha$ denote the least ordinal such that t_ρ is not defined. First suppose that $\rho < \alpha$. Let $t = \bigcup_{\zeta<\rho} t_\zeta$. Since ν is a limit, $N^{\alpha}_t \cap \operatorname{ran}(f) = \emptyset$. This defines a decomposition

$$\operatorname{ran}(f) = \bigcup_{\xi<\rho} \operatorname{ran}(f) \cap N^{\alpha}_{t_\xi^\frown\langle 0\rangle}$$

of $\operatorname{ran}(f)$. Let $\nu_\xi < \nu$ denote the order type of $\operatorname{ran}(f) \cap N_{t_\xi^\frown\langle 0\rangle}$. It is straightforward to define an injective function $G_f\colon \nu \to \alpha$ from ψ and F_{ν_ξ} for $\xi < \rho$. Let $F_\nu(f) = G_f$. The definition is analogous if $\rho = \alpha$. □

3.1. **The β-function.** In the proof of the following theorem and in many proofs to come, we will use the *β-function* β_h. The idea for the β-function is the comparison of the order of a tuple with another linear order. This is used in the definition of colourings as counterexamples to partition relations. The function β_h identifies the least difference of $x \neq y$ in ${}^{\alpha}2$ with an ordinal below κ.

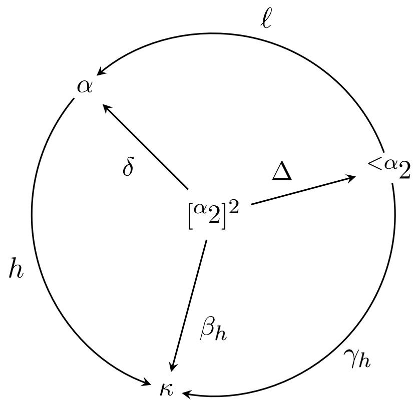


FIGURE 3. The functions $\Delta, \delta, \ell, h, \gamma_h$ and β_h

Definition 3.1.1. *Suppose that κ is an initial ordinal, $\kappa \leqslant \alpha < \kappa^+$ and $h : \alpha \longrightarrow \kappa$ is a bijection. Then $\beta_h\colon [{}^{\alpha}2]^2 \to \kappa$ is defined as $\beta(x,y) = \beta_h(x,y) = \beta_h(\{x,y\}) = h(\delta_{x,y})$ for $x <_{lex} y$. We will also write $\gamma_h = h \circ \ell$. We have the following:*

$$\beta_h = h \circ \delta = h \circ \ell \circ \Delta = \gamma_h \circ \Delta. \tag{Figure 3}$$

In the following we are going to say that a sequence *stabilises* if it is constant from some point onwards.

Theorem 3.1.2. *Let κ be an infinite initial ordinal and $\alpha < \kappa^+$. Then $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (2 + \kappa^* \vee \omega, \omega^* \vee \kappa + 2)^m$ for all $m \geqslant 3$.*

Proof. Suppose that $m = 3$. Let $h\colon \alpha \leftrightarrow \kappa$ be a bijection and β_h as in Definition 3.1.1. We consider the following colouring $f : [{}^{\alpha}2]^3 \to 2$. If $x, y, z \in {}^{\alpha}2$ and $x <_{lex} y <_{lex} z$, let $f(\{x,y,z\}) = 0$ if $\beta_h(y,z) < \beta_h(x,y)$.

In the first case, suppose that $X = \{x_\nu \mid \nu < \kappa + 2\} \in [{}^{\alpha}2]^{2+\kappa^*}$ and that $x_\gamma < x_\beta$ whenever $\beta < \gamma < \kappa + 2$. We distinguish two cases by considering the sequence $S \overset{df}{=} \langle \Delta(x_{\kappa+1}, x_\nu) \mid \nu < \kappa\rangle$ of splitting nodes. Either this sequence stabilises or not. In the first subcase, suppose that S stabilises at $s \in {}^{<\alpha}2$ from $\gamma < \kappa$ onwards. Then Lemma 3.0.2 implies that $|\{\delta(x_{\nu+1}, x_\nu) \mid \nu \in \kappa \setminus \gamma\}| = \kappa$. Since h is one-to-one, $|\{\beta_h(x_{\nu+1}, x_\nu) \mid \nu \in \kappa \setminus \gamma\}| = \kappa$, so we may choose a $\xi \in \kappa \setminus \gamma$ with $\beta_h(x_{\xi+1}, x_\xi) > \gamma_h(s)$. Then $f(\{x_{\kappa+1}, x_{\xi+1}, x_\xi\}) = 1$. Now suppose that S does not stabilise. The sequence $\langle \Delta(x_\nu, x_0) \mid \nu < \kappa\rangle$ stabilises at some s. Since S does not stabilise, Lemma 3.0.2 implies that $|\{\ell(\Delta(x_{\kappa+1}, x_\nu)) \mid \nu < \kappa\}| = \kappa$. Since h is one-to-one we have $|\{\beta_h(x_{\kappa+1}, x_\nu) \mid \nu < \kappa\}| = \kappa$, so we may choose a $\xi < \kappa$ with $\beta_h(x_{\kappa+1}, x_\xi) > \gamma_h(s)$. Then $f(\{x_{\kappa+1}, x_\xi, x_0\}) = 1$.

In the second case, consider a set $Y = \{x_i \mid i < \omega\} \in [{}^{\alpha}2]^{\omega}$ with $x_m < x_n$ for $m < n < \omega$. Assume towards a contradiction that Y were homogeneous in colour 0. Then for any $i < \omega$, we have $\beta_h(x_{i+1}, x_{i+2}) < \beta_h(x_i, x_{i+1})$, by considering the triple $\{x_i, x_{i+1}, x_{i+2}\}$. Then $\langle \beta_h(x_i, x_{i+1}) \mid i < \omega\rangle$ is an infinite decreasing sequence of ordinals, a contradiction.

The remaining cases in the proof for $m = 3$ are analogous.

The proof for $m \geqslant 4$ works similarly by considering the following colouring $f : [{}^{\alpha}2]^m \to 2$. If $\vec{x} \in [{}^{\alpha}2]^m$ and $x_0 <_{lex} \ldots < x_{m-1}$, let $f(\vec{x}) = 0$ if $\beta_h(x_0, x_1) < \beta_h(x_{m-2}, x_{m-1})$. □

Unlike for other results in this paper, assuming the Axiom of Choice, there is a linear ordering (even a well-ordering) that satisfies the partition relation in Theorem 3.1.2. In fact, by the Erdős-Rado-Theorem [956ER, Theorem 39] $(2^{2^\kappa})^+ \longrightarrow (\kappa^+)^3_\kappa$ holds. We do not know whether it is consistent with ZFC that there is a linear order L such that neither $\omega_2 \leqslant L$ nor $\omega_2^* \leqslant L$ and $L \longrightarrow (2 + \omega^* \vee \omega, \omega^* \vee \omega + 2)^3$.

3.2. **The classification.** The following result shows that the previous theorems solve the case of triple-colourings in the Cantor space completely, given that all sets of reals have the property of Baire.

We will only consider partition relations such that in no disjunction there are linear orders K, L with $K \leqslant L$, since in this case L can be omitted without changing the truth value of the partition relation.

Theorem 3.2.1. *Suppose that the principle of dependent choices* DC *holds true and all sets of reals have the property of Baire. Suppose that K_μ and L_ν are suborders of $\langle {}^{\omega}2, <_{lex}\rangle$ for all $\mu < \kappa$ and $\nu < \lambda$. Then the partition relation*

$$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow \left(\bigvee_{\nu<\kappa} K_\nu, \bigvee_{\nu<\lambda} M_\nu\right)^3$$

holds true if and only if one of the following cases applies.

(a) *$K_\xi \leqslant \omega + 1$ and $K_\rho \leqslant 1 + \omega^*$ for some $\xi, \rho < \kappa$,*
(b) *$M_\xi \leqslant 1 + \omega^*$ and $M_\rho \leqslant \omega + 1$ for some $\xi, \rho < \lambda$,*
(c) *$K_\xi, M_\rho \leqslant \omega + 1$ for some $\xi < \kappa$, $\rho < \lambda$,*
(d) *$K_\xi, M_\rho \leqslant 1 + \omega^*$ for some $\xi < \kappa$, $\rho < \lambda$.*

Moreover, if none of these cases applies, then the relation is inconsistent with ZF.

Proof. Note that $K_\xi = K_\rho$ is finite if $\xi = \rho$ in (a), and similarly in (b).

We first consider cases in which the partition relation fails. First assume that $K_\mu \nleqslant \omega + 1$ for all $\mu < \kappa$ and $M_\nu \nleqslant 1+\omega^*$ for all $\nu < \lambda$. We claim that the partition relation in question fails. Note that by DC, for any linear order K, $K \leqslant \omega + 1$ is equivalent to $\omega^* \nleqslant K \wedge \omega + 2 \nleqslant K$, and symmetrically, $K \leqslant 1 + \omega^*$ is equivalent to $\omega \nleqslant K \wedge 2 + \omega^* \nleqslant K$. Hence the partition relation in question implies $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\omega^* \vee \omega + 2, 2 + \omega^* \vee \omega)^3$, contradicting Theorem 3.1.2 for $\kappa = \omega$. Second, assume that $K_\mu \nleqslant 1 + \omega^*$ for all $\mu < \kappa$ and $M_\nu \nleqslant \omega + 1$ for all $\nu < \lambda$. This can be dealt with symmetrically.

The remaining cases are as follows, and in each case the partition relation holds. If there are $\xi, \rho < \kappa$ such that $K_\xi \leqslant \omega + 1$ and $K_\rho \leqslant 1 + \omega^*$, then the relation holds by Theorem 2.1.3. The argument is analogous if there are $\xi, \rho < \lambda$ such that $M_\xi \leqslant \omega + 1$ and $M_\rho \leqslant 1 + \omega^*$ If there are $\xi < \kappa$ and $\rho < \lambda$ with $K_\xi \leqslant \omega + 1$ and $M_\rho \leqslant \omega + 1$, then the relation holds by Theorem 2.1.5. An analogous argument works if there are $\xi < \kappa$ and $\rho < \lambda$ with $K_\xi \leqslant 1 + \omega^*$ and $M_\rho \leqslant 1 + \omega^*$. □

4. Negative partition relations for quadruples

In this section, we prove several negative partition theorems for partitions of $[{}^{\alpha}2]^4$ by providing colourings avoiding sets of certain order types in one colour and avoiding quintuples, sextuples, septuples, octuples or nonuples in the other. We first give an overview over the negative partition relations.

Summary 4.0.1. *If α is an ordinal, then the following statements hold.*

(Theorem 4.3.2) $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (5,\ \omega^* + \omega)^4$,

(Theorem 4.3.4) $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (5,\ \omega + \omega^*)^4$,

(Theorem 4.6.1) $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (7,\ \omega^* + \omega \vee \omega + \omega^*)^4$.

Summary 4.0.2. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then the following statements hold.*

(Theorem 4.3.6)	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (5,\ 2+\kappa^* \vee \kappa+2 \vee \eta)^4$,
(Theorem 4.3.7)	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (5,\ \omega^*+\omega \vee \kappa+2+\kappa^* \vee (\kappa 2)^* \vee \kappa 2)^4$,
(Theorem 4.5.2)	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (6,\ \omega^*+\omega \vee \kappa+\omega \vee \omega^*+\kappa^*)^4$,
(Theorem 4.5.3)	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (6,\ \omega+\omega^* \vee 2+\kappa^* \vee \kappa+2)^4$,
(Theorem 4.5.1 (a))	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (6,\ \kappa^*+\kappa \vee 2+\kappa^* \vee \kappa 2 \vee \omega\omega^*)^4$,
(Theorem 4.5.1 (b))	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (6,\ \kappa^*+\kappa \vee (\kappa 2)^* \vee \kappa+2 \vee \omega^*\omega)^4$,
(Theorem 4.6.3 (a))	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (7,\ \omega^*+\omega \vee 2+\kappa^* \vee \kappa+\omega)^4$,
(Theorem 4.6.3 (b))	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (7,\ \omega^*+\omega \vee \omega^*+\kappa^* \vee \kappa+2)^4$,
(Theorem 4.6.2)	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (7,\ \kappa^*+\kappa \vee \kappa+2 \vee 2+\kappa^* \vee \eta)^4$,
(Theorem 4.8.2)	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (8,\ \kappa^*+\omega \vee \omega^*+\kappa \vee 2+\kappa^* \vee \kappa+2 \vee \omega\omega^* \vee \omega^*\omega)^4$,
(Theorem 4.8.1)	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (8,\ \omega^*+\omega \vee \omega+\omega^* \vee (\kappa 2)^* \vee \kappa 2)^4$,
(Theorem 4.9.1)	$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (9,\ \omega^*+\omega \vee \omega+\omega^* \vee \kappa+2 \vee 2+\kappa^*)^4$.

The theorems collected it Summary 4.0.2 are consequences of certain order types enforcing the presence of certain kinds of quadruples. Long well-ordered (anti-well-ordered) sets, for instance, enforce the presence of many dextral (sinistral) combs. Every copy of the integers contains numerous candelabra. Every set of order type $\omega+\omega^*$ contains many a bouquet and so on.

These structural distinctions are, however, not yet sufficient to prove the theorems above. This can be easily seen for theorems involving well-ordered or anti-well-ordered sets since they neither need to contain a bouquet nor a candelabrum. Yet finite sets may contain only dextral or only sinistral combs so without further differentiation one would be unable to prove a negative partition relation with a well-ordered target in the first colour and a finite one in the second. By employing the functions γ and β we are able to prove the aforementioned statements. We provide figures (cf. page 30) for colourings employed in Summary 4.0.1, for reasons of space we only provide figures for the colouring of one of the theorems included in Summary 4.0.2, for Theorem 4.5.1.

4.1. **Lemmata in finite combinatorics.** The following lemma is needed in the proof of Theorem 4.5.1.

Lemma 4.1.1. *For all ordinals α every sextuple within $\langle {}^{\alpha}2, <_{lex}\rangle$ contains a cactus, lilac, sinistral bouquet, dextral olivillo or dextral grape (and, by symmetry, a cactus, lilac, dextral bouquet, sinistral olivillo or sinistral grape).*

Proof. Let $\vec{s} \in [{}^{\alpha}2]^6$ be given. Let $i \leqslant 4$ be such that $\delta(s_i, h_{i+1})$ is minimised. If $i \geqslant 3$ then $\{s_0, s_1, s_2, s_3, s_5\}$ is a sinistral cactus or there is a $j \in \{1,2\}$ such that $\{s_0, h_j, h_{j+1}, s_5\}$ is a sinistral bouquet.

If $i \in \{2,3\}$ then $\{s_0, s_1, s_2, s_5\}$ is a sinistral bouquet or $\{s_0, s_1, s_2, s_4, s_5\}$ is a sinistral lilac.

If $i = 1$ then $\{s_0, s_2, s_3, s_4, s_5\}$ is a dextral grape or there is an $i \in \{3,4\}$ such that $\{s_0, s_1, s_2, s_i, s_{i+1}\}$ is a dextral lilac.

So assume that $i = 0$ and consider $\vec{p} \overset{df}{=} \{s_1, \ldots, s_5\}$. Assuming that $\vec{p}$ is no cactus, lilac, dextral olivillo or dextral grape and does not contain a sinistral bouquet we may conclude that it is a dextral mistletoe or a dextral guinea flower. In the first case $\{s_0, s_1, s_3, s_4, s_5\}$ is a dextral cactus and in the second it is a dextral olivillo.

□

The following lemma is needed in the proof of Theorem 4.5.3.

Lemma 4.1.2. *For all ordinals α every sextuple within $\langle {}^{\alpha}2, <_{lex}\rangle$ contains a candelabrum, cactus, rose or grape.*

Proof. Let $\vec{s} \in [{}^{\alpha}2]^6$. Consider $\vec{p} \stackrel{df}{=} \{s_i \mid i \leqslant 4\}$. Since mistletoes, lilacs and guinea flowers contain candelabra we are finished unless $\vec{p}$ is an olivillo so suppose it is. If $\vec{p}$ is sinistral then $\{s_1, s_2, s_4, s_5\}$ is a candelabrum and if $\vec{p}$ is dextral then $\{s_i \mid 2 \leqslant i\}$ is one. $\square$

We leave one of the easier lemmata of this sort to the reader as an exercise. It is needed in the proof of Theorem 4.5.2.

Exercise 4.1.3. *For all ordinals α every sextuple within $\langle {}^{\alpha}2, <_{lex}\rangle$ contains a bouquet, cactus or antler.*

The following lemma is needed in the proof of Theorem 4.6.2.

Lemma 4.1.4. *For all ordinals α every septuple within $\langle {}^{\alpha}2, <_{lex}\rangle$ contains a cactus, rose, olivillo, grape or mistletoe.*

Proof. Let $\vec{s} \in [{}^{\alpha}2]^7$ be given and let $i < 7$ be such that $\delta(s_i, s_{i+1})$ is minimised. We may suppose without loss of generality that $i \leqslant 2$. Let $\vec{q} \stackrel{df}{=} \{s_j \mid 3 \leqslant j \leqslant 6\}$ and $\vec{p} \stackrel{df}{=} q \cup \{s_0\}$. Now if $\vec{q}$ is a dextral comb then $\vec{p}$ is a dextral cactus. If $\vec{p}$ is a sinistral comb, then $\vec{p}$ is a dextral grape. If $\vec{q}$ is a dextral bouquet, then $\vec{p}$ is a dextral olivillo. Furthermore, if $\vec{q}$ is a sinistral bouquet, then $\vec{p}$ is a dextral rose. Finally, if $\vec{q}$ is a candelabrum, then $\vec{p}$ is a dextral mistletoe. $\square$

The following lemma is needed in the proof of Theorem 4.6.3.

Lemma 4.1.5. *For all ordinals α every septuple within $\langle {}^{\alpha}2, <_{lex}\rangle$ contains an antler, cactus, dextral olivillo, dextral grape or sinistral bouquet (and, by symmetry, an antler, cactus, sinistral olivillo, sinistral grape or dextral bouquet).*

Proof. Let $\vec{s} \in [{}^{\alpha}2]^7$ be given and $i \leqslant 5$ such that $\delta(s_i, s_{i+1})$ is minimised. We consider several cases in turn:

If $i \geqslant 3$ then $\{s_0, s_1, s_2, s_6\}$ or $\{s_0, s_2, s_3, s_6\}$ is a sinistral bouquet or $\{s_0, \dots, s_3, s_6\}$ is a sinistral cactus.

If $i = 2$ then $\{s_0, \dots, s_3\}$ is a sinistral bouquet or $\{s_0, s_1, s_2, s_4, s_5, s_6\}$ or $\{s_0, \dots, s_5\}$ is an antler or $\{s_2, \dots, s_6\}$ is a dextral grape.

If $i \leqslant 1$ let j be such that $i < j \leqslant 5$ and $\delta(s_j, s_{j+1})$ is minimised. If $j \leqslant 3$ then $\{s_0, s_j, s_4, s_5, s_6\}$ is a dextral olivillo or a dextral grape and if $j \geqslant 4$ then $\{s_2, s_3, s_4, s_6\}$ is a sinistral bouquet or $\{s_0, s_2, s_3, s_4, s_6\}$ is a dextral grape. $\square$

The following lemma is needed in the proof of Theorem 4.8.1.

Lemma 4.1.6. *For all ordinals α every octuple within $\langle {}^{\alpha}2, <_{lex}\rangle$ contains a cactus, grape or lilac.*

Proof. Let α be an ordinal and $\{o_0, \dots, o_7\} \in \langle {}^{\alpha}2, <_{lex}\rangle$. Let $i < 8$ be such that $\delta(o_i, o_{i+1})$ is minimal. We may suppose without loss of generality that $i \leqslant 3$.

If $i \geqslant 1$ and there is a node $s \sqsupset \Delta(o_i, o_{i+1})^\frown\langle 1\rangle$ and j such that $o_j \sqsupset s^\frown\langle 0\rangle$ and $o_{j+1}, o_{j+2} \sqsupset s^\frown\langle 1\rangle$ then $\{o_0, o_1, o_j, o_{j+1}, o_{j+2}\}$ is a dextral lilac.

If there is no such node then $\{o_0, o_4, o_5, o_6, o_7\}$ is a dextral grape.

If $i = 0$ let $j < 8$ be such that $\delta(o_j, o_{j+1})$ is minimised. We distinguish two subcases:

First assume that $j \leqslant 3$. If $\{o_k \mid j < k \leqslant j + 4\}$ is a sinistral comb then $\{o_0, o_{j+1}, o_{j+2}, o_{j+3}, o_{j+4}\}$ is a dextral grape. Otherwise $\{o_0, o_1, o_{j+2}, o_{j+3}, o_{j+4}\}$ is a dextral cactus.

Now assume that $j \geqslant 4$. If $\{o_k \mid 1 \leqslant k \leqslant 4\}$ is dextral comb then $\{o_k \mid k \leqslant 4\}$ is a dextral cactus. Otherwise $\{o_0, o_1, o_2, o_3, o_7\}$ is a dextral grape. $\square$

The following lemma is needed in the proof of Theorem 4.8.2.

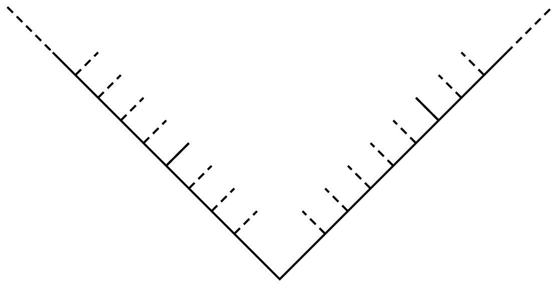


FIGURE 4. A candelabrum within a copy of the integers

Lemma 4.1.7. *For all ordinals α every octuple within $\langle {}^{\alpha}2, <_{lex}\rangle$ contains a cactus, rose, olivillo, grape or an antler.*

Proof. Let $\vec{o} \in {}^{\alpha}2$ be given and $i \leqslant 6$ be such that $\delta(o_i, o_{i+1})$ is minimised. We may assume without loss of generality that $i \leqslant 3$. Suppose that $\vec{o}$ does not contain an antler. Clearly $[\{o_0, \ldots, o_i\}]^3$ only contains dextral triples or $[\{o_{i+1}, \ldots, o_7\}]^3$ only contains sinistral triples.

If $i = 3$ then we may suppose without loss of generality that the latter is the case which implies that $\{o_3, \ldots, o_7\}$ is a dextral grape.

If $i \leqslant 2$ let j with $i < j \leqslant 6$ be such that $\delta(o_j, o_{j+1})$ is minimised. If $j \leqslant 4$ then $\{o_0, o_j, o_5, o_6, o_7\}$ is a dextral cactus or a dextral olivillo, otherwise $\{o_0, o_3, o_4, o_5, o_7\}$ is a dextral rose or a dextral grape. □

The following lemma is needed in the proof of Theorem 4.9.1.

Lemma 4.1.8. *For all ordinals α every nonuple within $\langle {}^{\alpha}2, <_{lex}\rangle$ contains a cactus, rose, olivillo or grape.*

Proof. If α is an ordinal and $N \in [\langle {}^{\alpha}2, <_{lex}\rangle]^9$ then there is an $x \in N$ such that there are pairwise different $x_0, x_1, x_2, x_3 \in N$ such that $\Delta(x, x_0) \sqsubseteq \Delta(x, x_1) \sqsubseteq \Delta(x, x_2) \sqsubseteq \Delta(x, x_3)$. Then $\{x, x_0, x_1, x_2, x_3\}$ is a cactus, rose, olivillo or grape. □

4.2. **Lemmata in infinite combinatorics.**

Lemma 4.2.1. *Suppose that α is an infinite ordinal. Then for every set $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$, there is a candelabrum in $\{z_0, z_1, z_2, z_3\}_{<_{lex}} \in [Z]^4$.*

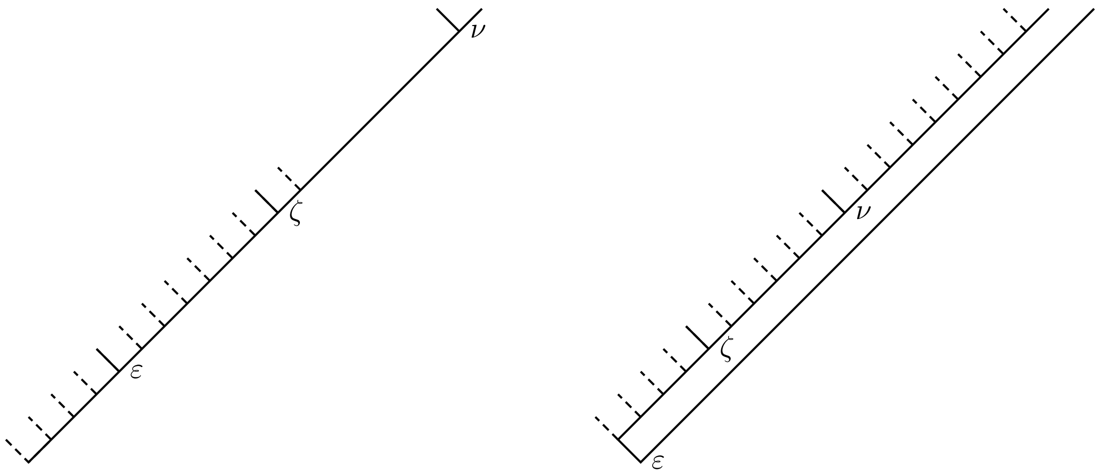
Proof. Note Figure 4. Let $\langle x_n \mid n < \omega\rangle$ be the order-reversing enumeration of the lower half of Z and $\langle y_n \mid n < \omega\rangle$ the order-preserving enumeration of its upper half such that $\Delta(x_0, y_0)$ is minimised. Then $\{x_1, x_0, y_0, y_1\}$ provides what was demanded. □

Lemma 4.2.2. *Suppose that α is an infinite ordinal and $h : \alpha \hookrightarrow |\alpha|$ is an injection.*

(1) *For every $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$, at least one of the following conditions hold.*
   (a) *There is a candelabrum $\vec{x} = \{x_0, x_1, x_2, x_3\} \in [Z]^4$ with $\beta_h(x_1, x_2) < \beta_h(x_0, x_1)$.*
   (b) *There is a sinistral comb $\vec{x} = \{x_0, x_1, x_2, x_3\} \in [Z]^4$ with $\beta_h(x_1, x_2) < \beta_h(x_0, x_1) < \beta_h(x_2, x_3)$.*

(2) *For every $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$, at least one of the following conditions hold.*
   (a) *There is a candelabrum $\vec{x} = \{x_0, x_1, x_2, x_3\} \in [Z]^4$ with $\beta_h(x_1, x_2) < \beta_h(x_2, x_3)$.*
   (b) *There is a dextral comb $\vec{x} = \{x_0, x_1, x_2, x_3\} \in [Z]^4$ with $\beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1)$.*

Proof. Let $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$ and $s \in {}^{<\alpha}2$ be the lowest splitting node of elements of Z. So let $\langle x_n \mid n < \omega\rangle$ be the enumeration of $\{x \in Z \mid x \sqsupset s^\frown\langle 0\rangle\}$ which is order-reversing. Let $y, z \in Z$ be such that $y, z \sqsupset s^\frown\langle 1\rangle$. If there is an $n < \omega$ for which $\beta_h(x_{n+1}, x_n) > \beta_h(x_n, y)$ then the candelabrum $\{x_{n+1}, x_n, y, z\}$ provides what was demanded. If not then by finitude of decreasing sequences of ordinals there has to be an $n < \omega$ such that $\beta_h(x_{n+2}, x_{n+1}) > \beta_h(x_{n+1}, x_n)$. Then the sinistral comb $\{x_{n+2}, x_{n+1}, x_n, y\}$ provides what was demanded. □



$\gamma_h(\nu) < \gamma_h(\varepsilon) < \gamma_h(\zeta)$ $\qquad$ $\gamma_h(\varepsilon) < \gamma_h(\zeta) < \gamma_h(\nu)$.

FIGURE 5. A dextral comb and a sinistral bouquet within sets of order type $\kappa + 2$

Lemma 4.2.3. *Suppose that α is an infinite ordinal and $h : \alpha \hookrightarrow |\alpha|$ is injective. Then for every $Q \in [{}^{\alpha}2]^{\eta}$ there is a bouquet $\vec{q} = \{q_0, q_1, q_2, q_3\}_{<_{lex}} \in [Q]^4$ such that*

(1) *$\beta_h(q_0, q_1) < \beta_h(q_2, q_3) < \beta_h(q_1, q_2)$ if $\vec{q}$ is dextral and*
(2) *$\beta_h(q_2, q_3) < \beta_h(q_0, q_1) < \beta_h(q_1, q_2)$ if $\vec{q}$ is sinistral.*

Proof. Consider a $Q \in [{}^{\alpha}2]^{\eta}$. Let $s \in {}^{<\alpha}2$ be such that there are $p_0, r_0 \in Q$ with $\Delta(p_0, r_0) = s$ and $\gamma_h(s)$ is minimised. Now inductively in step $n < \omega$ by density of Q there has to be a $t \in]p_n, r_n[\cap Q$. If $\Delta(p_n, t) = s$ then $p_{n+1} \overset{df}{=} p_n$ and $r_{n+1} \overset{df}{=} t$, otherwise $\Delta(t, r_n) = s$ and we define $p_{n+1} \overset{df}{=} t$ and $r_{n+1} \overset{df}{=} r_n$. At most one of the sequences $\vec{p} = \langle p_n \mid n < \omega\rangle$ and $\vec{r} = \langle r_n \mid n < \omega\rangle$ can stabilise. Suppose without loss of generality that $\vec{p}$ does not stabilise. Again without loss of generality suppose that $p_{n+1} = p_n$ for no $n < \omega$. Then there is an $n < \omega$ such that $\beta_h(p_n, p_{n+1}) < \beta_h(p_{n+1}, p_{n+2})$. Then $\{p_n, p_{n+1}, p_{n+2}, r_0\}$ is a sinistral bouquet and provides what was demanded. □

Lemma 4.2.4. *Suppose that α is an infinite ordinal and $h : \alpha \hookrightarrow |\alpha|$ in an injection.*

(1) *For every $A \in [{}^{\alpha}2]^{2+\kappa^*}$, there is a dextral bouquet or sinistral comb $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4$ with $\beta_h(q_0, q_1) < \beta_h(q_2, q_3) < \beta_h(q_1, q_2)$.*
(2) *For every $A \in [{}^{\alpha}2]^{\kappa+2}$, there is a sinistral bouquet or dextral comb $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4$ with $\beta_h(q_2, q_3) < \beta_h(q_0, q_1) < \beta_h(q_1, q_2)$.*

Proof. Note Figure 5. Since the first half of the lemma is a symmetric statement, only the second half is going to be proved.

First let $\kappa \overset{df}{=} |\alpha|$ and consider a $B \in [{}^{\alpha}2]^{\kappa+2}$. Let $\langle b_\nu \mid \nu < \kappa{+}2\rangle$ be the order-preserving enumeration of B. We distinguish two cases. First assume that the sequence $\vec{s} \overset{df}{=} \langle \Delta(b_\nu, b_{\kappa+1}) \mid \nu < \kappa\rangle$ stabilises, say at $s \in {}^{<\alpha}2$ from $\zeta < \kappa$ onwards. Since the domain and the range of h share their respective cardinality and by lemma 3.0.2 there has to be a $\rho \in \kappa \setminus \zeta$ such that $\beta_h(b_\rho, b_{\rho+1}) > \gamma_h(s)$ and $|\{\nu < \kappa \mid b_\nu \sqsupset \Delta(b_\rho, b_{\rho+1})\}| = \kappa$. Then choose a $\xi \in \kappa \setminus \rho$ such that $\beta_h(b_\xi, b_{\xi+1}) > \beta_h(b_\rho, b_{\rho+1})$. Now the sinistral bouquet $\{b_\rho, b_\xi, b_{\xi+1}, b_{\kappa+1}\}$ provides what was demanded.

So assume that $\vec{s}$ does not stabilise. Then, using lemma 3.0.2, pick a $\zeta < \kappa$ such that $\beta_h(b_\zeta, b_{\zeta+1}) > \beta_h(b_\kappa, b_{\kappa+1})$ and $b_\kappa \sqsupset \Delta(b_\zeta, b_{\zeta+1})$. After that again pick a $\rho \in \kappa \setminus \zeta$ with $\beta_h(b_\rho, b_{\rho+1}) > \beta_h(b_\zeta, b_{\zeta+1})$ and $b_\kappa \sqsupset \Delta(b_\rho, b_{\rho+1})$. Then the dextral comb $\{b_\zeta, b_\rho, b_\kappa, b_{\kappa+1}\}$ provides what was demanded. □

The following lemma is only used in the proof of Theorem 4.8.2.

Lemma 4.2.5. *Suppose that α is an infinite ordinal and $h : \alpha \hookrightarrow |\alpha|$ an injection.*

(1) *For every* $Z \in [{}^{\alpha}2]^{\kappa^*+\omega}$, *there is a candelabrum* $\{z_0, z_1, z_2, z_3\}_{<_{lex}} \in [Z]^4$ *with* $\beta_h(z_1, z_2) < \beta_h(z_0, z_1)$.
(2) *For every* $Z \in [{}^{\alpha}2]^{\omega^*+\kappa}$, *there is a candelabrum* $\{z_0, z_1, z_2, z_3\}_{<_{lex}} \in [Z]^4$ *with* $\beta_h(z_1, z_2) < \beta_h(z_2, z_3)$.

Proof. Since the two halves of the lemma are symmetric to one another, we are only going to prove the second one. So let Z be as in the lemma and let $s \in {}^{<\alpha}2$ be the minimal splitting node of elements of Z. Since Z has no least element there are $z_0, z_1 \sqsupset s^\frown\langle 0\rangle$. Let $\langle z_\nu \mid \nu < \kappa\rangle$ be the order-preserving enumeration of $\{z \in Z \mid z \sqsupset s^\frown\langle 1\rangle\}$. Let $\zeta < \kappa$ be such that $\beta_h(z_\zeta, z_{\zeta+1}) > \gamma_h(s)$. Then the candelabrum $\{z_0, z_1, z_\zeta, z_{\zeta+1}\}$ provides what was demanded. □

Lemma 4.2.6. *Suppose that* α *is an infinite ordinal and* $h : \alpha \hookrightarrow |\alpha|$ *is an injection. Then for every* $Z \in [{}^{\alpha}2]^{\kappa^*+\kappa}$, *there is a candelabrum* $\{z_0, z_1, z_2, z_3\}_{<_{lex}} \in [Z]^4$ *with* $\beta_h(z_1, z_2) < \min(\beta_h(z_0, z_1), \beta_h(z_2, z_3))$.

Proof. Let Z be as in the lemma and let $s \in {}^{<\alpha}2$ be the minimal splitting node of elements of Z. Let $\langle x_\nu \mid \nu < \kappa\rangle$ be an order-reversing enumeration of elements of Z extending $s^\frown\langle 0\rangle$ and let $\langle y_\nu \mid \nu < \kappa\rangle$ be an order-presering enumeration of elements of Z extending $s^\frown\langle 1\rangle$. Then let $\zeta, \rho < \kappa$ be such that $\beta_h(x_{\zeta+1}, x_\zeta) > \gamma_h(s)$ and $\beta_h(y_\rho, y_{\rho+1}) > \gamma_h(s)$. Now the candelabrum $\{x_{\zeta+1}, x_\zeta, y_\rho, y_{\rho+1}\}$ provides what was demanded. □

The following lemma is only used in the proof of Theorem 4.8.2.

Lemma 4.2.7. *Suppose that* α *is an infinite ordinal and* $h : \alpha \hookrightarrow |\alpha|$ *is an injection.*
(1) *For every* $X \in [{}^{\alpha}2]^{\omega\omega^*}$, *there is a candelabrum* $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [X]^4$ *with* $\beta_h(x_1, x_2) < \beta_h(x_0, x_1)$.
(2) *For every* $X \in [{}^{\alpha}2]^{\omega^*\omega}$, *there is a candelabrum* $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [X]^4$ *such that* $\beta_h(x_1, x_2) < \beta_h(x_2, x_3)$.

Proof. Note Figure 6. Since the two halves of the lemma are symmetric to each other we only need to prove the first one. So let $X \in [{}^{\alpha}2]^{\omega\omega^*}$. Let s_0 be the first splitting node of elements of X and for every $k < \omega$ let s_{k+1} be the first splitting node of elements of X extending $s_k^\frown\langle 0\rangle$.

$$\{k < \omega \mid \operatorname{otyp}\{x \in X \mid x \sqsupset s_k^\frown\langle 1\rangle\} \geqslant \omega\} \text{ is infinite.} \tag{1}$$

This is the case because any initial segment of X has the same order type as X itself so if (1) fails there is an element of X which for no $k < \omega$ extends $s_k^\frown\langle 1\rangle$. Then it has to extend $\lim_{k<\omega} s_k$ in which case $1 + \omega^*$ embeds into X which is a contradiction.

So let $\langle k_i \mid i < \omega\rangle$ be an enumeration of the set in (1). Since there is no decreasing sequence of ordinals there has to be an $i < \omega$ such that $\gamma_h(s_{k_{i+1}}) > \gamma_h(s_{k_i})$. So pick an $a \in X$ with $a \sqsupset s_{k_{i+1}}^\frown\langle 0\rangle$, some $b \in X$ such that $b \sqsupset s_{k_{i+1}}^\frown\langle 1\rangle$ and $\{c, d\}_{<_{lex}} \in [X]^2$ satisfying $c, d \sqsupset s_{k_i}^\frown\langle 1\rangle$. Now clearly the candelabrum $\{a, b, c, d\}$ provides what was demanded. □

The following lemma is only used in the proof of Theorem 4.5.1.

Lemma 4.2.8. *Suppose that* α *is an infinite ordinal and* $h : \alpha \hookrightarrow |\alpha|$ *is an injection.*
(1) *For every* $X \in [{}^{\alpha}2]^{\omega\omega^*}$, *at least one of the following conditions hold.*
 (a) *There is a candelabrum* $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [X]^4$ *with* $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3))$.
 (b) *There is a dextral comb* $\vec{x} = \{x_0, x_1, x_2, x_4\}_{<_{lex}} \in [X]^4$ *with* $\beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1)$.
(2) *For every* $X \in [{}^{\alpha}2]^{\omega^*\omega}$, *at least one of the following conditions hold.*
 (a) *There is a candelabrum* $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [X]^4$ *with* $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3))$.

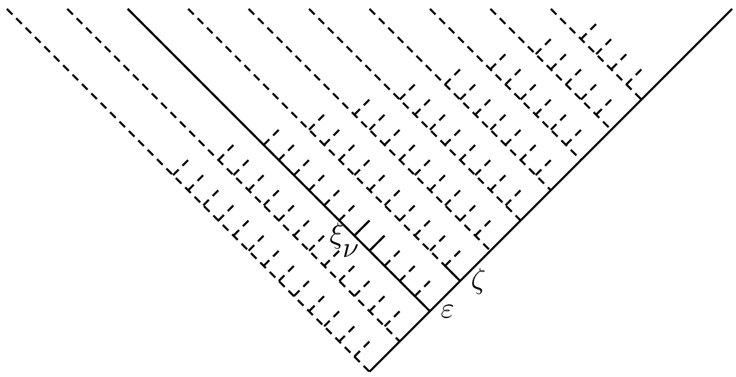


FIGURE 6. A sinistral comb and a candelabrum in a set of order type $\omega^*\omega$

(b) *There is a sinistral comb* $\vec{x} = \{x_0, x_1, x_2, x_4\}_{<_{lex}} \in [X]^4$ *with* $\beta_h(x_1, x_2) < \beta_h(x_0, x_1) < \beta_h(x_2, x_3)$.

Proof. Note Figure 6. Since the two halves of the lemma are symmetric, it suffices only to prove the first one. Suppose that $X \in [{}^\alpha 2]^{\omega\omega^*}$. Let s_0 be the first splitting node of elements of X and for every $k < \omega$ let s_{k+1} be the first splitting node of elements of X extending $s_k^\frown\langle 0\rangle$. Note that as in the proof of Lemma 4.2.7 for infinitely many $k < \omega$ we have otyp $\{x \in X \mid x \sqsupset s_k^\frown\langle 1\rangle\} \geqslant \omega$. So let $\langle k_i \mid i < \omega\rangle$ be an enumeration of these k. Since there is no decreasing sequence of ordinals there has to be an $i < \omega$ such that $\gamma_h(s_{k_{i+1}}) > \gamma_h(s_{k_i})$. If there are $c, d \sqsupset s_{k_i}^\frown\langle 1\rangle$ with $\beta_h(c, d) > \gamma_h(s_{k_i})$ then for $a \sqsupset s_{k_{i+1}}^\frown\langle 0\rangle$ and $b \sqsupset s_{k_{i+1}}^\frown\langle 1\rangle$ the candelabrum $\{a, b, c, d\}$ provides what was demanded. So suppose now that for all $c, d \sqsupset s_{k_i}$ we have $c = d$ or $\beta_h(c, d) < \gamma_h(s_{k_i})$. Let $\langle c_i \mid i < \omega\rangle$ be an ascending enumeration of elements of $\{x \in X \mid x \sqsupset s_{k_i}^\frown\langle 1\rangle\}$. The finitude of decreasing sequences of ordinals implies that there has to be an $n < \omega$ such that $\beta_h(c_n, c_{n+1}) < \beta_h(c_{n+1}, c_{n+2})$. But then for any $b \sqsupset s_{k_{i+1}}$ the dextral comb $\{b, c_n, c_{n+1}, c_{n+2}\}$ provides what was demanded. □

Lemma 4.2.9. *Suppose that α is an infinite ordinal and $h : \alpha \hookrightarrow |\alpha|$ an injection.*

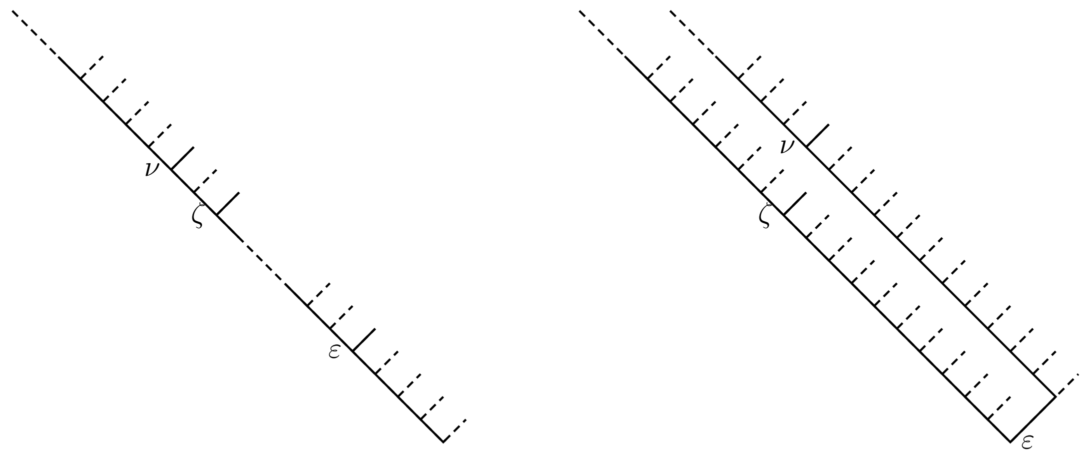
(1) *For every $A \in [{}^\alpha 2]^{\omega^*+\kappa^*}$, at least one of the following conditions hold.*
   (a) *There is a sinistral comb $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4$ with $\beta_h(a_1, a_2) < \beta_h(a_0, a_1) < \beta_h(a_2, a_3)$.*
   (b) *There is a dextral bouquet $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4$ with $\beta_h(a_0, a_1) < \beta_h(a_2, a_3) < \beta_h(a_1, a_2)$ and there is a candelabrum $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4$ with $\beta_h(a_1, a_2) < \beta_h(a_0, a_1)$.*

(2) *For every $B \in [{}^\alpha 2]^{\kappa+\omega}$, at least one of the following conditions hold.*
   (a) *There is a dextral comb $\{b_0, b_1, b_2, b_3\}_{<_{lex}} \in [B]^4$ with $\beta_h(b_1, b_2) < \beta_h(b_2, b_3) < \beta_h(b_0, b_1)$.*
   (b) *There is a sinistral bouquet $\{b_0, b_1, b_2, b_3\}_{<_{lex}} \in [B]^4$ with $\beta_h(b_2, b_3) < \beta_h(b_0, b_1) < \beta_h(b_1, b_2)$ and there is a candelabrum $\{b_0, b_1, b_2, b_3\}_{<_{lex}} \in [B]^4$ with $\beta_h(b_1, b_2) < \beta_h(b_2, b_3)$.*

Proof. Since both halves of the the Lemma are symmetric to each other we are only going to prove the second one. First suppose that there is an $s \in {}^{<\alpha}2$ such that otyp$(B_0) \geqslant \kappa$ and otyp$(B_1) \geqslant \omega$ where $B_i \overset{df}{=} \{b \in B \mid b \sqsupset s^\frown\langle i\rangle\}$ for $i < 2$. Let $\langle x_\nu \mid \nu < \kappa\rangle$ be an ascending enumeration of elements of B_0 and $\langle y_n \mid n < \omega\rangle$ an ascending enumeration of elements of B_1. Then, using Lemma 3.0.2 one can pick a $\zeta < \kappa$ such that $\beta_h(x_\zeta, x_{\zeta+1}) > \gamma_h(s)$ and $\{b \in B_0 \mid b \sqsupset \Delta(x_\zeta, x_{\zeta+1})\}$ has size κ. After that one can choose a $\rho \in \kappa \setminus \zeta$ such that $\beta_h(x_\rho, x_{\rho+1}) > \beta_h(x_\zeta, x_{\zeta+1})$. Then for any $y, z \in B_1$ the candelabrum $\{x_\nu, x_{\nu+1}, y, z\}$ and the sinistral bouquet $\{x_\zeta, x_\rho, x_{\rho+1}, y\}$ provide what was demanded.

Now assume that there is no such s. The nonexistence of infinite decreasing sequences of ordinals yields $m, n < \omega$ such that $\Delta(y_m, y_{m+1})^\frown\langle 1\rangle \sqsubseteq \Delta(y_n, y_{n+1})$ and $\beta_h(y_m, y_{m+1}) < \beta_h(y_n, y_{n+1})$. Now using Lemma 3.0.2 one can find a $\zeta < \kappa$ such that $\Delta(x_\zeta, x_{\zeta+1})^\frown\langle 1\rangle \sqsubseteq \Delta(y_n, y_{n+1})$ and $\beta_h(x_\zeta, x_{\zeta+1}) > \beta_h(y_n, y_{n+1})$. Now the dextral comb $\{x_\zeta, y_m, y_n, y_{n+1}\}$ provides what was demanded. □

Lemma 4.2.10. *Suppose that α is an infinite ordinal and $h : \alpha \hookrightarrow |\alpha|$ is an injection.*



Your preferred relation between $\gamma_h(\varepsilon)$, $\gamma_h(\zeta)$ and $\gamma_h(\nu)$ in a sinistral comb.

$\gamma_h(\varepsilon) < \min(\gamma_h(\zeta)), \gamma_h(\nu))$ in a dextral bouquet.

FIGURE 7. A sinistral comb and a candelabrum in sets of order type $(\kappa 2)^*$

(1) *At least one of the following conditions holds.*
   (a) *For any $A \in [{}^{\alpha}2]^{(\kappa 2)^*}$, there is some candelabrum $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [A]^4$ such that $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3))$.*
   (b) *For each of the following β_h-relations, there is a sinistral comb $\{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [A]^4$ that satisfyies it.*
      (i) $\beta_h(x_1, x_2) < \beta_h(x_0, x_1) < \beta_h(x_2, x_3)$,
      (ii) $\beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1)$,
      (iii) $\beta_h(x_2, x_3) < \beta_h(x_0, x_1) < \beta_h(x_1, x_2)$,
      (iv) $\beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2)$.
(2) *At least one of the following conditions holds.*
   (a) *For any $B \in [{}^{\alpha}2]^{\kappa 2}$, there is a candelabrum $\vec{x} = \{x_0, x_1, x_2, x_3\} \in [B]^4$ such that $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1) < \beta_h(x_2, x_3))$.*
   (b) *For each of the β_h-relations in* (1) (b) *there is a dextral comb $\{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [B]^4$ that satisfies it.*

Proof. Note Figure 7. As the two halves of the lemma are symmetric to each other, it suffices to prove the second one. So let $B \in [{}^{\alpha}2]^{\kappa 2}$ and suppose that for all candelabra $\{t_0, t_1, t_2, t_3\}_{<_{lex}} \in [B]^4$ there is an $i < 2$ with $\beta_h(t_{2i}, t_{2i+1}) < \beta_h(t_1, t_2)$. Via Lemma 3.0.2 this implies that there is a $\{b_\nu \mid \nu < \kappa 2\}_{<_{lex}} \in [B]^{\kappa 2}$ such that $\Delta(b_\zeta, b_{\zeta+1})^\frown\langle 1\rangle \sqsubseteq \Delta(b_\rho, b_{\rho+1})$ for every $\{\zeta, \rho\}_< \in [\kappa 2]^2$. Now for every β_h-relation mentioned above it is easy to choose ζ, ν, ξ, ρ such that $\{b_\zeta, b_\nu, b_\xi, b_\rho\}$ provides what was demanded. □

Lemma 4.2.11. *Let α be an infinite ordinal. Then for every $X \in [{}^{\alpha}2]^{\omega+\omega^*}$, there is a bouquet $\vec{x} = \{x_0, x_1, x_2, x_3\} \in [X]^4$. Moreover, if $h : \alpha \hookrightarrow |\alpha|$ is an injection then $\vec{x}$ may be chosen such that*

(1) *$\beta_h(x_2, x_3) < \beta_h(x_1, x_2)$ if $\vec{x}$ is dextral and*
(2) *$\beta_h(x_0, x_1) < \beta_h(x_1, x_2)$ if $\vec{x}$ is sinistral.*

Proof. Let $X \in [{}^{\alpha}2]^{\omega+\omega^*}$ and let $s \in {}^{<\alpha}2$ be the splitting node of minimal height of elements of X. Then with $X_j \overset{df}{=} \{x \in X \mid x \sqsupset s^\frown\langle j\rangle\}$ we have $\mathrm{otyp}(X_0) \geqslant \omega$ or $\mathrm{otyp}(X_1) \geqslant \omega^*$. Suppose the former holds and let $\langle x_n \mid n < \omega\rangle$ be an ascending enumeration of elements in X_0. There is an $I \in [\omega]^\omega$ such that $\Delta(x_l, x_{l+1}) \sqsupset \Delta(x_k, x_{k+1})^\frown\langle 1\rangle$ for all $\{k, l\}_< \in [I]^2$. The finitude of decreasing sequences of ordinals implies that there is a pair $\{m, n\}_< \in [I]^2$ with $\beta_h(x_m, x_{m+1}) < \beta_h(x_n, x_{n+1})$. Now for any $y \in X_1$ the sinistral bouquet $\{x_m, x_n, x_{n+1}, y\}$ provides what was demanded. □

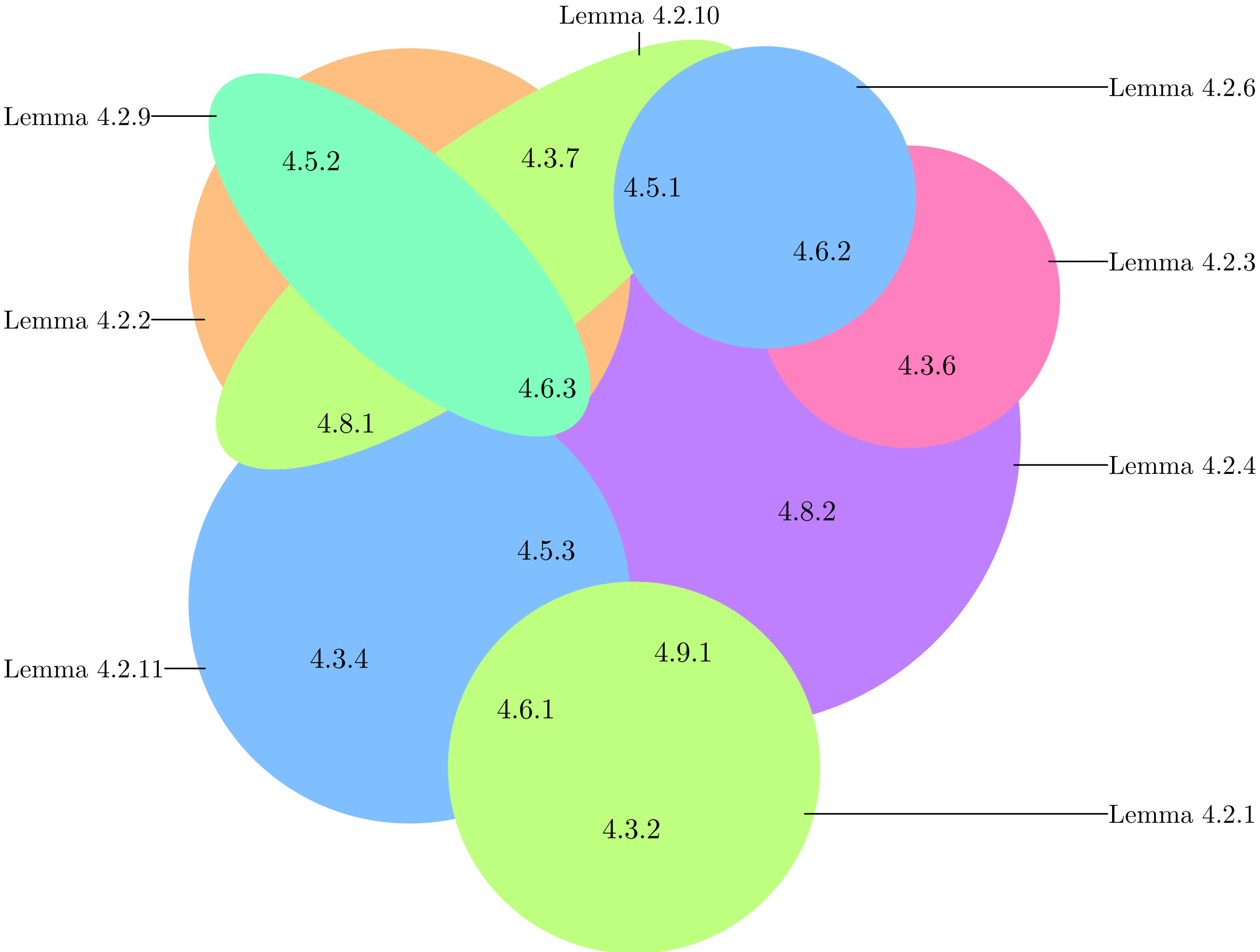


Figure 8. How Lemmata and Theorems relate

Note that Figure 4.2 shows for which theorems of the following sections the lemmata of this section which are quoted at least twice in the proof of a theorem are used. Every ellipse corresponds to a lemma and every numerical code within an ellipse to a theorem in which this lemma is used. Although it somehow looks like one, it is not in the strict sense a Venn diagram (cf. [969Ba]) since whether or not two ellipses overlap fails to have any significance on its own.

4.3. **Quintuples.** In this section, we prove several negative partition relations with 5 on one side of the relation. These results are used in the classification in Section 4.7. We start with a lemma in the light of which we may extend Theorems 4.3.2 and 4.3.4 to higher exponents.

Lemma 4.3.1. *Suppose that κ and λ are ordinals, $\{\sigma_\xi \mid \xi < \lambda\}$ and $\{\tau_\xi \mid \xi < \nu\}$ are families of order types and $\kappa < \lambda$ and $\mu < \nu$ are cardinals such that neither a σ_ξ with $\xi < \kappa$ nor a τ_ξ with $\xi < \mu$ has a last element. Now if n is a natural number and ρ is an order type such that*

$$\rho \not\to \left(\bigvee_{\xi<\kappa} \sigma_\xi \vee \bigvee_{\xi\in\lambda\setminus\kappa} \sigma_\xi, \bigvee_{\xi<\mu} \tau_\xi \vee \bigvee_{\xi\in\nu\setminus\mu} \tau_\xi\right)^n, \textit{ then}$$

$$\rho \not\to \left(\bigvee_{\xi<\kappa} \sigma_\xi \vee \bigvee_{\xi\in\lambda\setminus\kappa} (\sigma_\xi+1), \bigvee_{\xi<\mu} \tau_\xi \vee \bigvee_{\xi\in\nu\setminus\mu} (\tau_\xi+1)\right)^{n+1}.$$

Proof. Suppose that the statement above would fail. Then there is a colouring χ of $[\tau]^n$ which witnesses the failure of the first partition relation. We define $\bar\chi$ by $\{x_0,\dots,x_n\} \mapsto \chi(\{x_0,\dots,x_{n-1}\})$.

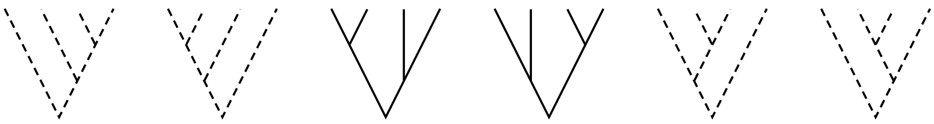


FIGURE 9. Colouring of the splitting types for the proof of Theorem 4.3.2.

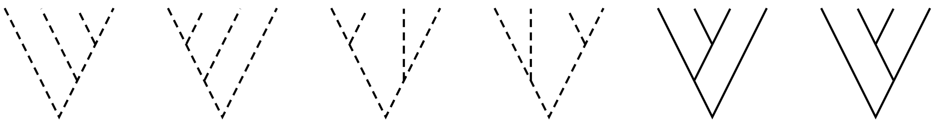


FIGURE 10. Colouring of the splitting types for the proof of Theorem 4.3.4.

Since the latter partition relation holds true there is a homogeneous set H for $\bar{\chi}$. We may suppose without loss of generality that H is homogeneous in colour 0. Then we may distinguish two cases:

First suppose that H has order type σ_ξ with $\xi < \kappa$. Let $\{x_0, \ldots, x_{n-1}\} = \vec{x} \in [H]^n$ be such that $\chi(\vec{x}) = 1$. Since H has no last element we may choose an $x_n \in H$ such that $x_n > x_{n-1}$. Then $\bar{\chi}(\{x_0, \ldots, x_n\}) = 1$, a contradiction.

Second suppose that H has order type $\sigma_\xi + 1$ with $\xi \in \lambda \setminus \kappa$. Let x_n be the last element of H. Let $\{x_0, \ldots, x_{n-1}\} = \vec{x} \in [H \setminus \{x\}]^n$ be such that $\chi(\vec{x}) = 1$. Then $\bar{\chi}(\{x_0, \ldots, x_n\}) = 1$, a contradiction.

□

Theorem 4.3.2. *If α is an ordinal, then $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (5,\ \omega^* + \omega)^4$*

Proof. Suppose that $\vec{x} = (x_0, ..., x_3)$ is a tuple in $[{}^{\kappa}2]^4$ with $x_0 <_{lex} x_1 \cdots <_{lex} x_3$. For the first claim, let $f(\vec{x}) = 1$ if $\vec{x}$ is a candelabrum and $f(\vec{x}) = 0$ otherwise. We claim that there is no homogeneous set for f.

By Lemma 4.2.1 there is no set of order type $\omega^* + \omega$ homogeneous in colour 0.

Suppose that there is a quintuple homogeneous in colour 1. Suppose that $H = \{q_i \mid i < 5\}$ with $q_i <_{lex} q_j$ for $i < j < 5$. If $\delta_{q_2,q_3} < \delta_{q_1,q_2}$, then $\{q_i \in H \mid i < 5\}$ has colour 0, contradicting the assumption. If $\delta_{q_1,q_2} < \delta_{q_2,q_3}$, then $\{q_i \in H \mid i \neq 0\}$ has colour 0, contradicting the assumption. □

By Lemma 4.3.1 we have the following Corollary.

Corollary 4.3.3. *If α is an ordinal and $m > n \geqslant 4$, then $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (m,\ \omega^* + \omega)^n$.*

Theorem 4.3.4. *If α is an ordinal, then $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (5,\ \omega + \omega^*)^4$.*

Proof. Let $g(\vec{x}) = 1$ if $\vec{x}$ is a bouquet and $g(\vec{x}) = 0$ otherwise. We claim that there is no homogeneous set for g.

By Lemma 4.2.11 there is no set of order type $\omega + \omega^*$ homogeneous in colour 0.

So suppose that there is a quintuple H homogeneous in colour 1. Suppose that $H = \{q_i \mid i < 5\}$ with $q_i <_{lex} q_j$ for $i < j < 5$. Since $\{q_i \mid i \neq 4\}$ has colour 1, $\delta_{q_2,q_3} < \delta_{q_1,q_2}$. Then $\{q_i \mid i \neq 0\}$ has colour 0, contradicting the assumption. □

By observing that $\omega + \omega^* = \omega + \omega^* + 1$ and considering Lemma 4.3.1 once more we again have a corollary.

Corollary 4.3.5. *If α is an ordinal and $m > n \geqslant 4$, then $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (m,\ \omega + \omega^*)^n$.*

Theorem 4.3.2 implies that Theorem 2.1.4 does not lift to higher exponents. In the following, we weaken the requirement of an infinite homogeneous set in colour 0 to the requirement that the set has one of two, three, four, five and, in the case of Theorem 4.8.2, six given order types.

Theorem 4.3.6. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (5,\ 2 + \kappa^* \vee \kappa + 2 \vee \eta)^4.$$

Proof. We write $\vec{x}$ for $(x_0, \ldots, x_3)$ with $x_0 <_{lex} \cdots <_{lex} x_3$. Let

$$f : [\langle {}^{\alpha}2, <_{lex} \rangle]^4 \longrightarrow 2$$

$$\vec{x} \longmapsto \begin{cases} 1 \text{ if and only if} & \vec{x} \text{ is a dextral comb or a sinistral bouquet and} \\ & \beta_h(x_2, x_3) < \beta_h(x_0, x_1) < \beta_h(x_1, x_2), \\ \qquad\qquad\qquad \text{or} & \vec{x} \text{ is a dextral bouquet or a sinistral comb and} \\ & \beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2); \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4.2.4 there are no sets of order type $2+\kappa^*$ or $\kappa+2$ that are homogeneous for f in colour 0. By Lemma 4.2.3 every copy of the rationals includes a quadruple getting colour 1.

Claim. *There is no quintuple that is homogeneous for f in colour* 1.

Proof. Suppose that $\vec{p} = \{p_0, \ldots, p_5\}$ with $p_0 <_{lex} \cdots <_{lex} p_4$ is homogeneous for f in colour 1. We consider four cases. Each of the cases consist of two symmetric subcases.

In the first case, suppose that $\vec{p}$ is a cactus. We can assume that $\vec{p}$ is a dextral cactus. Then $\{p_j \mid j < 4\}$ and $\{p_j \mid j > 0\}$ are dextral combs, hence $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the second case, suppose that $\vec{p}$ is an olivillo. We can assume that $\vec{p}$ is a dextral olivillo. Then $\{p_i \mid j < 4\}$ is a dextral comb and $\{p_j \mid j > 0\}$ is a dextral bouquet. Hence $\beta_h(p_2, p_3) < \beta_h(p_0, p_1) < \beta_h(p_1, p_2)$ and $\beta_h(p_1, p_2) < \beta_h(p_3, p_4) < \beta_h(p_2, p_3)$. Then $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the third case, suppose that $\vec{p}$ is a grape. We can assume that $\vec{p}$ is a dextral grape. Then $\{p_j \mid j < 4\}$ is a dextral bouquet and $\{p_j \mid j > 0\}$ is a sinistral comb. Hence $\beta_h(p_0, p_1) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$ and $\beta_h(p_1, p_2) < \beta_h(p_3, p_4) < \beta_h(p_2, p_3)$. Then $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the final case, suppose that $\vec{p}$ is a rose. We can assume that $\vec{p}$ is a dextral rose. Then $\{p_j \mid j < 4\}$ is a dextral comb and and $\{p_j \mid j > 0\}$ is a sinistral bouquet. Hence $\beta_h(p_2, p_3) < \beta_h(p_0, p_1) < \beta_h(p_1, p_2)$ and $\beta_h(p_3, p_4) < \beta_h(p_1, p_2) < \beta_h(p_2, p_3)$. Then $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction. □

This completes the proof of Theorem 4.3.6. □

Theorem 4.3.7. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex} \rangle \not\longrightarrow (5,\ \omega^* + \omega \vee \kappa + 2 + \kappa^* \vee (\kappa 2)^* \vee \kappa 2)^4.$$

Proof. Suppose that there is an infinite initial ordinal κ and an $\alpha < \kappa^+$ such that this partition property holds. Suppose that $h \colon \alpha \leftrightarrow \kappa$ is bijective and let β_h be as in Definition 3.1.1. If $\alpha = \kappa$, we can choose $\beta_h(x, y) = \delta(x, y)$ and obtain a simplified version of the following proof. We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$.

$$f : [\langle {}^{\alpha}2, <_{lex} \rangle]^4 \longrightarrow 2$$

$$\vec{x} \longmapsto \begin{cases} 1 \text{ if and only if} & \vec{x} \text{ is a dextral comb or a sinistral bouquet and} \\ & \beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1), \\ \qquad\qquad\qquad \text{or} & \vec{x} \text{ is a sinistral comb or a dextral bouquet and} \\ & \beta_h(x_1, x_2) < \beta_h(x_0, x_1) < \beta_h(x_2, x_3), \\ \qquad\qquad\qquad \text{or} & \vec{x} \text{ is a candelabrum and} \\ & \beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3)); \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that there is no homogeneous set of the required type for f.

To see that there are no sets which are homogeneous for f in colour 0 of order type $\omega^*+\omega$, see Lemma 4.2.2. In order to see that there are no such sets of order type $(\kappa 2)^*$ or $\kappa 2$ use Lemma 4.2.10.

Now consider a $C \in [{}^{\alpha}2]^{\kappa+2+\kappa^*}$. We distinguish three cases. First assume that there is an $s \in {}^{<\alpha}2$ such that $\kappa \leqslant \operatorname{otyp}(\{t \in C \mid t \sqsupset s^\frown\langle 0\rangle\})$ and $\kappa^* \leqslant \operatorname{otyp}(\{t \in C \mid t \sqsupset s^\frown\langle 1\rangle\})$. Then one proceeds essentially as in the proof of Lemma 4.2.6 and finds a candelabrum $\{q_0, q_1, q_2, q_3\}_< \in [C]^4$ with $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3)$. Then, again, $f(Q) = 1$.

For the second case, assume that there is no such s. Let $\langle c_\nu \mid \nu < \kappa+1\rangle$ be an ascending enumeration of the left half of C and let $\langle d_\nu \mid \nu < \kappa+1\rangle$ be a descending enumeration of its right half. Then, using Lemma 3.0.2 it is easy to choose $\{\nu, \zeta\} \in [\kappa]^2$ such that $B \overset{df}{=} \{c_\nu, c_\kappa, d_\kappa, d_\zeta\}$ is a bouquet and $f(B) = 1$.

Finally consider a $\vec{p} = \{p_0, \ldots, p_4\}_{<_{lex}} \in [{}^{\alpha}2]^5$. Assume towards a contradiction that $f[[\vec{p}]^4] = \{1\}$. There are fourteen cases to check half of which are mirror images of the other half.

We assume in the first case that $\vec{p}$ is a cactus. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\}$ are combs of the same chirality as $\vec{p}$ and by definition of f we have $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the second case, we assume that $\vec{p}$ is a olivillo. Then $\{\{p_j \mid j < 4\}, \{p_j \mid j > 0\}\}$ consists of a comb and a bouquet, both of the same chirality as $\vec{p}$. Then $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the third case, assume that $\vec{p}$ is a grape. Then $\{\{p_j \mid j < 4\}, \{p_j \mid j > 0\}\}$ contains a bouquet of the same chirality as $\vec{p}$ and a comb of the opposite one. This implies $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the fourth case, assume that $\vec{p}$ is a rose. Then $\{\{p_j \mid j < 4\}, \{p_j \mid j > 0\}\}$ contains a comb of the same chirality as $\vec{p}$ and a bouquet of the opposite one. Hence $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the fifth case, assume that $\vec{p}$ is a lilac. Then $\{\{p_j \mid j < 4\}, \{p_j \mid j > 0\}\}$ consists of a comb of the same chirality as $\vec{p}$ and a candelabrum. Suppose without loss of generality that $\vec{p}$ is dextral. Then $\beta_h(p_1, p_2) < \min(\beta_h(p_0, p_1), \beta_h(p_2, p_3)) \leqslant \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the sixth case, assume that $\vec{x}$ is a guinea flower. Then $\{\{p_j \mid j > 0\}, \{p_j \mid j < 4\}\}$ consists of a bouquet of the same chirality as $\vec{p}$ and a candelabrum. Suppose without loss of generality that $\vec{p}$ is dextral. Then $\beta_h(p_1, p_2) < \min(\beta_h(p_0, p_1), \beta_h(p_2, p_3)) \leqslant \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the final case, assume that $\vec{x}$ is a mistletoe. This means that $\{\{p_j \mid j < 4\}, \{p_j \mid j > 0\}\}$ consists of a bouquet of the same chirality as $\vec{p}$ and a candelabrum. Suppose without loss of generality that $\vec{x}$ is dextral. Then $\beta_h(p_2, p_3) < \min(\beta_h(p_1, p_2), \beta_h(p_3, p_4)) \leqslant \beta_h(p_1, p_2) < \beta_h(p_2, p_3)$, a contradiction. □

4.4. **Choice, after all.** The following result shows that Theorem 2.2.2 fails in ZFC.

Theorem 4.4.1. *Suppose that the Axiom of Choice holds and $\alpha < \omega_1$. Then*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (5,\ \omega^*+\omega \vee 2+\omega^* \vee \omega+2)^4.$$

Proof. Suppose that $\alpha < \omega_1$ and that $g : {}^{\alpha}2 \hookrightarrow \gamma$ is an injective function into some ordinal γ. Suppose that $h : \alpha \hookrightarrow \omega$ is injective and that β_h is defined according to Definition 3.1.1. For any $\vec{q} = \{q_0, q_1, q_2, q_3\}_{<_{lex}} \in [{}^{\alpha}2]^4$ let $f(\vec{q}) = 1$ if and only if

(a) $\vec{q}$ is a candelabrum, $\beta_h(q_1, q_2) < \min(\beta_h(q_0, q_1), \beta_h(q_2, q_3)), g(q_1) < g(q_0)$ and $g(q_2) < g(q_3)$ or
(b) $\vec{q}$ is a sinistral bouquet, $\beta_h(q_2, q_3) < \beta_h(q_0, q_1) < \beta_h(q_1, q_2)$ and $g(q_0) < g(q_1) < g(q_2)$ or
(c) $\vec{q}$ is a dextral bouquet, $\beta_h(q_0, q_2) < \beta_h(q_2, q_3) < \beta_h(q_1, q_2)$ and $g(q_3) < g(q_2) < g(q_1)$ or
(d) $\vec{q}$ is a sinistral comb, $\beta_h(q_0, q_1) < \beta_h(q_2, q_3) < \beta_h(q_1, q_2)$ and $g(q_3) < g(q_2)$ or
(e) $\vec{q}$ is a dextral comb, $\beta_h(q_2, q_3) < \beta_h(q_0, q_1) < \beta_h(q_1, q_2)$ and $g(q_0) < g(q_1)$.

Note that the definition of f is symmetric.

Claim. *There is no set of order type $\omega^*+\omega$ that is homogeneous for f in colour* 0.

Proof. Suppose that $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$. We have to find a $\vec{q} = \{q_0, q_1, q_2, q_3\}_{<_{lex}} \in [Z]^4$ for which $f(\vec{q}) = 1$. To this end, let $\langle z^0_n \mid n < \omega\rangle$ the order-reversing enumeration of the lower half of Z and let $\langle z^1_n \mid n < \omega\rangle$ be the order-preserving enumeration of its upper half. Suppose without loss of generality that for both $i < 2$ the sequence $\langle g(z^i_n) \mid n < \omega\rangle$ is ascending. Note that there has to be an $k < \omega$ such that for all $m \in \omega \setminus k$ and both $i < 2$ one has $\Delta(z^0_m, z^1_m) \sqsubseteq \Delta(z^i_m, z^i_{m+1})$. Furthermore, observe that there is an $m \in \omega \setminus k$ such that for all $n \in \omega \setminus m$ and both $i < 2$ one has $\beta_h(q_1, q_2) < \beta_h(q_{2i}, q_{2i+1})$. Let $\vec{q} \overset{df}{=} \{z^0_{m+1}, z^0_m, z^1_m, z^1_{m+1}\}$. Then $f(\vec{q}) = 1$ by (a). □

Claim. *There is no set of order type $2 + \omega^*$ that is homogeneous for f in colour 0.*

Proof. Now let $A \in [{}^{\alpha}2]^{2+\omega^*}$ and let $\langle a_\gamma \mid \gamma < \omega + 2\rangle$ be its order-reversing enumeration. We distinguish two cases. First assume that the sequence $\vec{s} \overset{df}{=} \langle \Delta(a_n, a_{\omega+1}) \mid n < \omega\rangle$ is stabilising, say at $s \in {}^{<\alpha}2$ from $k < \omega$ onwards. Because there is no infinite decreasing sequence of ordinals there is an $A_0 \in [A \setminus \{a_{\omega+1}\}]^{\omega^*}$ such that $g(c) < g(b)$ for any $\{b, c\}_< \in [A_0]^2$. Then there is an $A_1 \in [A_0]^{\omega^*}$ such that $\Delta(b, c) \sqsupset \Delta(c, d)^\frown\langle 0\rangle$ for any $\{b, c, d\}_< \in [A_1]^3$. One can find an $A_2 \in [A_1]^{\omega^*}$ such that $\beta_h(b, c) > \gamma_h(s)$ for all $\{b, c\}_< \in [A_2]^2$. Finally there is a $\{b, c, d\}_< \in [A_2]^3$ such that $\beta_h(c, d) < \beta_h(b, c)$. Then for the sinistral bouquet $\vec{q} \overset{df}{=} \{a_{\omega+1}, b, c, d\}$ we have $f(\vec{q}) = 1$ by (b).

Second assume that $\vec{s}$ does not stabilise. Then there is is an $A_0 \in [A \setminus \{a_{\omega+1}\}]^{\omega^*}$ such that $\Delta(b, c) \sqsupset \Delta(c, d)^\frown\langle 0\rangle$ for all $\{b, c, d\}_< \in [A_0]^3$. There is an $A_1 \in [A_0]^{\omega^*}$ such that $g(c) < g(b)$ for every $\{b, c\}_< \in [A_1]^2$. Since $\beta_h(a_{\omega+1}, a_\omega)$ is finite there is an $\{b, c\}_< \in [A_1]^2$ such that $\beta_h(b, c) > \beta_h(a_{\omega+1}, a_\omega)$. Then for the sinistral comb $\vec{q} \overset{df}{=} \{a_{\omega+1}, a_\omega, b, c\}$ we have $f(\vec{q}) = 1$ by (d). □

Since the definition of f is symmetric, the case of order type $\omega + 2$ is symmetric.

Claim. *There is no quintuple that is homogeneous for f in colour 1.*

Proof. Let $\vec{p} = \{p_0, p_1, p_2, p_3, p_4\}_{<_{lex}} \in [{}^{\alpha}2]^5$. We distinguish seven cases.

First assume that $\vec{p}$ is a cactus. Applying (c) to both $\{p_k \mid k < 4\}$ and $\{p_k \mid k \in 5 \setminus 1\}$ one gets $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Second suppose that $\vec{p}$ is a olivillo. We may assume that $\vec{p}$ is sinistral. Applying (b) to the sinistral bouquet $\{p_k \mid k < 4\}$ and (d) to the sinistral comb $\{p_k \mid k \in 5\setminus\{2\}\}$ one gets $\beta_h(p_3, p_4) < \beta_h(p_1, p_3) = \beta_h(p_1, p_2) < \beta_h(p_3, p_4)$, a contradiction.

Third assume that $\vec{p}$ is a rose. We may assume that $\vec{p}$ is dextral. Applying (e) to the dextral comb $\{p_k \mid k < 4\}$ and (c) to the dextral bouquet $\{p_k \mid k \neq 1\}$ one gets $\beta_h(p_2, p_3) < \beta_h(p_0, p_1) = \beta_h(p_0, p_2) < \beta_h(p_2, p_3)$, a contradiction.

Fourth suppose $\vec{p}$ is a grape. We may assume that $\vec{p}$ is sinistral. Applying (e) to the dextral comb $\{p_k \mid k < 4\}$ and (b) to the sinistral bouquet $\{p_k \mid k \neq 1\}$ yields $\beta_h(p_2, p_3) < \beta_h(p_0, p_1) = \beta_h(p_0, p_2) < \beta_h(p_2, p_3)$, a contradiction.

Fifth assume that $\vec{p}$ is a mistletoe. We may assume that $\vec{p}$ is sinistral. Applying (a) to the candelabrum $\{p_k \mid k < 4\}$ and (d) to the sinistral comb $\{p_k \mid k \neq 2\}$ yields $\beta_h(p_0, p_1) < \beta_h(p_1, p_3) = \beta_h(p_1, p_2) < \beta_h(p_0, p_1)$, a contradiction.

Sixth assume that $\vec{p}$ is a lilac. We may assume that $\vec{p}$ is sinistral. Applying (d) to the sinistral comb $\{p_k \mid k < 4\}$ and (a) to the candelabrum $\{p_k \mid k \neq 2\}$ yields $\beta_h(p_0, p_1) < \beta_h(p_2, p_3) = \beta_h(p_1, p_3) < \beta_h(p_0, p_1)$, a contradiction.

Last assume that $\vec{p}$ is a guinea flower. We may assume that $\vec{p}$ is sinistral. Applying (b) to the sinistral bouquet $\{p_k \mid k < 4\}$ and (a) to the candelabrum $\{p_k \mid k \neq 2\}$ yields $g(p_0) < g(p_1) < g(p_0)$, a contradiction. □

This completes the proof. □

Note that since λ and $\langle {}^{\omega}2, <_{lex}\rangle$ are mutually embeddable, Theorem 4.4.1 is a strengthening of [956ER, Theorem 28] which states that $\lambda \not\to (5,\ \omega + 2)^4$.

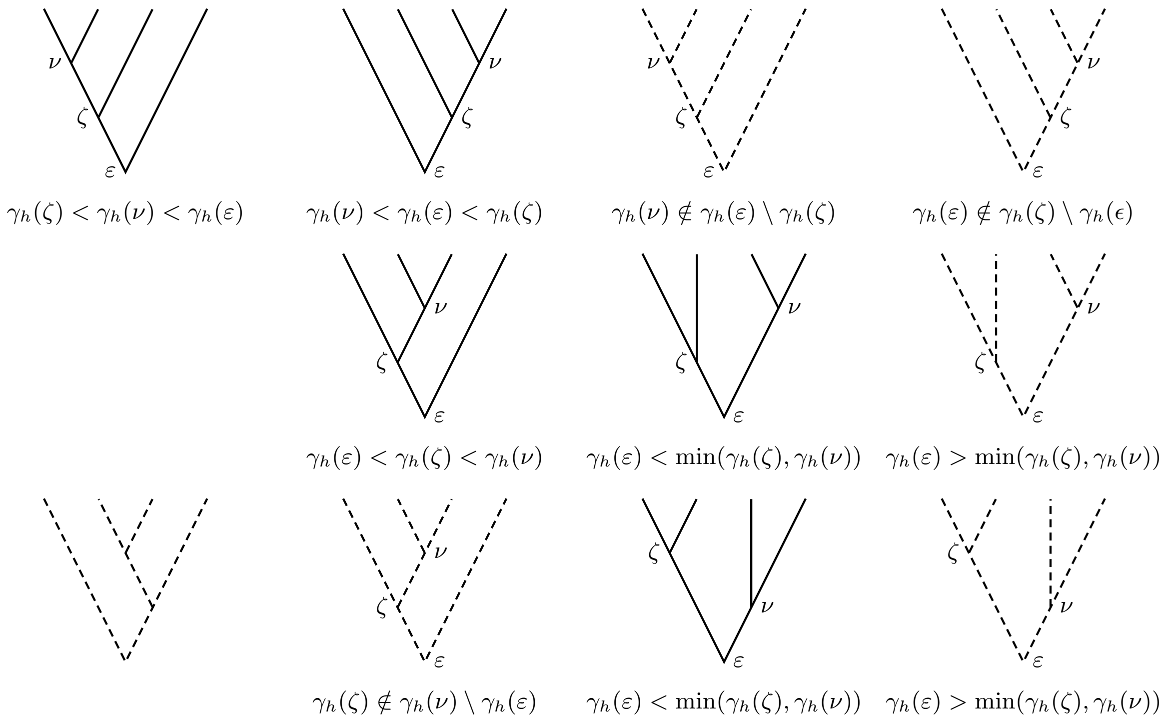


FIGURE 11. Colouring for Theorem 4.5.1(b)

In ZFC the statement of Theorem 2.1.5 is also provable, but Theorems 2.2.1 and 2.2.2 are falsified there by Theorem 4.4.1. For Theorem 2.2.2 this can also be shown using Theorem 1.2.3.

4.5. **Sextuples.** In this section, we prove several negative partition relations with 6 on one side of the relation. Most of these results are used in the classification in Section 4.7.

Theorem 4.5.1. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*
(a) $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (6,\ \kappa^* + \kappa \vee 2 + \kappa^* \vee \kappa 2 \vee \omega\omega^*)^4$ *and*
(b) $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (6,\ \kappa^* + \kappa \vee (\kappa 2)^* \vee \kappa + 2 \vee \omega^*\omega)^4$.

Proof. The statements (a) and (b) are mirror images of each other which is why we are only going to prove (a).

Suppose that κ is as in the theorem and there is an ordinal α such that the first partition property holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective and let β_h be defined as in Definition 3.1.1. If $\alpha = \kappa$, we can choose $h = \mathrm{id}$ (thus $\beta_h = \delta$) and obtain a simplified version of the following proof. We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$f : [\langle {}^{\alpha}2, <_{lex}\rangle]^4 \longrightarrow 2$$

$$\vec{x} \longmapsto \begin{cases} 1 \text{ if and only if} & \vec{x} \text{ is a dextral comb and } \beta_h(x_1,x_2) < \beta_h(x_2,x_3) < \beta_h(x_0,x_1), \\ \qquad\qquad\text{or} & \vec{x} \text{ is a sinistral comb or a dextral bouquet} \\ & \text{and } \beta_h(x_0,x_1) < \beta_h(x_2,x_3) < \beta_h(x_1,x_2), \\ \qquad\qquad\text{or} & \vec{x} \text{ is a candelabrum and} \\ & \beta_h(x_1,x_2) < \min(\beta_h(x_0,x_1), \beta_h(x_2,x_3)); \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that there is no homogeneous set of the required type for f.

To see that there is no set of order type $\kappa^* + \kappa$ which is homogeneous for f in colour 0 consult Lemma 4.2.6. To show the nonexistence of such sets of order type $2 + \kappa^*$ consider the first half of Lemma 4.2.4 and for the proof that f does not admit homogeneous sets in colour 0 of order type $\kappa 2$

use the second half of Lemma 4.2.10 Finally, to see that there is no $X \in [{}^{\alpha}2]^{\omega\omega^*}$ homogeneous for f in colour 0 consider Lemma 4.2.8.

We consider sets homogeneous for f in colour 1. Assume towards a contradiction that $\vec{s} \in [{}^{\alpha}2]^6$ is homogeneous for f in colour 1. Since $[\vec{s}]^4$ does not contain a sinistral bouquet, by Lemma 4.1.1 there is a quintuple $\{p_0 \dots, p_4\}_{<_{lex}} \in [\vec{s}]^5$ for which one of the following six cases applies.

First assume that $\vec{p}$ is a cactus. Then $\{p_j \mid j < 4\}$ and $\{p_j \mid j > 0\}$ are combs of the same chirality as $\vec{p}$ so $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Second assume that $\vec{p}$ is a dextral olivillo. Then $\vec{q}$ is a dextral comb and $\{p_j \mid j \neq 1\}$ is a dextral bouquet from which we get $\beta_h(p_2, p_3) < \beta_h(p_0, p_1) = \beta_h(p_0, p_2) < \beta_h(p_2, p_3)$, a contradiction.

Fourth assume that $\vec{p}$ is a dextral grape. Then $\{p_j \mid j < 4\}$ is a dextral bouquet and $\vec{q}$ is a sinistral comb. It follows that $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Fifth assume that $\vec{x}$ is a sinistral lilac. Then $\{p_j \mid j \neq 2\}$ is a candelabrum and $\{p_j \mid j < 4\}$ is a sinistral comb. It follows that $\beta_h(p_1, p_3) < \min(\beta_h(p_0, p_1), \beta_h(p_3, p_4)) \leqslant \beta_h(p_0, p_1) < \beta_h(p_2, p_3) = \beta_h(p_1, p_3)$, a contradiction.

Finally assume that $\vec{x}$ is a dextral lilac. This means that $\{p_j \mid j < 4\}$ is a candelabrum and $\{p_j \mid j > 0\}$ is a dextral comb. Then $\beta_h(p_1, p_2) < \min(\beta_h(p_0, p_1), \beta_h(p_2, p_3)) \leqslant \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Once more the second part of the theorem follows immediately by consideration of symmetry. $\square$

Theorem 4.5.2. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle{}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (6,\ \omega^* + \omega \vee \omega^* + \kappa^* \vee \kappa + \omega)^4.$$

Proof. Suppose that κ is as in the theorem and there is an ordinal $\alpha < \kappa^+$ such that the theorem holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective and let β_h be defined as in Definition 3.1.1. If $\alpha = \kappa$, we can choose $h = \mathrm{id}$ (thus $\beta_h = \delta$) and obtain a simplified version of the following proof. We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$f : [\langle{}^{\alpha}2, <_{lex}\rangle]^4 \longrightarrow 2$$

$$\vec{x} \longmapsto \begin{cases} 1 \text{ if and only if} & \vec{x} \text{ is a dextral comb and } \beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1), \\ \qquad\qquad\text{or} & \vec{x} \text{ is a sinistral comb and } \beta_h(x_1, x_2) < \beta_h(x_0, x_1) < \beta_h(x_2, x_3), \\ \qquad\qquad\text{or} & \vec{x} \text{ is a candelabrum and} \\ & \beta_h(x_1, x_2) < \max(\beta_h(x_0, x_1), \beta_h(x_2, x_3)); \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that there is no homogeneous set of the required type for f.

To see that there are no homogeneous sets of order type $\omega^* + \omega$ in colour 0, consider Lemma 4.2.2. For sets of order type $\omega^* + \kappa^*$ or $\kappa + \omega$, apply Lemma 4.2.9.

Finally consider a sextuple $\vec{s} = \{s_0, \dots, s_5\}_{<_{lex}} \in [{}^{\alpha}2]^6$ homogeneous in colour 1. Since $[\vec{s}]^4 \subseteq [{}^{\alpha}2]^4$ does not contain bouquets, by Exercise 4.1.3 to consider the following cases:

First suppose that $\vec{s}$ is an antler. Then $\{h_j \mid j \notin \{4j, 4j+1\}\}$ are combs for $j < 2$ and $\{s_0, s_1, s_4, s_5\}$ is a candelabrum. Then $\beta_h(h_{4j}, h_{4j+1}) < \beta_h(s_2, s_3)$ for $j < 2$ hence $\max(\beta_h(s_0, s_1), \beta_h(s_4, s_5)) < \beta_h(s_2, s_3) = \beta_h(s_1, s_4)$. But $\{s_0, s_1, s_4, s_5\}$ is a candelabrum, a contradiction.

Now suppose that there is a cactus $\vec{p} \in [\vec{s}]^5$. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\}$ are combs of the same chirality as $\vec{s}$ so $\beta_h(p_2, p_3) < \beta_h(p_1, p_2) < \beta_h(p_2, p_3)$, a contradiction. $\square$

Theorem 4.5.3. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle{}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (6,\ \omega + \omega^* \vee 2 + \kappa^* \vee \kappa + 2)^4.$$

Proof. Suppose that κ is as in the theorem and there is an ordinal $\alpha < \kappa^+$ such that the theorem holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective and let β_h be defined as in Definition 3.1.1. If $\alpha = \kappa$, we can choose $h = \mathrm{id}$ (thus $\beta_h = \delta$) and obtain a simplified version of the following proof.

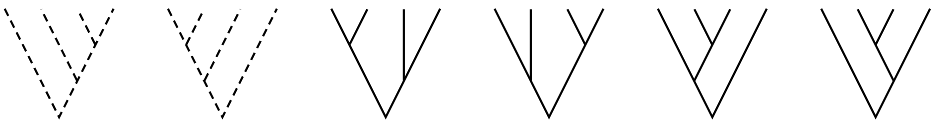


FIGURE 12. Colouring of the splitting types for the proof of Theorem 4.6.1.

We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$f : [\langle {}^{\alpha}2, <_{lex}\rangle]^4 \longrightarrow 2$$

$$\vec{x} \longmapsto \begin{cases} 1 \text{ if and only if} & \vec{x} \text{ is a dextral comb and } \beta_h(x_2, x_3) < \beta_h(x_0, x_1) < \beta_h(x_1, x_2), \\ \qquad\qquad \text{or} & \vec{x} \text{ is a dextral bouquet and } \beta_h(x_2, x_3) < \beta_h(x_1, x_2), \\ \qquad\qquad \text{or} & \vec{x} \text{ is a sinistral bouquet and } \beta_h(x_0, x_1) < \beta_h(x_1, x_2), \\ \qquad\qquad \text{or} & \vec{x} \text{ is a sinistral comb and} \\ & \beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2)); \\ 0 & \text{otherwise.} \end{cases}$$

To see that there are no sets of order type $\omega + \omega^*$ which are homogeneous for f in colour 0 consider Lemma 4.2.11. In order to show that there are no such sets of order type $2 + \kappa^*$ or $\kappa + 2$, see Lemma 4.2.4.

So consider a sextuple $\vec{s} \in [{}^{\alpha}2]^6$ and suppose towards a contradiction that it were homogeneous for f in colour 1. Then clearly $[\vec{s}]^4$ does not contain a candelabrum. Thus, by Lemma 4.1.2 for some quintuple $\{p_0, \ldots, p_4\}_{<_{lex}} \in [\vec{s}]^5$ one of the following three cases holds.

First assume that $\vec{p}$ is a cactus. Then $\{p_j \mid j < 4\}$ and $\{p_j \mid j > 0\}$ are both combs of the same chirality as $\vec{p}$ and hence $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Second assume that $\vec{p}$ is a grape. Then $\{\{p_j \mid j < 4\}, \{p_j \mid j > 0\}\}$ consists of a bouquet of the same chirality as $\vec{p}$ and a comb of the opposite one. This implies $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Last assume that $\vec{x}$ is a rose. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\}$ consists of a comb of the same chirality as $\vec{x}$ and a bouquet of the opposite one. This implies $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) = \beta_h(p_1, p_2)$, a contradiction. □

4.6. **Septuples.** In this section, we prove several negative partition relations with 7 on one side of the relation. Most of these results are used in the classification in Section 4.7.

Theorem 4.6.1. *If α is an ordinal, then* $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (7,\ \omega^* + \omega \vee \omega + \omega^*)^4$.

Proof. Suppose that (x_0, x_1, x_2, x_3) is a tuple in $[{}^{\alpha}2]^4$ with $x_0 <_{lex} x_1 <_{lex} x_3 <_{lex} x_4$. We define $g(x_0, x_1, x_2, x_3) = 1$ if $f_0(x_0, x_1, x_2, x_3) = 1$ or $f_1(x_0, x_1, x_2, x_3) = 1$, where f_0 is the colouring in the proof of Theorem 4.3.2 and f_1 is the colouring in the proof of Theorem 4.3.4. Otherwise we define $g(x_0, x_1, x_2, x_3) = 0$. By Lemma 4.2.1 there are no sets with order type $\omega^* + \omega$ homogeneous for g in colour 0 and by Lemma 4.2.11 there are no such sets of order type $\omega + \omega^*$.

Suppose that $H \in [{}^{\alpha}2]^7$ is homogeneous for g in colour 1. Suppose that $H = \{x_i \mid i < 7\}$ and $x_i <_{lex} x_j$ for $i < j < 7$. Choose $i \leqslant 5$ such that $\delta_{x_i, x_{i+1}}$ is least in $\{\delta_{x_j, x_{j+1}} \mid j \leqslant 5\}$. We can assume that $n \leqslant 2$. If $\delta_{x_6, x_5} < \delta_{x_5, x_4} < \delta_{x_4, x_3}$, then $g(\{x_j \mid 3 \leqslant j \leqslant 6\}) = 0$, contradicting the choice of H. Otherwise, there is some j with $3 \leqslant j \leqslant 5$ and $\delta_{x_j, x_{j+1}} < \delta_{x_{j+1}}, \delta_{x_{j+2}}$. Then $g(\{x_i, x_j, x_{j+1}, x_{j+2}\}) = 0$, contradicting the choice of H. □

A variation of this theorem is the following.

Theorem 4.6.2. *If κ is an initial ordinal number and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (7,\ \kappa^* + \kappa \vee \kappa + 2 \vee 2 + \kappa^* \vee \eta)^4.$$

Proof. Suppose that there is an ordinal α such that the first partition property holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective and let β_h be defined as in Definition 3.1.1. If $\alpha = \kappa$, we can choose $h = \mathrm{id}$ (thus $\beta_h = \delta$) and obtain a simplified version of the following proof.

We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$f : [\langle{}^{\alpha}2, <_{lex}\rangle]^4 \longrightarrow 2$$

$$\vec{x} \longmapsto \begin{cases} 1 \text{ if and only if} & \vec{x} \text{ is a dextral comb or a sinistral bouquet and} \\ & \beta_h(x_2,x_3) < \beta_h(x_0,x_1) < \beta_h(x_1,x_2), \\ \qquad\qquad\quad \text{or} & \vec{x} \text{ is a sinistral comb or a dextral bouquet and} \\ & \beta_h(x_0,x_1) < \beta_h(x_2,x_3) < \beta_h(x_1,x_2), \\ \qquad\qquad\quad \text{or} & \vec{x} \text{ is a candelabrum and} \\ & \beta_h(x_1,x_2) < \min(\beta_h(x_0,x_1), \beta_h(x_2,x_3)); \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that there is no homogeneous set of the required type for f.

We can use Lemma 4.2.6 to show that there are no sets of order type $\kappa^* + \kappa$ which are homogeneous for f in colour 0, Lemma 4.2.4 to see that there are no such sets of order type $2 + \kappa^*$ or $\kappa + 2$ and Lemma 4.2.3 to see that there are no such sets of order type η.

Finally consider some $S \in [{}^{\alpha}2]^7$. Let $\langle s_i \mid i < 7\rangle$ be the order-preserving enumeration of S. By Lemma 4.1.4 it contains a cactus, rose, olivillo, grape or mistletoe. We examine these cases in turn. Let $\vec{p} \in [\vec{s}]^5$.

If $\vec{p}$ is a cactus then $\{p_i \mid i \leqslant 3\}, \{p_i \mid 1 \leqslant i \leqslant 4\}$ are combs so $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

If $\vec{p}$ is a rose then $\{\{p_i \mid i \leqslant 3\}, \{p_i \mid 1 \leqslant i \leqslant 4\}\}$ consists of a comb of the same chirality as $\vec{p}$ and a bouquet of the opposite one. It follows that $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

If $\vec{p}$ is a olivillo then $\{\{p_i \mid i \leqslant 3\}, \{p_i \mid 1 \leqslant i \leqslant 4\}\}$ consists of a comb and a bouquet both of the same chirality as $\vec{p}$. It follows that $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

If $\vec{p}$ is a grape then $\{\{p_i \mid i \leqslant 3\}, \{p_2 \mid 1 \leqslant i \leqslant 4\}\}$ consists of a bouquet of the same chirality as $\vec{p}$ and a comb of the opposite one. It follows that $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

If $\vec{p}$ is a dextral mistletoe then $\{p_i \mid 1 \leqslant i \leqslant 4\}$ is a candelabrum while $\{p_0,p_1,p_3,p_4\}$ is a dextral comb. It follows that $\beta_h(p_3,p_4) < \beta_h(p_1,p_3) = \beta_h(p_2,p_3) < \beta_h(p_3,p_4)$, a contradiction.

If $\vec{p}$ is a sinistral mistletoe then $\{p_i \mid i \leqslant 3\}$ is a candelabrum while $\{p_0,p_1,p_3,p_4\}$ is a sinistral comb. It follows that $\beta_h(p_0,p_1) < \beta_h(p_1,p_3) = \beta_h(p_1,p_2) < \beta_h(p_0,p_1)$, a contradiction.

□

Theorem 4.6.3. *If κ is an initial ordinal and $\alpha < \kappa^+$, then*

(a) $\langle{}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (7,\ \omega^* + \omega \vee 2 + \kappa^* \vee \kappa + \omega)^4$ *and*
(b) $\langle{}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (7,\ \omega^* + \omega \vee \omega^* + \kappa^* \vee \kappa + 2)^4$.

Proof. Suppose that there is an ordinal α such that the first partition property holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective and let β_h be defined as in Definition 3.1.1. If $\alpha = \kappa$, we can choose $h = \mathrm{id}$ (thus $\beta_h = \delta$) and obtain a simplified version of the following proof.

We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$f : [\langle {}^{\alpha}2, <_{lex}\rangle]^4 \longrightarrow 2$$

$$\vec{x} \longmapsto \begin{cases} 1 \text{ if and only if} & \vec{x} \text{ is a dextral comb and } \beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1), \\ \qquad\qquad\quad \text{or} & \vec{x} \text{ is a dextral bouquet or a sinistral comb} \\ & \text{and } \beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2), \\ \qquad\qquad\quad \text{or} & \vec{x} \text{ is a candelabrum and} \\ & \beta_h(x_1, x_2) < \max(\beta_h(x_0, x_1), \beta_h(x_2, x_3)); \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that there is no homogeneous set of the required type for f.

It follows from Lemma 4.2.2 (2) that there are no sets of order type $\omega^* + \omega$ which are homogeneous for f in colour 0, from Lemma 4.2.4 (1) that there are no such sets of order type $2 + \kappa^*$ and the second half of Lemma 4.2.9 that there are no such sets of order type $\kappa + \omega$.

Finally consider some $\vec{s} \in [{}^{\alpha}2]^7$ and assume towards a contradiction that it were homogeneous for f in colour 1. Note that $\vec{s} \in [{}^{\alpha}2]^4$ does not contain a sinistral bouquet. By Lemma 4.1.5 one of the following cases has to apply:

First suppose that there is an antler $\{\bar{s}_0, \ldots, \bar{s}_5\}_{<_{lex}} \in [\vec{s}]^6$. Then $\{\bar{s}_0, \ldots, \bar{s}_3\}$ is a sinistral comb, $\{\bar{s}_2, \ldots, \bar{s}_5\}$ is a dextral comb and $\{\bar{s}_0, \bar{s}_1, \bar{s}_4, \bar{s}_5\}$ is a candelabrum. Together this implies $\beta_h(\bar{s}_1, \bar{s}_4) < \max(\beta_h(\bar{s}_0, \bar{s}_1), \beta_h(\bar{s}_4, \bar{s}_5)) \leqslant \beta_h(\bar{s}_2, \bar{s}_3) = \beta_h(\bar{s}_1, \bar{s}_4)$, a contradiction.

Now consider some $\{p_0, \ldots, p_4\}_{<_{lex}} \in [\vec{s}]^5$.

Second suppose that $\vec{p}$ is a dextral olivillo. Then $\{p_0, p_1, p_2, p_3\}$ is a dextral comb while $\{p_0, p_2, p_3, p_4\}$ is a dextral bouquet so $\beta_h(p_0, p_1) < \beta_h(p_2, p_3) < \beta_h(p_0, p_1)$, a contradiction.

Third suppose that $\vec{p}$ is a dextral grape. Then $\{p_0, p_1, p_2, p_3\}$ is a dextral bouquet while $\{p_1, p_2, p_3, p_4\}$ is a sinistral comb so $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Finally suppose that $\vec{p}$ is a cactus. Then $\{p_0, p_1, p_2, p_3\}$ and $\{p_1, p_2, p_3, p_4\}$ are combs of the same chirality as $\vec{p}$ so $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

The second half of the theorem can be proved in an analogous way. □

4.7. **The classification.** We will determine which partition relations of the forms

$$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (K, L)^n$$

$$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow \left(\bigvee_{\nu<\lambda} K_\nu, \bigvee_{\nu<\mu} L_\nu \right)^4$$

for linear orders K, L, K_ν, L_ν are consistent with $\mathsf{ZF} + \mathsf{DC}$. By Chapter 2 we have—under the assumption that all sets of reals have the property of Baire—the following positive relations.

(Theorem 2.1.5)	$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\omega + 1)^4_2$,
(Theorem 2.2.2)	$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (5,\ 1 + \omega^* + \omega + 1 \vee \omega + 1 + \omega^*)^4$,
(Theorem 2.2.1)	$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (6,\ 1 + \omega^* + \omega + 1 \vee m + \omega^* \vee \omega + n)^4$.

We have the following negative relations by Chapter 4 (cf. Summary 4.0.2).

(Theorem 4.3.7) $\langle {}^{\omega}2, <_{lex}\rangle \not\longrightarrow (5,\ \omega^* + \omega \vee \kappa + 2 + \kappa^* \vee (\kappa 2)^* \vee \kappa 2)^4$,

(Theorem 4.3.6) $\langle {}^{\omega}2, <_{lex}\rangle \not\longrightarrow (5,\ 2 + \kappa^* \vee \kappa + 2 \vee \eta)^4$,

(Theorem 4.5.1 (a)) $\langle {}^{\omega}2, <_{lex}\rangle \not\longrightarrow (6,\ \kappa^* + \kappa \vee 2 + \kappa^* \vee \kappa 2 \vee \omega\omega^*)^4$,

(Theorem 4.5.1 (b)) $\langle {}^{\omega}2, <_{lex}\rangle \not\longrightarrow (6,\ \kappa^* + \kappa \vee (\kappa 2)^* \vee \kappa + 2 \vee \omega^*\omega)^4$,

(Theorem 4.6.2) $\langle {}^{\omega}2, <_{lex}\rangle \not\longrightarrow (7,\ \kappa^* + \kappa \vee \kappa + 2 \vee 2 + \kappa^* \vee \eta)^4$.

Theorem 4.7.1. *Suppose that the principle of dependent choices* DC *holds true and all sets of reals have the property of Baire. Suppose that K and L are suborders of $\langle {}^{\omega}2, <_{lex}\rangle$ and $n \geqslant 4$. Then the partition relation*

$$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (K, M)^n$$

holds true if and only if ($K \leqslant n$ and $M \leqslant \langle {}^{\omega}2, <_{lex}\rangle$) or ($M \leqslant n$ and $K \leqslant \langle {}^{\omega}2, <_{lex}\rangle$) or $K, M \leqslant \omega+1$ or $K, M \leqslant 1 + \omega^*$. Otherwise the relation is inconsistent with ZF.

Proof. Suppose that $K \not\leqslant \omega + 1$ and $L \not\leqslant 1 + \omega^*$. Then $\omega + 2 \leqslant K$ or $\omega^* \leqslant K$ and $1 + \omega^* \leqslant M$ or $\omega \leqslant M$, using DC. Then the partition relation fails by Theorem 3.1.2. If $K \not\leqslant 1 + \omega^*$ and $L \not\leqslant \omega + 1$, again the partition relation fails by Theorem 3.1.2.

If $K \leqslant \omega + 1$ and $L \leqslant \omega + 1$, then the relation holds by Theorem 2.1.5. Similarly, if $K \leqslant 1 + \omega^*$ and $L \leqslant 1 + \omega^*$, then the relation holds by Theorem 2.1.5.

In the other cases $K \leqslant \omega + 1$ and $K \leqslant 1 + \omega^*$, so that K is finite, or in the remaining symmetric case that M is finite, which we omit. Suppose that $|K| = n + 1$. We can assume that none of the previous cases applies, so $\omega + 2 \leqslant M$, $2 + \omega^* \leqslant M$, or $\omega^* + \omega \leqslant M$. If $\omega^* + \omega \leqslant M$, then the relation fails by Theorem 4.3.2. If $\omega + 2 \leqslant M$ or $2 + \omega^* \leqslant M$, then the relation fails by Theorem 4.3.6. □

The following result shows that the previous theorems solve the case of quadruple-colourings in the Cantor space completely, given that all sets of reals have the property of Baire. We will only consider partition relations such that in no disjunction there are linear orders K, L with $K \leqslant L$, since in this case L can be omitted without changing the truth value of the partition relation.

Theorem 4.7.2. *Suppose that the principle of dependent choices* DC *holds true and all sets of reals have the property of Baire. Suppose that K_μ and L_ν are suborders of $\langle {}^{\omega}2, <_{lex}\rangle$ for all $\mu < \kappa$ and $\nu < \lambda$. Then the partition relation*

$$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow \left(\bigvee_{\mu<\kappa} K_\mu, \bigvee_{\nu<\lambda} M_\nu \right)^4$$

holds true if and only if one of the following cases applies.

(a) *$K_\xi, M_\rho \leqslant \omega + 1$ for some $\xi < \kappa$, $\rho < \lambda$,*
(b) *$K_\xi, M_\rho \leqslant 1 + \omega^*$ for some $\xi < \kappa$, $\rho < \lambda$.*
(c) *$\kappa = 1$, $K_0 \leqslant 6$, $\lambda = 3$, and for some $i, j, k < 3$ and some m, n*

$$M_i \leqslant 1 + \omega^* + \omega + 1,\ M_j \leqslant \omega + m,\ M_k \leqslant n + \omega^*.$$

(d) *$\lambda = 1$, $M_0 \leqslant 6$, $\kappa = 3$, and for some $i, j, k < 3$ and some m, n*

$$K_i \leqslant 1 + \omega^* + \omega + 1,\ K_j \leqslant \omega + m,\ K_k \leqslant n + \omega^*.$$

(e) *$\kappa = 1$, $K_0 \leqslant 5$, $\lambda = 2$, and for some $i, j < 2$*

$$M_i \leqslant 1 + \omega^* + \omega + 1,\ M_j \leqslant \omega + 1 + \omega^*.$$

(f) *$\lambda = 1$, $M_0 \leqslant 5$, $\kappa = 2$, and for some $i, j < 2$*

$$K_i \leqslant 1 + \omega^* + \omega + 1,\ K_j \leqslant \omega + 1 + \omega^*.$$

Moreover, if none of these cases applies, then the relation is inconsistent with ZF.

Proof. Suppose that $K_\mu \nleqslant \omega+1$ and $M_\nu \nleqslant 1+\omega^*$ for all $\mu<\kappa$ and $\nu<\lambda$. Then $\omega+2 \leqslant K_\mu$ or $\omega^* \leqslant K_\mu$ for all $\mu<\kappa$, and $2+\omega^* \leqslant M_\nu$ or $\omega \leqslant M_\nu$ for all $\nu<\lambda$, using DC. Then the partition relation fails by Theorem 3.1.2.

If $K_\mu \nleqslant 1+\omega^*$ and $M_\nu \nleqslant \omega+1$ for all $\mu<\kappa$ and $\nu<\lambda$, again the partition relation fails by Theorem 3.1.2.

If $K_\mu \leqslant \omega+1$ for some $\mu<\kappa$ and $M_\nu \leqslant \omega+1$ for some $\nu<\lambda$, then the relation holds by Theorem 2.1.5. Similarly, if $K_\mu \leqslant 1+\omega^*$ for some $\mu<\kappa$ and $M_\nu \leqslant 1+\omega^*$ for some $\nu<\lambda$, then the relation holds by Theorem 2.1.5. These are the first two cases in the classification.

It follows that $K_\mu \leqslant \omega+1$ and $K_\nu \leqslant 1+\omega^*$ for some $\mu,\nu<\kappa$, or the symmetric case for M_μ, M_ν and $\mu,\nu<\lambda$, which we omit. We can assume that none of the previous cases applies.

We first suppose that $\mu \neq \nu$, or that $\mu=\nu$ and $K_\mu \geqslant 7$. Let us consider the linear orders on the right side of the relation. Since none of the previous cases applies, the linear orders are neither embeddable into $1+\omega^*$ nor into $\omega+1$. Hence for each $\nu<\lambda$, $\omega+2 \leqslant M_\nu$, $2+\omega^* \leqslant M_\nu$, or $\omega^*+\omega \leqslant M_\nu$. Then the relation fails by Theorem 4.6.2.

Second, we suppose that $\kappa=1$ and $K_0=6$. Again, we consider the linear orders on the right. If every linear order contains $\omega+2$ or $2+\omega^*$, then the relation fails by Theorem 4.3.6. If every linear order contains $\omega^*+\omega$, $2+\omega^*$ or $\omega 2$ then the relation fails by Theorem 4.5.1(a). If every linear order contains $\omega^*+\omega$, $\omega+2$ or $(\omega 2)^*$ then the relation fails by Theorem 4.5.1(b). Any linear order which neither contains $2+\omega^*$ nor $\omega+2$, is contained in $1+\omega^*+\omega+1$, any linear order which neither contains $\omega^*+\omega$ nor $\omega+2$ nor $(\omega 2)^*$ is contained in $n+\omega^*$ for some natural number n and any linear order which neither contains $\omega^*+\omega$ nor $2+\omega^*$ nor $\omega 2$ is contained in $\omega+n$ for some natural number n. Hence the linear orders on the right side of the relation are contained in $1+\omega^*+\omega+1$, $\omega+n$, and $n+\omega^*$ for some natural number n. Then the partition relation holds by Theorem 2.2.1. This is the third case in the classification. The fourth case is symmetric and occurs when we exchange the left and right sides of the relation.

Finally, we consider the case $\kappa=0$ and $K_0=5$. Again, if every linear order contains $\omega+2$ or $2+\omega^*$, then the relation fails by Theorem 4.3.6. If every linear order contains $\omega^*+\omega$, $\omega+2+\omega^*$, $\omega 2$, or $(\omega 2)^*$, then the relation fails by Theorem 4.3.7.

Otherwise, there are $\mu,\nu<\lambda$ such that M_μ contains neither $\omega^*+\omega$, $\omega+2+\omega^*$, $\omega 2$, nor $(\omega 2)^*$, and M_ν contains neither $2+\omega^*$ nor $\omega+2$. Then M_μ is embeddable into $\omega+1+\omega^*$ and M_ν is contained in $1+\omega^*+\omega+1$. If $\mu\neq\nu$, the relation holds by Theorem 2.2.2. This is the fifth case in the classification. The sixth case occurs symmetrically when the left and right sides in the relation are exchanged. If $\mu=\nu$, then M_μ embeds into $\omega+1$ or $1+\omega^*$, so the relation holds by Theorem 2.1.3. This is one of the first two cases of the classification. □

4.8. **Octuples.** In the remaining sections, we prove three negative partition relations for octuples and nonuples. These relations follow from Theorem 4.6.2 for $\kappa=\omega$, but are new for $\kappa>\omega$.

Theorem 4.8.1. *if κ is an initial ordinal and $\alpha<\kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (8,\ \omega^*+\omega \vee \omega+\omega^* \vee (\kappa 2)^* \vee \kappa 2)^4.$$

Proof. Suppose that there is an ordinal α such that the first partition property holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective and let β_h be defined as in Definition 3.1.1. If $\alpha=\kappa$, we can choose $h=\mathrm{id}$ (thus $\beta_h=\delta$) and obtain a simplified version of the following proof.

We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$f : [\langle {}^{\alpha}2, <_{lex}\rangle]^4 \longrightarrow 2$$

$$\vec{x} \longmapsto \begin{cases} 1 \text{ if and only if} & \vec{x} \text{ is a sinistral comb and } \beta_h(x_1,x_2) < \beta_h(x_0,x_1) < \beta_h(x_2,x_3), \\ \qquad\qquad\quad \text{or} & \vec{x} \text{ is a dextral comb and } \beta_h(x_1,x_2) < \beta_h(x_2,x_3) < \beta_h(x_0,x_1), \\ \qquad\qquad\quad \text{or} & \vec{x} \text{ is a sinistral bouquet and } \beta_h(x_0,x_1) < \beta_h(x_1,x_2), \\ \qquad\qquad\quad \text{or} & \vec{x} \text{ is a dextral bouquet and } \beta_h(x_2,x_3) < \beta_h(x_1,x_2), \\ \qquad\qquad\quad \text{or} & \vec{x} \text{ is a candelabrum and} \\ & \beta_h(x_1,x_2) < \min(\beta_h(x_0,x_1), \beta_h(x_2,x_3)); \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that there is no homogeneous set of the required type for f. One can use Lemma 4.2.2 to show that there are no sets of order type $\omega^* + \omega$ which are homogeneous for f in colour 0, Lemma 4.2.10 to see that there are no such sets of order type $(\kappa 2)^*$ or $\kappa 2$ and Lemma 4.2.11 to see that there are no such sets of order type $\omega + \omega^*$.

Finally consider some $\vec{o} \in [{}^{\alpha}2]^8$ and assume towards a contradiction that it were homogeneous for f in colour 1. Then by Lemma 4.1.6 there is a quintuple $\vec{p} = \{p_0, \dots, p_4\}_{<_{lex}} \in [\vec{o}]^5$ for which one of the following three cases obtains:

First assume that $\vec{p}$ is a cactus. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\}$ are combs so $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

Second assume that $\vec{p}$ is a grape. Suppose without loss of generality that $\vec{p}$ is sinistral. Then $\{p_j \mid j \neq 1\}$ is a sinistral bouquet and $\{p_j \mid j < 4\}$ is a dextral comb. We get $\beta_h(p_2,p_3) < \beta_h(p_0,p_1) = \beta_h(p_0,p_2) < \beta_h(p_2,p_3)$, a contradiction.

Last assume that $\vec{p}$ is a lilac. Suppose without loss of generality that $\vec{x}$ is dextral. Then $\{p_j \mid j < 4\}$ is a candelabrum and $\{p_j \mid j > 0\}$ is a dextral comb so $\beta_h(p_1,p_2) < \min(\beta_h(p_0,p_1), \beta_h(p_2,p_3)) \leqslant \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction. □

Theorem 4.8.2. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle {}^{\kappa}2, <_{lex}\rangle \not\longrightarrow (8,\ \kappa^* + \omega \vee \omega^* + \kappa \vee 2 + \kappa^* \vee \kappa + 2 \vee \omega\omega^* \vee \omega^*\omega)^4.$$

Proof. Let α be any ordinal. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective and let β_h be defined as in Definition 3.1.1. If $\alpha = \kappa$, we can choose $h = \mathrm{id}$ (thus $\beta_h = \delta$) and obtain a simplified version of the following proof.

$$f : [\langle {}^{\alpha}2, <_{lex}\rangle]^4 \longrightarrow 2$$

$$\vec{x} \longmapsto \begin{cases} 1 \text{ if and only if } \vec{x} & \text{is a dextral comb or a sinistral bouquet and} \\ & \beta_h(x_2,x_3) < \beta_h(x_0,x_1) < \beta_h(x_1,x_2), \\ \qquad\qquad\qquad \text{or } \vec{x} & \text{is a sinistral comb or a dextral bouquet and} \\ & \beta_h(x_0,x_1) < \beta_h(x_2,x_3) < \beta_h(x_1,x_2), \\ \qquad\qquad\qquad \text{or } \vec{x} & \text{is a candelabrum and} \\ & \beta_h(x_1,x_2) < \max(\beta_h(x_0,x_1), \beta_h(x_2,x_3)); \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that there is no homogeneous set of the required type for f.

In order to see that there is no homogeneous set of the required type in colour 1, consider the Lemmata 4.2.5, 4.2.4 and 4.2.7.

Now consider an octuple $O \in [{}^{\alpha}2]^8$ with order-preserving enumeration $\langle o_i \mid i < 8\rangle$. By Lemma 4.1.7 the octuple O has to contain one of the following types.

First suppose that $\{s_0, \ldots, s_5\}_{<_{lex}} \in [O]^6$ is an antler. Then $\{s_i \mid i < 4\}$ is a sinistral comb and $\{s_i \mid i \in 6 \setminus 2\}$ is a dextral comb. This implies $\beta_h(s_0, s_1) < \beta_h(s_2, s_3)$ and $\beta_h(s_4, s_5) < \beta_h(s_2, s_3)$ so $\max(\beta_h(s_0, s_1), \beta_h(s_4, s_5)) < \beta_h(s_2, s_3)$. But $\{s_i \mid i \in 6 \setminus \{1, 4\}\}$ is a candelabrum which is a contradiction.

Second suppose that there is a cactus $\vec{p}$. Then $\{p_i \mid i < 4\}, \{p_i \mid i > 0\}$ are combs so $\beta_h(p_2, p_3) < \beta_h(p_1, p_2) < \beta_h(p_2, p_3)$, a contradiction.

Third assume that there is a olivillo $\vec{p}$. Then $\{\{p_i \mid i < 4\}, \{p_i \mid i > 0\}\}$ consists of a comb and a bouquet of the same chirality as $\vec{p}$ so $\beta_h(p_2, p_3) < \beta_h(p_1, p_2) < \beta_h(p_2, p_3)$, a contradiction.

Fourth suppose that there is a grape $\vec{x}$. Then $\{\{p_i \mid i < 4\}, \{p_i \mid i > 0\}\}$ consists of a bouquet of the same chirality as $\vec{x}$ and a comb of the opposite one. It follows that $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Finally assume that there is a rose $\vec{p}$. Then $\{\{p_i \mid i < 4\}, \{p_i \mid i > 0\}\}$ consists of a comb of the same chirality as $\vec{p}$ and a bouquet of the opposite one, so $\beta_h(p_2, p_3) < \beta_h(p_1, p_2) < \beta_h(p_2, p_3)$, a contradiction. □

4.9. **Nonuples.** In the final section of this chapter, we prove a negative partition relation for nonuples. This relation follows from Theorem 4.6.2 for $\kappa = \omega$, but is new for $\kappa > \omega$.

Theorem 4.9.1. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex} \rangle \not\longrightarrow (9, \ \omega^* + \omega \vee \omega + \omega^* \vee \kappa + 2 \vee 2 + \kappa^*)^4.$$

Proof. Suppose that κ is as in the theorem and there is an ordinal $\alpha < \kappa^+$ such that the Theorem holds. Suppose that $h \colon \alpha \leftrightarrow \kappa$ is bijective and let β_h be defined as in Definition 3.1.1. If $\alpha = \kappa$, we can choose $h = \mathrm{id}$ (thus $\beta_h = \delta$) and obtain a simplified version of the following proof.

We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$f(\vec{x}) = \begin{cases} 1 & \text{if and only if } \vec{x} \text{ is a candelabrum,} \\ & \text{or } \vec{x} \text{ is a dextral comb or a sinistral bouquet and } \beta_h(x_2, x_3) < \beta_h(x_0, x_1) < \beta_h(x_1, x_2), \\ & \text{or } \vec{x} \text{ is a sinistral comb or a dextral bouquet and } \beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2); \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that there is no homogeneous set of the required type for f. By Lemmata 4.2.1 and 4.2.4, there are no homogeneous sets of the order types $\omega^* + \omega$, $2 + \kappa^*$, and $\kappa + 2$ in colour 0.

In the first case, suppose that there is some $Y \in [{}^{\alpha}2]^{\omega + \omega^*}$ that is homogeneous for f in colour 0. We distinguish three cases.

First suppose that there is some $s \in {}^{<\alpha}2$ such that there are $y_{2i} <_{lex} y_{2i+1}$ extending $s^\frown\langle i \rangle$ for $i < 2$. Then $\{y_0, y_1, y_2, y_3\}$ is a candelabrum.

Now suppose that there is no such s. This implies that all splitting nodes lie on a single branch. Let $\langle y^0_n \mid n < \omega \rangle$ be the ascending enumeration of the lower half of Y and $\langle y^1_n \mid n < \omega \rangle$ the descending one of the upper half. Let $\gamma_i \overset{df}{=} \sup_{n<\omega} \delta(y^i_n, y^i_{n+1})$ and $\zeta_i \overset{df}{=} \limsup_{n<\omega} \beta_h(y^i_n, y^i_{n+1})$ for $i < 2$.

Second suppose that $\gamma_0 \leqslant \gamma_1 \Leftrightarrow \zeta_1 \leqslant \zeta_0$. Let $i < 2$ be such that $\gamma_i \leqslant \gamma_{1-i}$. Now choose $m < \omega$ such that $\delta(y^{1-i}_m, y^{1-i}_{m+1}) > \delta(y^i_0, y^i_1)$ and $\beta_h(y^{1-i}_m, y^{1-i}_{m+1}) \in \zeta_{1-i} \setminus \beta_h(y^i_0, y^i_1)$. We choose an $n \in \omega \setminus m$ such that $\beta_h(y^{1-i}_n, y^{1-i}_{n+1}) > \beta_h(y^{1-i}_m, y^{1-i}_{m+1})$. Then $\{y^i_0, y^{1-i}_m, y^{1-i}_n, y^{1-i}_{n+1}\}$ is a dextral comb $(i = 0)$ or sinistral bouquet $(i = 1)$, providing what was demanded.

Third suppose that $\gamma_0 \leqslant \gamma_1 \veebar \zeta_1 \leqslant \zeta_0$. Let $k < \omega$ be such that $\delta(y^{1-i}_k, y^{1-i}_{k+1}) > \gamma_i$. Then choose $m < \omega$ such that $\beta_h(y^i_m, y^i_{m+1}) \in \zeta_i \setminus \beta_h(y^i_k, y^i_{k+1})$ and finally $n \in \omega \setminus m$ such that $\beta_h(y^i_n, y^i_{n+1}) > \beta_h(y^i_m, y^i_{m+1})$. Then $\{y^i_m, y^i_n, y^{1-i}_k, y^{1-i}_{k+1}\}$ is a sinistral comb $(i = 0)$ or dextral bouquet $(i = 1)$, providing what was demanded.

In the second case, suppose that there is a nonuple $N \in [{}^{\alpha}2]^9$ that is homogeneous for f in colour 1. We consider the following four cases. By Lemma 4.1.8 these four pairs of cases exhaust all possibilities and hence this completes the proof.

First suppose that there is some cactus $\vec{p} = \{p_0, \dots, p_4\}_{<_{lex}} \in [N]^5$. Now $\{p_j \mid j < 4\}, \{p_j \mid j > 0\}$ are combs so $\beta_h(p_1, q_2) < \beta_h(q_2, q_3) < \beta_h(q_1, q_2)$, a contradiction.

Second suppose that there is an olivillo $\vec{p}$. Then $\{\{p_j \mid j < 4\}, \{p_j \mid j > 0\}\}$ contains a comb and a bouquet, both of the same chirality as $\vec{p}$. It follows that $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Third suppose that there is a rose $\vec{p}$. Then $\{\{p_j \mid j < 4\}, \{p_j \mid j > 0\}\}$ contains a comb of the same chirality as $\vec{x}$ and a bouquet of the opposite one. It follows that $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(q_1, q_2)$, a contradiction.

Finally suppose that there is a grape $\vec{p}$. Then $\{\{p_j \mid j < 4\}, \{p_j \mid j > 0\}\}$ contains a bouquet of the same cardinality as $\vec{x}$ and a comb of the opposite one. It follows that $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction. □

5. Questions

We conclude this paper with the main open questions. The strong partition property for ω_1 implies $\langle {}^{\omega_1}2, <_{lex}\rangle \longrightarrow (\langle {}^{\omega_1}2, <_{lex}\rangle)^1_2$. This motivates the following question.

Question 5.0.1. *Does the axiom of determinacy imply $\langle {}^{\omega_1}2, <_{lex}\rangle \longrightarrow (\langle {}^{\omega_1}2, <_{lex}\rangle)^2_2$?*

The following question asks about an uncountable analogue of Blass' theorem. This seems necessary to generalise the positive partition results from $\langle {}^{\omega}2, <_{lex}\rangle$ to $\langle {}^{\kappa}2, <_{lex}\rangle$.

Question 5.0.2. *Is it consistent that $\kappa = \kappa^{<\kappa} > \omega$ and $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow_t (\langle {}^{\kappa}2, <_{lex}\rangle)^m_n$ for all m, n?*

We ask whether the classifications in Theorems 3.2.1 and 4.7.2 generalise to exponent 5.

Question 5.0.3. *Which partition relations of the form*

$$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow \left(\bigvee_{\nu<\lambda} K_\nu, \bigvee_{\nu<\mu} L_\nu\right)^5$$

hold if all subsets of $\langle {}^{\omega}2, <_{lex}\rangle$ have the property of Baire?

It seems harder to generalise the classification to uncountable κ.

Question 5.0.4. *Which partition relations of the form*

$$\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow \left(\bigvee_{\nu<\lambda} K_\nu, \bigvee_{\nu<\mu} L_\nu\right)^n$$

for $n \geqslant 3$ are (jointly) consistent with ZF ($+\,\mathsf{DC}_\kappa$), and which of the relations for $\kappa = \omega_1$ are provable in the theories $\mathsf{ZF} + \mathsf{AD} + [V = L(\mathbb{R})]$ and $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R}$?

Theorems 1.4.2, 2.1.2, 2.1.3 and 2.2.2 suggest that models of determinacy are good candidates for obtaining positive partition relations. In particular $L(\mathbb{R})$ is a canonical model of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$, provided that there are infinitely many Woodin cardinals and a measurable cardinal above them all, cf. [988MS].

The partition relations in Question 5.0.4 for which all K_ν for $\nu < \lambda$ are well-ordered hold for large ordinals on the left side of the relation by the Erdős-Rado Theorem. On the other hand it is unclear whether the existence of linear orderings K such that $K \longrightarrow (5,\ 2 + \omega^* + \omega \vee \omega + \omega^*)^4$, $K \longrightarrow (5,\ \omega^* + \omega \vee \omega + 2 + \omega^*)^4$ or $K \longrightarrow (6,\ \omega^* + \omega \vee \omega + \omega^*)^4$ is consistent with ZF. The relations fail in ZFC by Theorem 1.2.3. Moreover, if one of the relations holds for a linear order K of the form $K = \langle {}^{\gamma}2, <_{lex}\rangle$, then $\gamma \geqslant \omega_1$ by Summary 4.0.2.

Finally, we ask about partition relations in the context of strong failures of the Axiom of Choice. The assumption in the following question is consistent from a proper class of strongly compact cardinals by [980Gi].

Question 5.0.5. *Which partition relations for linear orders hold if all uncountable cardinals are singular?*

6. Closing Remarks

The results in Sections 3 and 4 were proved by the last author together with the second author and extend results from [014We2]. We would like to thank Paul Larson for letting us include his Theorem 1.5.2. We would also like to thank Jean Larson for letting us include her Lemma 1.3.5 on faithful embeddings of skew subtrees and for many useful comments on a previous version. Finally we would like to thank the referee for detailed reading and suggestions of improvements.

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