CALLIGRAPHY CONCERNING CASUALLY COMPILED CARDINAL CHARACTERISTIC COMPARISONS

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Abstract. We establish several inequalities between cardinal characteristics of the continuum. In particular, it is shown that the partition splitting number is not larger than the uniformity of the meagre ideal; not all sets of reals having the cardinality of an (the?) ε-almost bisecting number are of strong measure zero; no fewer sets of strong measure zero than indicated by the statistically reaping number suffice to cover the reals; the pair-splitting number is not smaller than the evasion number; the cofinally bisecting number is not larger than both the unbounding number and the splitting number; and the subseries number is neither smaller than the pair-splitting number nor than the minimum of the unbounding number and the unbisecting number.

Moreover we provide a diagram putting these results into context and give a brief historical account.

Contents

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1. Preliminaries

1.1. **Introduction.**

1.1.1. *Prologue.* Cardinal characteristics of the continuum are infinite cardinals, usually uncountable and no larger than the continuum, given by a specific definition. They are often derived from considerations not belonging to set theory proper and sometimes from investigations in combinatorial set theory for their own sake. There are two kinds of possible results about them: Often one can show metamathematically that it is consistent with the axioms of set theory to assume that one particular characteristic is smaller than another; sometimes one can straightforwardly show that one is no larger than another. This paper contains seven results of the latter sort, in general related to work of Blass, Hrušák, Laflamme, Meza-Alcántara, A. Miller, Minami, Raghavan and Steprans, [9, 7, 1, 31, 32, 24, 33] and is particularly inspired by the more recent work of J. Brendle, Brian, Halbeisen, Hamkins, Klausner, Lischka, and Shelah, [14, 13]. Its purpose is twofold. On the one hand it shall disseminate the results presented among those specialising—among possibly other fields—in the theory of cardinal characteristics of the continuum. On the other hand it shall give a broader audience a small overview over parts of the subject and serve as a possible starting point for enquiries. While thus having aspects of a survey, it does not aim to be an exposition. For a more expository treatment, see [39, 5, 11, 8, 22]. For a thorough historical account, see [37]. Combining these objectives seems appropriate as, firstly, none of the results presented are of a metamathematical nature, so the reader does not need to know any mathematical logic, and, secondly, they are somewhat scattered about within the subject of cardinal characteristics of the continuum. A Hasse diagram is provided on page 8. It includes all cardinal characteristics involved in new results presented here—see the Summary on page 8—and then some. The general aim was to included as many cardinal characteristics as possible without rendering the resulting diagram incomprehensible. In particular, if a characteristic appears in the diagram, known relations to other characteristic in the diagram appear as well; moreover, lines are not crossing. These conditions explain why $\mathfrak{b}$ does appear in the diagram within both a minimum and a maximum but not on its own, as $\mathfrak{h}_{\mathbb{Q}} \leqslant \mathfrak{b}$. No reader should infer that a cardinal characteristic is generally wider known or in some objective sense more interesting than another only because in contrast to the latter, the former appears in the diagram. A line between two characteristics means an inequality applies with the characteristic higher up on the page being no smaller than the one further down on it.

All proofs presented in this paper are by contradiction and written unwaveringly in this manner. This is to say, that the contradiction is not reached before the end of the proof. Their style might strike the readers as terse but the author hopes that they will (grow to) like it. It may be less suited to give readers a rough idea but it probably is all the more suited to thoroughly check for correctness which, in the absence of a coauthor, is an important feature.

1.1.2. *History.* It only gradually became a custom to name cardinal characteristics when they appeared in a theorem. This is, by the way, the reason that the legend on page 8 contains two columns with references: One to a place in which an inequality between characteristics can be said to have appeared earliest in a history of ideas, and one to a source where explicit definitions and proofs in modern terminology can be found, would this not be possible in the the original source. Many characteristics were named by van Douwen in [39] but appeared in mathematics earlier. It later became customary to depict characteristics thus named in a Hasse diagram called *van Douwen's diagram.*

The unbounding number $\mathfrak{b}$, cf. Definition 1, goes back to Rothberger and Sierpiński, [35, 36], the splitting number $\mathfrak{s}$, cf. Definition 10, to Booth, [10] and the distributivity number $\mathfrak{h}$, cf. Definition 3, to Balcar, Pelant, and Simon, [2]. The study of cardinal characteristics related to measure and category became oriented around what, following Fremlin, [21], became known as Cichoń's diagram. Again, investigations had begun before that, at the latest with A. Miller's [30]. The groupwise density number $\mathfrak{g}$, cf. Definition 2 was introduced by Blass and Laflamme, [9]. In [7], Blass defined the *evasion number* $\mathfrak{e}$, cf. Definition 19 and the linear evasion number $\mathfrak{e}_\ell$, and used them to analyse what he termed the *Specker phenomenon* thus continuing earlier investigations by Eda, [20]. A subgroup G of the Baer-Specker-group $\mathbb{Z}^\omega$, exhibits the *Specker phenomenon* if it contains a sequence $\langle a_i \,|\, i < \omega\rangle$ of linearly independent elements such that for every homomorphism $h : G \longrightarrow \mathbb{Z}$ the set $\{i < \omega \,|\, h(a_i) \neq 0\}$ is finite. Blass showed that there is a group no larger than $\min(\mathfrak{b}, \mathfrak{e})$ exhibiting the Specker phenomenon and that the smallest group exhibiting it can be no smaller than $\mathfrak{e}_\ell$. Brendle, cf. [15], went on to show that the smallest group exhibiting it is in fact of cardinality $\mathfrak{e}_\ell$ which he proved to equal $\min(\mathfrak{b}, \mathfrak{e})$.

Another development starting in the nineties of the last century with a paper by Majcher, [29], is the dualisation of van Douwen's diagram. The notion of duality here is the same as in the dual form of Ramsey's Theorem, proved by Carlson and Simpson, cf. [17]: One switches from subsets of a set to partitions of it and as subsets are induced by ranges of injections, partitions are induced by domains of surjections. These dual characteristics were subsequently analysed by several people: In [32], Minami analysed the dual splitting number and the dual reaping number $\mathfrak{r}_d$ and to that end introduced the notions of pair-splitting and pair-reaping.

The characteristic $\mathfrak{eq}$ was introduced officially by A. Miller and Steprans, [31, p. 57, Def. 16], but one could argue that it was in the air since Miller proved that it is a combinatorial description of the minimal size of a set of reals failing to have strong measure zero, cf. [30, p. 97, Thm. 2.3]. Note, however, that this characterisation holds for $\mathbb{R}$ and the Cantor space but fails for the Baire space as discussed by Hrušák, Wohofsky, and Zindulka, cf. [25].

The cardinal characteristics in Definition 22 were first considered under different names in [16] and subsequently analysed in [23, 34, 33].

Another variation of van Douwen's diagram comes from studying the structure $(\mathrm{Dense}(\mathbb{Q}), \subset_{\mathrm{nwd}})$ instead of $(\mathcal{P}(\omega), \subset^*)$. This was done by Balcar, Brendle, Cichoń, Hernández-Hernández, and Hrušák, cf. [19, 1, 12].

Topological selection principles inspired the definition of the linear refinement number, [38, Definition 61] and the linear excluded middle number, [28, Definition 1.2].

Most recently, Blass, Brendle, Brian, Hamkins, Hardy, and Larson analysed the number of manipulations needed to change the limit of a conditionally convergent series, by respectively rearranging, cf. [6], and dropping some summands, cf. [13]. Meanwhile Brendle, Halbeisen, Klausner, Lischka, and Shelah investigated variants of the splitting and unsplitting numbers derived from requiring splitting to happen equitably in various ways.

1.2. **Notation, Definitions, a Summary, and a Diagram.**

1.2.1. *Notation and Definitions.* For a set A, we write $\mathcal{P}(A)$ for the power set of A; $[A]^\kappa$ for the families of subsets of A having cardinality κ; and $[A]^{<\kappa}$ for the families of subsets of A having fewer than κ elements. For sets A and B, by writing ${}^A B$, we refer to the collection of all functions from A to B. We employ von-Neumann-ordinals, which means that every ordinal number is equal to the set of its predecessors. So 0 denotes the number zero but also the empty set and ω denotes the smallest transfinite ordinal but also the set of all natural numbers. Moreover, this means that for two ordinals α and β, the set $\alpha \setminus \beta$ is the half-open interval from β (and if $\beta < \alpha$ including it) to α but excluding it. By writing $X \subset Y$, we mean that X is a subset of Y, not necessarily a proper subset. Writing $X \subset^* Y$ means that $Y \setminus X$ is finite while for sets X and Y of rational numbers, $X \subset_{\mathrm{nwd}} Y$ means that $Y \setminus X$ is nowhere dense in $\mathbb{Q}$.

We denote by $\mathcal{M}$ the ideal of meagre (or first category) sets, by $\mathcal{N}$ the ideal of sets of Lebesgue measure 0, and by $\mathcal{SN}$ the ideal of sets of strong measure zero. Recall that a set X is of *strong measure zero* if for every sequence $\langle \varepsilon_i \,|\, i < \omega \rangle$ there is a sequence $\langle I_n \,|\, n < \omega \rangle$ of open intervals with I_n having length at most ε_n for every natural number n such that $X \subset \bigcup_{n<\omega} I_n$. Finally, $\mathcal{Z}_0$ denotes the ideal of sets of natural numbers that have asymptotic density 0.

Definition 1. *The* unbounding number *(sometimes called the* bounding number*)* $\mathfrak{b}$ *is the minimal cardinality of a family $\mathcal{F}$ of functions $f \in {}^\omega\omega$ such that there is no function $g \in {}^\omega\omega$ satisfying $f \leqslant^* g$ for all $f \in \mathcal{F}$.*

A family $\mathcal{G} \subset [\omega]^\omega$ is *groupwise dense* if it is open and for every interval partition $\langle J_i \,|\, i < \omega \rangle$ there is an infinite set A of natural numbers such that $\bigcup_{i \in A} J_i \in \mathcal{G}$.

Definition 2. *The* groupwise density number $\mathfrak{g}$ *is the minimal cardinality of a collection $\mathcal{C}$ of groupwise dense families such that $\bigcap \mathcal{C} = 0$.*

A family $\mathcal{F} \subset [\omega]^\omega$ is called *open* if for every $X \in \mathcal{F}$ we have $Y \in \mathcal{F}$ for all $Y \subset^* X$. It is called *dense* if for every $X \in [\omega]^\omega$ there is a $Y \in \mathcal{P}(X) \cap \mathcal{F}$.

Definition 3. *The* distributivity number $\mathfrak{h}$ *is the minimal cardinality of a collection $\mathcal{C}$ of dense open families of sets of natural numbers such that $\bigcap \mathcal{C} = 0$.*

A family $\mathcal{F} \subset \mathrm{Dense}(\mathbb{Q})$ is called $\mathbb{Q}$-*open* if for every $X \in \mathcal{F}$ we have $Y \in \mathcal{F}$ for all $Y \subset_{\mathrm{nwd}} X$. Moreover, a family $\mathcal{D} \subset \mathrm{Dense}(\mathbb{Q})$ is called $\mathbb{Q}$-*dense* if for every $X \in \mathrm{Dense}(\mathbb{Q})$ there is a $Y \in \mathcal{P}(X) \cap \mathcal{D}$.

Definition 4. *The* distributivity number *of* $(\mathrm{Dense}(\mathbb{Q}), \subset_{\mathrm{nwd}})$*, written* $\mathfrak{h}_{\mathbb{Q}}$ *is the minimal cardinality of a collection $\mathcal{C}$ of $\mathbb{Q}$-dense $\mathbb{Q}$-open families of sets of rationals such that $\bigcap \mathcal{C} = 0$.*

Definition 5. *For infinite sets S and X of natural numbers and $\varepsilon \in]0, 1/2[$, we say*

$$S \text{ splits } X \textit{ if both } X \cap S \textit{ and } X \setminus S \textit{ are infinite,}$$

$$S \text{ bisects } X \textit{ if } \lim_{n \nearrow \infty} \frac{|S \cap X \cap n|}{|X \cap n|} = \frac{1}{2},$$

$$S \text{ infinitely often}^1 \text{ bisects } X \textit{ if } \frac{|S \cap X \cap n|}{|X \cap n|} = \frac{1}{2} \textit{ for infinitely many natural numbers } n,$$

$$S\ \varepsilon\text{-almost bisects } X \textit{ if } \left\{ n < \omega \,\middle|\, \frac{|S \cap X \cap n|}{|X \cap n|} \notin \left] \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon \right[\right\} \textit{ is finite.}$$

A set X of natural numbers is called *moderate* if

$$0 < \liminf_{n \nearrow \infty} \frac{|X \cap n|}{n} \leqslant \limsup_{n \nearrow \infty} \frac{|X \cap n|}{n} < 1.$$

We also say that a set A of pairs of natural numbers is *unbounded* if $A \not\subset [\omega]^2 \setminus [\omega \setminus k]^2$ for every natural number k. Moreover, an infinite set X of natural numbers *pair-splits* A if $A \setminus ([X]^2 \cup [\omega \setminus X]^2)$ is infinite.

Definition 6. *A family $\mathcal{F}$ of infinite sets of natural numbers is called*

splitting *if for every infinite set X of natural numbers there is a member of $\mathcal{F}$ splitting X,*

pair-splitting *if for every unbounded $A \subset [\omega]^2$ there is a member of $\mathcal{F}$ pair-splitting it.*

unbisected *if there is no set S of natural numbers bisecting every member of $\mathcal{F}$,*

statistically refining[2] *if there is no moderate $S \in [\omega]^\omega$ such that* $\lim_{n \nearrow \infty} \frac{n \cdot |S \cap X \cap n|}{|S \cap n| \cdot |X \cap n|} = 1$ *for all $X \in \mathcal{F}$,*

bisecting *if for every infinite set X of natural numbers there is a member of $\mathcal{F}$ bisecting X,*

cofinally bisecting *if for every $X \in [\omega]^\omega$ there is a member of $\mathcal{F}$ cofinally bisecting X,*

ε-almost bisecting *for an $\varepsilon \in]0, 1/2[$ if for every $X \in [\omega]^\omega$ there is a member of $\mathcal{F}$ ε-almost bisecting X.*

The following notion of splitting number was introduced by Raghavan, cf. [33].

Definition 7. *Let P be an infinite partition of ω and a an infinite set of natural numbers. We say that P* splits *a if $a \cap x$ is infinite for every $x \in P$. An family $\mathcal{F}$ of partitions of ω is called a* splitting family of partitions *if for every $a \in [\omega]^\omega$ there is a $P \in \mathcal{F}$ splitting a. The* partition splitting number $\mathfrak{s}(\mathfrak{pr})$ *is the minimal cardinality of a splitting family of partitions.*

It is currently open whether $\mathfrak{s}(\mathfrak{pr})$ can be different from $\mathfrak{s}$, cf. [33, p. 2].

Definition 8. *For sets $X, Y \in$ Dense$(\mathbb{Q})$ we say that X $\mathbb{Q}$-splits Y if both $Y \cap X$ and $Y \setminus X$ are dense in $\mathbb{Q}$. A family $\mathcal{F}$ of dense sets of rational numbers is called* $\mathbb{Q}$-splitting *if for every dense set X of rational numbers there is a member of $\mathcal{F}$ which $\mathbb{Q}$-splits X.*

Definition 9. *The* statistically refining[3] number $\mathfrak{r}_*$ *is the minimal cardinality of a statistically refining family.*

Definition 10. *The* splitting number $\mathfrak{s}$ *is the minimal cardinality of a splitting family.*

Definition 11. *The* $\mathbb{Q}$-splitting number $\mathfrak{s}_{\mathbb{Q}}$ *is the minimal cardinality of a* $\mathbb{Q}$*-splitting family.*

Definition 12. *The* pair-splitting number $\mathfrak{s}_{pair}$ *is the minimal cardinality of a pair-splitting family.*

Definition 13. *The* bisecting number $\mathfrak{s}_{1/2}$ *is the minimal cardinality of a bisecting family.*

Definition 14. *The* unbisecting (or semirefining[4]) number $\mathfrak{r}_{1/2}$ *is the minimal cardinality of an unbisected family.*

Definition 15. *The* cofinally bisecting number $\mathfrak{s}_{1/2}^{\infty}$ *is the minimal cardinality of a cofinally bisecting family.*

Definition 16. *The* ε-almost bisecting number $\mathfrak{s}_{1/2\pm\varepsilon}$ *is the minimal cardinality of an* ε*-almost bisecting family.*

Recall that a series of real numbers $\sum_{i\nearrow\infty} r_i$ is *conditionally convergent* if it is convergent but there is a set A of natural numbers such that $\sum_{i\in A} r_i$ fails to be. Moreover we say that $\sum_{i\nearrow\infty} r_i$ *diverges by oscillation* if for every natural number k and every real number r there are $\ell, m, n \in \omega \setminus k$ such that

$$\sum_{i<\ell} r_i \notin]r - 1/k, r + 1/k[, \tag{1}$$

$$\sum_{i<m} r_i > -k, \tag{2}$$

$$\sum_{i<n} r_i < k. \tag{3}$$

Definition 17. *The* subseries number ß *is the minimal cardinality of a family $\mathcal{I}$ of sets of natural numbers such that for every conditionally convergent series $\sum_{i\nearrow\infty} r_i$ there is an index set $I \in \mathcal{I}$ such that $\sum_{j\in I} r_j$ fails to converge to a real number.*

Definition 18. *The* oscillating subseries number ß_o *is the minimal cardinality of a family $\mathcal{I}$ of sets of natural numbers such that for every conditionally convergent series $\sum_{i\nearrow\infty} r_i$ there is an index set $I \in \mathcal{I}$ such that $\sum_{j\in I} r_j$ diverges by oscillation.*

We call a pair $\pi = (D, \langle \pi_n \,|\, n \in D\rangle)$ where D is an infinite set of natural numbers and $\pi_n : {}^n\omega \longrightarrow \omega$ for every $n \in D$, a *predictor*. We say that π *predicts* a sequence $s \in {}^\omega\omega$ if $\{n \in D \,|\, \pi_n(s \restriction n) \neq s(n)\}$ is finite, otherwise s *evades* π.

[3] or statistically reaping
[4] or semireaping

Definition 19. *The* evasion number $\mathfrak{e}$ *is the minimal cardinality of a family $\mathcal{E}$ of functions in ${}^{\omega}\omega$ such that every predictor is evaded by a member of $\mathcal{E}$.*

Definition 20. $\mathfrak{eq}$ *is the minimal cardinality of a bounded family $\mathcal{F} \subset {}^{\omega}\omega$ such that for every $g \in {}^{\omega}\omega$ there is an $f \in \mathcal{F}$ such that $f(n) \neq g(n)$ for all natural numbers n.*

Definition 21. *Given a proper ideal $\mathcal{I}$ on a set X,*

$$\begin{aligned}
\operatorname{add}(\mathcal{I}) & \textit{ is the minimal cardinality of a subfamily of } \mathcal{I} \textit{ whose union is not in } \mathcal{I}, \\
\operatorname{cov}(\mathcal{I}) & \textit{ is the minimal cardinality of a subfamily of } \mathcal{I} \textit{ whose union is } X, \\
\operatorname{non}(\mathcal{I}) & \textit{ is the minimal cardinality of a an element of } \mathcal{P}(X) \setminus \mathcal{I}, \\
\operatorname{cof}(\mathcal{I}) & \textit{ is the minimal cardinality of a base of } \mathcal{I}.
\end{aligned}$$

We have $\operatorname{add}(\mathcal{I}) \leqslant \operatorname{cov}(\mathcal{I})$*. If $\mathcal{I}$ contains all singletons then, clearly,* $\operatorname{add}(\mathcal{I}) \leqslant \operatorname{non}(\mathcal{I})$*, also, the union of any base is then X, hence* $\operatorname{cov}(\mathcal{I}) \leqslant \operatorname{cof}(\mathcal{I})$*. Also,* $\operatorname{non}(\mathcal{I}) \leqslant \operatorname{cof}(\mathcal{I})$ *for otherwise one could inductively in step α pick an $x_\alpha \in X \setminus (B_\alpha \cup \{x_\beta \mid \beta < \alpha\})$ for any base $\{B_\alpha \mid \alpha < \operatorname{cof}(\mathcal{I})\} \in [\mathcal{I}]^{\operatorname{cof}(\mathcal{I})}$ which would yield $\{x_\alpha \mid \alpha < \operatorname{cof}(\mathcal{I})\} \in [X]^{\operatorname{cof}(\mathcal{I})} \setminus \mathcal{I}$.*

Definition 22. *Given an ideal $\mathcal{I}$ on the set of natural numbers,*

$$\begin{aligned}
\operatorname{add}^*(\mathcal{I}) & \textit{ is the minimal size of an } \mathcal{F} \subset \mathcal{I} \textit{ such that for all } x \in \mathcal{I} \textit{ there is a } y \in \mathcal{F} \textit{ with } |x \setminus y| = \aleph_0. \\
\operatorname{non}^*(\mathcal{I}) & \textit{ is the minimal size of an } \mathcal{F} \subset [\omega]^\omega \textit{ such that for all } x \in \mathcal{I} \textit{ there is a } y \in \mathcal{F} \textit{ with } |x \cap y| < \aleph_0. \\
\operatorname{cov}^*(\mathcal{I}) & \textit{ is the minimal size of an } \mathcal{F} \subset \mathcal{I} \textit{ such that for all } x \in [\omega]^\omega \textit{ there is a } y \in \mathcal{F} \textit{ with } |y \setminus x| < \aleph_0. \\
\operatorname{cof}^*(\mathcal{I}) & \textit{ is the minimal size of an } \mathcal{F} \subset \mathcal{I} \textit{ such that for all } x \in \mathcal{I} \textit{ there is a } y \in \mathcal{F} \textit{ with } |x \cap y| < \aleph_0.
\end{aligned}$$

We now consider partitions of ω. We say that a partition R *almost refines* another partition X if

$$\Big\{A \in X \,\big|\, \forall B \subset R : \bigcup B \neq A\Big\} \text{ is finite.}$$

We say that S *dual-splits* X if S is not almost refining X yet there is an infinite partition of the natural numbers almost refining both S and X.

Definition 23. *The* dual reaping number $\mathfrak{r}_d$ *is the minimal cardinality of a family $\mathcal{F}$ of partitions of ω for which there is no single partition of ω dual-splitting all members of $\mathcal{F}$.*

A family of sets is called *linear*, if it is linearly ordered by $\subset^*$ and a family $\mathcal{G}$ is a *refinement* of a family $\mathcal{F}$ if for every $A \in \mathcal{F}$ there is a $B \in \mathcal{G}$ with $B \subset^* A$.

Definition 24. *The* linear refinement number $\mathfrak{lr}$ *is the minimal cardinality of a family $\mathcal{F} \subset [\omega]^\omega$ without a linear refinement such that for every finite $\mathcal{A} \subset \mathcal{F}$ the set $\bigcap \mathcal{A}$ is infinite.*

Definition 25. *The* linear excluded middle number $\mathfrak{lx}$ *is the minimal cardinality of a family $\mathcal{F} \subset {}^{\omega}\omega$ such that for every $h \in {}^{\omega}\omega$ for which $\mathcal{S} := \big\{ \{n < \omega \mid f(n) \leqslant h(n)\} \,\big|\, f \in \mathcal{F}\big\} \subset [\omega]^\omega$, the family $\mathcal{S}$ has no linear refinement.*

1.2.2. *A Summary and a Diagram.*

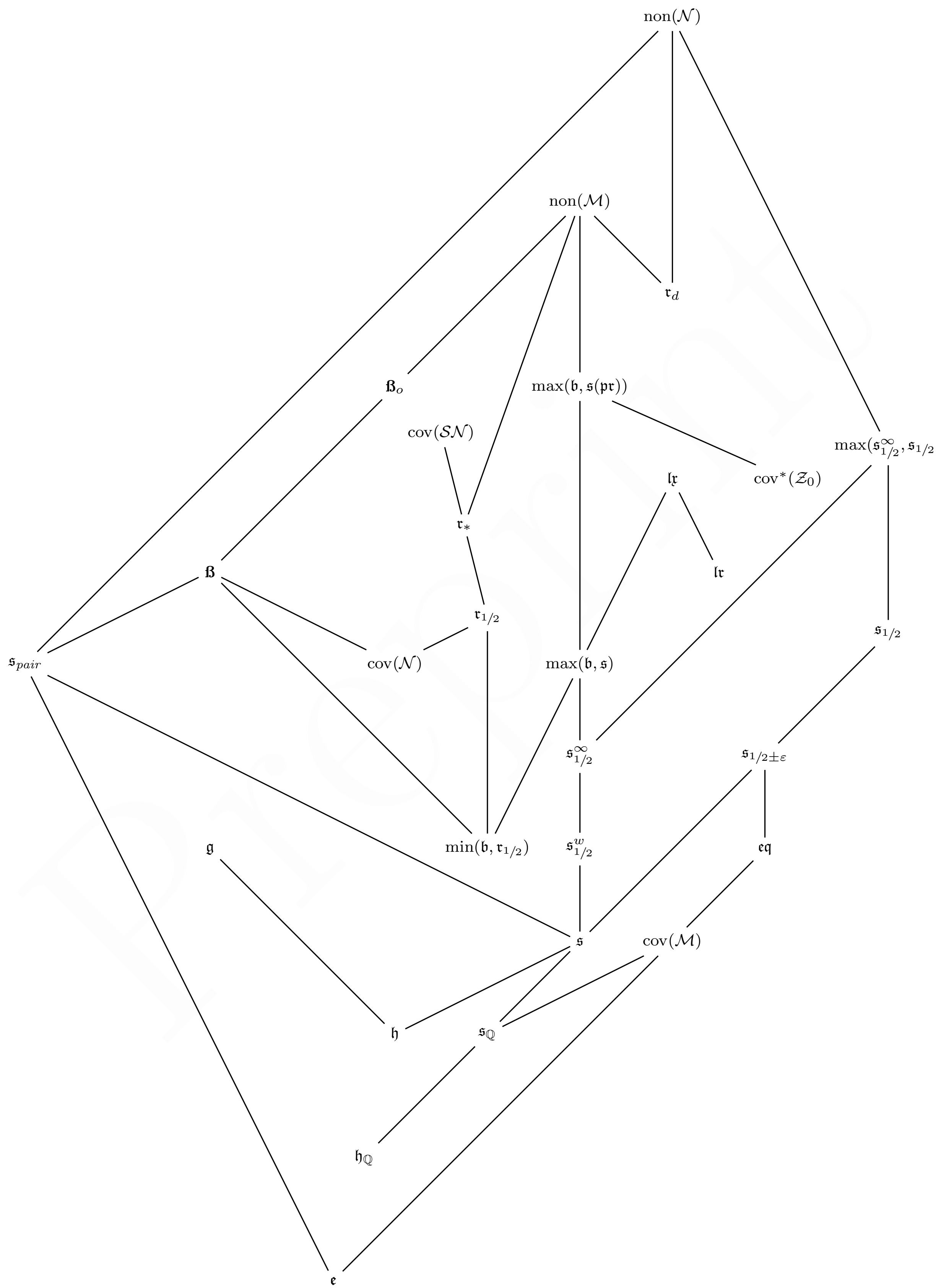
Summary. *The following statements are being proved:*

$\mathfrak{s}^{\infty}_{1/2} \leqslant \max(\mathfrak{b}, \mathfrak{s})$	Theorem	1
$\mathfrak{eq} \leqslant \inf_{\varepsilon \nearrow 1/2} (\mathfrak{s}_{1/2 \pm \varepsilon})$	Theorem	2
$\mathfrak{r}_* \leqslant \operatorname{cov}(\mathcal{SN})$	Theorem	3
$\min(\mathfrak{b}, \mathfrak{r}_{1/2}) \leqslant \text{ß}$	Theorem	4
$\mathfrak{s}(\mathfrak{pr}) \leqslant \operatorname{non}(\mathcal{M})$	Theorem	5
$\mathfrak{e} \leqslant \mathfrak{s}_{pair}$	Theorem	6
$\mathfrak{s}_{pair} \leqslant \text{ß}$	Theorem	7

Note that the Mathias model witnesses both the consistency of $\mathfrak{h} = \aleph_2$, [8, Section 11.8], and the one of the Borel conjecture,[5, Thm. 8.3.4]. So in particular, the Mathias model believes in $\mathfrak{eq} = \operatorname{non}(\mathcal{SN}) = \aleph_1$ thus witnessing that the inequalities in Theorems 2, and 6 are consistently strict. Analogously to the proof of Theorem 5 and even more straightforwardly it is possible to show that $\mathfrak{s}(\mathfrak{pr}) \leqslant \mathfrak{d}$. See [8, Thm. 3.3] and [18, Thm. 2.2] to get an idea about proving it. Furthermore, the Random model witnesses that $\max(\mathfrak{b}, \mathfrak{d}, \operatorname{non}(\mathcal{N})) < \operatorname{cov}(\mathcal{N})$ is consistent, [8, Section 11.4], so a fortiori the inequalities in Theorems 4, 5, and 7 are consistently strict.

Legend.

Inequalities	**Origin**	**Proof**
$\operatorname{cov}(\mathcal{N}) \leqslant \text{ß} \leqslant \text{ß}_o \leqslant \operatorname{non}(\mathcal{M})$	[13]	
$\operatorname{cov}(\mathcal{N}) \leqslant \mathfrak{r}_{1/2} \leqslant \mathfrak{r}_* \leqslant \operatorname{non}(\mathcal{M})$	[14]	
$\mathfrak{s} \leqslant \mathfrak{s}_{1/2 \pm \varepsilon} \leqslant \mathfrak{s}_{1/2} \leqslant \operatorname{non}(\mathcal{N})$	[14]	
$\mathfrak{s} \leqslant \mathfrak{s}^{w}_{1/2} \leqslant \mathfrak{s}^{\infty}_{1/2} \leqslant \operatorname{non}(\mathcal{N})$	[14]	
$\mathfrak{s} \leqslant \mathfrak{s}_{pair} \leqslant \operatorname{non}(\mathcal{N})$	[32]	
$\mathfrak{r}_d \leqslant \operatorname{non}(\mathcal{M}), \operatorname{non}(\mathcal{N})$	*J. Brendle*	[32, p. 503, Prop. 6]
$\mathfrak{s}_{\mathbb{Q}} \leqslant \operatorname{add}(\mathcal{M}), \mathfrak{s}$	[12, p. 57]	
$\mathfrak{h}_{\mathbb{Q}} \leqslant \mathfrak{s}_{\mathbb{Q}}$	[1, p. 73]	
$\mathfrak{e} \leqslant \operatorname{cov}(\mathcal{M})$	[26]	
$\mathfrak{h} \leqslant \mathfrak{g}$	[9, p. 53]	[8, p. 433, Prop. 6.27]
$\mathfrak{h} \leqslant \mathfrak{s}$	[3, p. 354]	[8, p. 427, Thm. 6.9]
$\operatorname{cov}(\mathcal{M}) \leqslant \mathfrak{eq}$	[4]	[8, p. 420, Thm. 5.9]
$\mathfrak{b} \leqslant \operatorname{non}(\mathcal{M})$	[4, p. 237]	[8, p. 419, Prop. 5.5]
$\mathfrak{s} \leqslant \operatorname{non}(\mathcal{M})$	*Folklore*	[8, p. 424, Thm. 5.19]
$\max(\mathfrak{b}, \mathfrak{s}, \mathfrak{lr}) \leqslant \mathfrak{lz}$	[28, Cor. 2.13]	
$\operatorname{cov}^*(\mathcal{Z}_0) \leqslant \max\big(\mathfrak{b}, \mathfrak{s}(\mathfrak{pr})\big)$	[33, Cor. 30]	



$\mathrm{non}(\mathcal{N})$
$\mathrm{non}(\mathcal{M})$
$\mathfrak{r}_d$
$\mathfrak{B}_o$
$\max(\mathfrak{b}, \mathfrak{s}(\mathfrak{pr}))$
$\mathrm{cov}(\mathcal{SN})$
$\max(\mathfrak{s}^\infty_{1/2}, \mathfrak{s}_{1/2})$
$\mathfrak{lr}$
$\mathrm{cov}^*(\mathcal{Z}_0)$
$\mathfrak{r}_*$
$\mathfrak{B}$
$\mathfrak{lr}$
$\mathfrak{r}_{1/2}$
$\mathfrak{s}_{1/2}$
$\mathfrak{s}_{pair}$
$\mathrm{cov}(\mathcal{N})$
$\max(\mathfrak{b}, \mathfrak{s})$
$\mathfrak{s}^\infty_{1/2}$
$\mathfrak{s}_{1/2\pm\varepsilon}$
$\mathfrak{g}$
$\min(\mathfrak{b}, \mathfrak{r}_{1/2})$
$\mathfrak{s}^w_{1/2}$
$\mathfrak{eq}$
$\mathfrak{s}$
$\mathrm{cov}(\mathcal{M})$
$\mathfrak{h}$
$\mathfrak{s}_\mathbb{Q}$
$\mathfrak{h}_\mathbb{Q}$
$\mathfrak{e}$

2. Results

2.1. Inequalities Involving Halfway New Cardinal Characteristics.

2.1.1. *Splitting.*

Theorem 1. $\mathfrak{s}^\infty_{1/2} \leqslant \max(\mathfrak{b}, \mathfrak{s})$.

This proof is to a certain degree inspired by the work of Kamburelis and Węglorz, [27].

Proof. Assume towards a contradiction that $\max(\mathfrak{b}, \mathfrak{s}) < \mathfrak{s}^\infty_{1/2}$ and let $\mathcal{S} \subset [\omega]^\omega$, $\mathcal{U} \subset {}^\omega\omega$ be families which are respectively splitting, of size $\mathfrak{s}$, and unbounded, of size $\mathfrak{b}$. We may assume without loss of generality that all $f \in \mathcal{U}$ are properly increasing and satisfy $f(0) > 0$. For any function $f : \omega \longrightarrow \omega$, let $g_f : \omega \longrightarrow \omega$ be the function given inductively by $g_f(0) := 0$ and $g_f(n+1) := f(g_f(n))$. Now we define

$$b(S, f) := \bigcup_{n \in S} g_f(n+1) \setminus g_f(n) \text{ for } S \in \mathcal{S} \text{ and } f \in \mathcal{U},$$
$$\mathcal{B} := \Big\{ b(S, f) \,\Big|\, \{S, \omega \setminus S\} \not\subset [\omega]^\omega \setminus \mathcal{S} \wedge f \in \mathcal{U} \Big\}.$$

Clearly, $|\mathcal{B}| \leqslant 2 \cdot \mathfrak{b} \cdot \mathfrak{s} = \max(\mathfrak{b}, \mathfrak{s})$. As $\max(\mathfrak{b}, \mathfrak{s}) < \mathfrak{s}^\infty_{1/2}$, the family $\mathcal{B}$ cannot be infinitely often bisecting, so let $X \in [\omega]^\omega$ witness this, i.e. X is not infinitely often bisected by any member of $\mathcal{B}$. Now define a function $h : \omega \longrightarrow \omega$ by $n \longmapsto \min(\{i < \omega \,|\, |X \cap i \setminus n| > n\})$ and pick an $A \in \mathcal{U}$ which is unbounded by h. Let this be witnessed by $Y \in [\omega]^\omega$, i.e. $h(n) < A(n)$ for all $n \in Y$. Let $Z \in \mathcal{S}$ split Y. As $B := b(Z, A)$ fails to infinitely bisect X, there is a natural number k such that $2|B \cap X \cap n| \neq |X \cap n|$ for all $n \in \omega \setminus k$. Note that under these circumstances, $2|B \cap X \cap k| < |X \cap k|$ implies $2|B \cap X \cap n| < |X \cap n|$ for all $n \in \omega \setminus k$ and $2|B \cap X \cap k| > |X \cap k|$ implies $2|B \cap X \cap n| > |X \cap n|$ for all $n \in \omega \setminus k$, in other words, the density of B in X up to n cannot, as n gets larger, change between being larger than one half and smaller than one half without assuming the value of one half itself. As $\{\Xi \,|\, \omega \setminus \Xi \in \mathcal{B}\} = \mathcal{B}$, we may suppose without loss of generality that

$$2|B \cap X \cap n| < |X \cap n| \text{ for all } n \in \omega \setminus k. \tag{4}$$

Now let $\ell \in Y \cap Z$ be minimal such that $g_A(\ell) \geqslant k$. Note that $g_A(\ell+1) \setminus g_A(\ell) \subset B$ and $h(g_A(\ell)) < g_A(\ell+1)$. We have

$$g_A(\ell) < |X \cap h(g_A(\ell)) \setminus g_A(\ell)|,$$
$$X \cap g_A(\ell+1) \subset (B \cap X \cap g_A(\ell+1)) \cup g_A(\ell) \text{ and}$$
$$X \cap h(g_A(\ell)) \setminus g_A(\ell) \subset B \cap X \cap g_A(\ell+1),$$

contradicting (4) $\square$

Theorem 2. $\mathfrak{eq} \leqslant \inf_{\varepsilon \nearrow 1/2} (\mathfrak{s}_{1/2 \pm \varepsilon})$.

The following proof uses an partition of the natural numbers into intervals whose lengths are growing sufficiently quickly such that one can define an eventually different sequence from a bisecting sequence.

Proof. Suppose towards a contradiction that $\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{eq}$ for some $\varepsilon \in\,]0, 1/2[$ and let $\mathcal{S}$ be an ε-almost bisecting family of cardinality $\mathfrak{s}_{1/2\pm\varepsilon}$. Let $\langle I_k \mid k < \omega\rangle$ be the interval partition of the natural numbers defined by $\min(I_0) = 0$, $\min(I_{k+1}) = \max(I_k) + 1$ and $\max(I_k) = k + 2\lceil 2\min(I_k)/(1-2\varepsilon)\rceil$ for all natural numbers k. Note that every interval has an odd length. For every natural number k, choose a function $h_k : \mathcal{P}(I_k) \longrightarrow 2^{\max(I_k)-\min(I_k)}$ such that $h_k(A) = h_k(B)$ if and only if $A \in \{B, I_k \setminus B\}$. Now we can define

$$\mathcal{E} := \{e \in {}^{\omega}\omega \mid \exists S \in \mathcal{S}\, \exists k \in \omega\, \forall \ell \in \omega \setminus k : e(\ell) = h_\ell(S \cap I_\ell)\}.$$

We have $|\mathcal{E}| = \aleph_0 \cdot |\mathcal{S}| = \aleph_0 \cdot \mathfrak{s}_{1/2\pm\varepsilon} = \mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{eq}$. As $\mathcal{E}$ is bounded, there is an $f \in {}^{\omega}\omega$ such that for all $e \in \mathcal{E}$ there is a natural number ℓ such that $e(\ell) = f(\ell)$. Now, for every natural number k, let g_k be the right-inverse of h_k such that $|g_k(\ell)| > |I_k \setminus g_k(\ell)|$ for all $\ell < 2^{\max(I_k)-\min(I_k)}$. We consider the set

$$X := \bigcup_{\ell<\omega} g_\ell(\max(f(\ell), 2^{\max(I_\ell)-\min(I_\ell)} - 1)).$$

Now there is an $S \in \mathcal{S}$ which is ε-almost bisecting X. Let m be a natural number such that

$$\frac{|S \cap X \cap n|}{|X \cap n|} \in \left]\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right[\text{ for all } n \in \omega \setminus m.$$

Let k be the least natural number such that $I_k \subset \omega \setminus m$. We define a function,

$$\begin{aligned} d : \omega &\longrightarrow \omega \\ n &\longmapsto \begin{cases} f(n) + 1 & \text{if } n < k, \\ h_n(S \cap I_n) & \text{else.} \end{cases} \end{aligned}$$

Clearly, $d \in \mathcal{E}$. Now there is a natural number ℓ such that $d(\ell) = f(\ell)$. By definition of d we have $\ell \in \omega \setminus k$, moreover

$$f(\ell) = d(\ell) = h_\ell(S \cap I_n) < 2^{\max(I_\ell)-\min(I_\ell)}.$$

Next we distinguish two cases. First assume that $2|S \cap I_\ell| < |I_\ell|$. Then

$$X \cap I_\ell = g_\ell(\max(f(\ell), 2^{\max(I_\ell)-\min(I_\ell)} - 1)) = g_\ell(f(\ell)) = g_\ell(d(\ell)) = g_\ell(h_\ell(I_\ell \cap S)) = I_\ell \setminus S,$$

and therefore $S \cap X \cap \min(I_{\ell+1}) \subset \min(I_\ell)$. We have

$$\begin{aligned} \frac{|X \cap S \cap \min(I_{\ell+1})|}{|X \cap \min(I_{\ell+1})|} &= \frac{|X \cap S \cap \min(I_\ell)|}{|X \cap \min(I_{\ell+1})|} \leqslant \frac{\min(I_\ell)}{|X \cap \min(I_{\ell+1})|} \\ &< \frac{2\min(I_\ell)}{\min(I_{\ell+1})} < \frac{2\min(I_\ell)}{\max(I_\ell)} \leqslant \frac{2\min(I_\ell)(1-2\varepsilon)}{\ell(1-2\varepsilon) + 4\min(I_\ell)} \leqslant \frac{1}{2} - \varepsilon, \end{aligned}$$

implying $\ell < k$, a contradiction.

Now we assume $|I_\ell| < 2|S \cap I_\ell|$, we have

$$X \cap I_\ell = g_\ell(\max(f(\ell), 2^{\max(I_\ell)-\min(I_\ell)} - 1)) = g_\ell(f(\ell)) = g_\ell(d(\ell)) = g_\ell(h_\ell(I_\ell \cap S)) = I_\ell \cap S,$$

and therefore $S \cap I_\ell \subset X \cap S$. Then

$$\frac{|X \cap S \cap \min(I_{\ell+1})|}{|X \cap \min(I_{\ell+1})|} \geqslant \frac{|S \cap I_\ell|}{|X \cap \min(I_{\ell+1})|} \geqslant \frac{|S \cap I_\ell|}{\min(I_\ell) + |X \cap I_\ell|} = \frac{|S \cap I_\ell|}{\min(I_\ell) + |S \cap I_\ell|}$$ and as,

for nonnegative values, the function $\dfrac{x}{a+x}$ is monotonically increasing in x,

$$\geqslant \frac{|I_\ell|}{2\min(I_\ell) + |I_\ell|} = \frac{\min(I_{\ell+1}) - \min(I_\ell)}{\min(I_{\ell+1}) + \min(I_\ell)} > \frac{\max(I_\ell) - \min(I_\ell)}{\max(I_\ell) + \min(I_\ell)}$$

$$\geqslant \frac{4\min(I_\ell) + (1-2\varepsilon)(\ell - \min(I_\ell))}{4\min(I_\ell) + (1-2\varepsilon)(\ell + \min(I_\ell))}$$ and as,

for nonnegative values, the function $\dfrac{a+b(x-1)}{a+b(x+1)}$ is monotonically increasing in x,

$\geqslant \dfrac{3+2\varepsilon}{5-2\varepsilon} > \dfrac{1}{2} + \varepsilon$, implying $\ell < k$, a contradiction.

□

2.1.2. *Reaping.*

Theorem 3. $\mathfrak{r}_* \leqslant \operatorname{cov}(\mathcal{SN})$.

The following proof uses the fact that a bisecting real can be used to code avoidance of a sequence of sufficiently small open intervals.

Proof. Assume towards a contradiction that $\operatorname{cov}(\mathcal{SN}) < \mathfrak{r}_*$ and let $\mathcal{C}$ be a family of $\operatorname{cov}(\mathcal{SN})$ sets of strong measure zero covering the Cantor space. Let $\langle J_i \mid i < \omega\rangle$ be an interval partition defined by $J_i := 1 + i + (i+2)! \setminus \bigcup_{k<i} J_k$. Note that all parts of this partition are of odd length! Furthermore, let $\langle P_i \mid i < \omega\rangle$ be a partition of ω into infinitely many infinite parts with $P_i \subset \omega \setminus i$ for all natural numbers i. For every $Z \in \mathcal{C}$ let $\langle s^Z_j \mid j < \omega\rangle$ be a sequence of $s^Z_j \in {}^{\bigcup_{i\leqslant j} J_i}2$ such that for all natural numbers k and all $x \in Z$ there is a $j \in P_k$ such that $x \restriction \bigcup_{i\leqslant j} J_i = s^Z_j$. This is possible as Z is of strong measure zero. Now we can define

$$\mathcal{D} := \{X_Z \mid Z \in \mathcal{C}\}$$

$$\text{where } X_Z := \bigcup_{i<\omega} F^Z_i$$

and F^Z_i is the larger of the two sets $\left\{k \in J_i \mid s^Z_i(k) = 0\right\}$ and $\left\{k \in J_i \mid s^Z_i(k) = 1\right\}$.

As $|\mathcal{D}| \leqslant |\mathcal{C}| = \mathrm{cov}(\mathcal{SN}) < \mathfrak{r}_*$, there is a moderate $S \in [\omega]^\omega$ such that $\lim_{n\nearrow\infty} \frac{n \cdot |S \cap X \cap n|}{|S \cap n| \cdot |X \cap n|} = 1$ for all $X \in \mathcal{D}$. As $\mathcal{C}$ covers the Cantor space, we can fix a $Z \in \mathcal{C}$ containing χ_S. Let $\varepsilon \in]0, 1/2[$ be such that

$$\varepsilon < \liminf_{n\nearrow\infty} \frac{|S \cap n|}{n} \text{ and } \limsup_{n\nearrow\infty} \frac{|S \cap n|}{n} < 1 - \varepsilon.$$

By the choice of S we then also have $\varepsilon < \liminf_{n\nearrow\infty} \frac{|S \cap X_Z \cap n|}{|X_Z \cap n|}$ and $\limsup_{n\nearrow\infty} \frac{|S \cap X_Z \cap n|}{|X_Z \cap n|} < 1 - \varepsilon$.

Now choose a natural number $k > 3/\varepsilon$ large enough such that $\frac{|S \cap X_Z \cap n|}{|X_Z \cap n|} \in]\varepsilon, 1-\varepsilon[$ for all $n \in \omega \setminus k$. Choose a natural number j such that $P_j \subset \omega \setminus k$ and let $\ell \in P_j$ be such that $\chi_S \restriction \bigcup_{i\leqslant\ell} J_i = s^Z_\ell$. Note that $2|X_Z \cap \bigcup_{i\leqslant\ell} J_i| > |\bigcup_{i\leqslant\ell} J_i|$. Also note that, as $P_j \ni \ell$ consists of natural numbers no smaller than k, we have $\ell \geqslant k > 3/\varepsilon$. We now distinguish two cases. First assume, that $F^Z_\ell = \{k \in J_\ell \mid s^Z_\ell(k) = 0\}$. Then $S \cap X_Z \cap \bigcup_{i\leqslant\ell} J_i \subset \bigcup_{i<\ell} J_i$ and hence

$$\begin{aligned}\frac{|S \cap X_Z \cap \bigcup_{i\leqslant\ell} J_i|}{|X_Z \cap \bigcup_{i\leqslant\ell} J_i|} &\leqslant \frac{2|\bigcup_{i<\ell} J_i|}{|\bigcup_{i\leqslant\ell} J_i|} = \frac{2(\ell + (\ell+1)!)}{1 + \ell + (\ell+2)!} < \frac{2(\ell + (\ell+1)!)}{\ell + (\ell+2)!} = \frac{2(1 + (\ell+1)(\ell-1)!)}{1 + (\ell+1)(\ell+2)(\ell-1)!} \\ &< \frac{2(1 + (\ell+1)(\ell-1)!)}{(\ell+1)(\ell+2)(\ell-1)!} \leqslant \frac{2(\ell+2)(\ell-1)!}{(\ell+1)(\ell+2)(\ell-1)!} = \frac{2}{\ell+1} < \frac{2}{\ell} < \frac{2}{3/\varepsilon} = \frac{2\varepsilon}{3} < \varepsilon.\end{aligned}$$

Now assume, that $F^Z_\ell = \{k \in J_\ell \mid s^Z_\ell(k) = 1\}$. Then $X_Z \cap J_\ell \subset S \cap X_Z \cap \bigcup_{i\leqslant\ell} J_i$ and hence

$$\frac{|S \cap X_Z \cap \bigcup_{i\leqslant\ell} J_i|}{|X_Z \cap \bigcup_{i\leqslant\ell} J_i|} \geqslant \frac{|X_Z \cap J_\ell|}{|X_Z \cap \bigcup_{i\leqslant\ell} J_i|} = \frac{|X_Z \cap J_\ell|}{|X_Z \cap \bigcup_{i<\ell} J_i| + |X_Z \cap J_\ell|}.$$

As $X_Z \cap J_\ell = F^Z_\ell$ and $2|F^Z_\ell| > |J_\ell|$ and for nonnegative values, the function

$$\frac{x}{a+x} \text{ is monotonically increasing in } x,$$

$$\begin{aligned}&\frac{|X_Z \cap J_\ell|}{|X_Z \cap \bigcup_{i<\ell} J_i| + |X_Z \cap J_\ell|} > \frac{|J_\ell|}{2|X_Z \cap \bigcup_{i<\ell} J_i| + |J_\ell|} \geqslant \frac{|J_\ell|}{2|\bigcup_{i<\ell} J_i| + |J_\ell|} \\ &= \frac{1 + (\ell+2)! - (\ell+1)!}{(\ell+2)! + (\ell+1)! + 2\ell + 1} > \frac{(\ell+2)! - (\ell+1)!}{(\ell+2)! + 2(\ell+1)!} = \frac{\ell+1}{\ell+4} > \frac{3+\varepsilon}{3+4\varepsilon} > 1 - \varepsilon.\end{aligned}$$

In both cases we reach a contradiction thus concluding the proof. □

Theorem 4. $\min(\mathfrak{b}, \mathfrak{r}_{1/2}) \leqslant ß$.

The idea of the following proof is to use a bound for the distances between elements of the index set of a subseries and observe that a bisecting sequence implies convergence of the respective subseries.

Proof. Assume towards a contradiction that $ß < \min(\mathfrak{b}, \mathfrak{r}_{1/2})$. Let $\mathcal{I}$ be a $ß$-sized family of infinite index sets of natural numbers such that for every conditionally convergent series $\sum_{i\to\infty} r_i$ there is an index set $I \in \mathcal{I}$ such that $\sum_{i\in I} r_i$ is diverging. As $|\mathcal{I}| = ß < \mathfrak{r}_{1/2}$, there is an $S \in [\omega]^\omega$ bisecting all members of $\mathcal{I}$. We define a family

$$\begin{aligned}
&\mathcal{B} := \{f_I \mid I \in \mathcal{I}\}\,, \text{ where}\\
&f_I := \langle n_j^I \mid j < \omega\rangle, \text{ and}\\
&n_j^I := \min\left(\left\{k < \omega \,\middle|\, \forall \ell \in \omega \setminus k : \frac{|S \cap I \cap \ell|}{|I \cap \ell|} \in \left]\frac{1}{2} - \frac{1}{2j}, \frac{1}{2} + \frac{1}{2j}\right[\right\}\right).
\end{aligned}$$

As $|\mathcal{B}| \leqslant |\mathcal{I}| = \mathfrak{ß} < \mathfrak{b}$, there is a $g \in {}^{\omega}\omega$ such that $f \leqslant^* g$ for all $f \in \mathcal{B}$. We may suppose without loss of generality that $g(i)2(i+1) \leqslant g(i+1)$ for all natural numbers i. We define a conditionally convergent series

$$\begin{aligned}
&\sum_{k \to \infty} t_k\\
&\text{with } t_k = \begin{cases} \frac{(-1)^i}{g(i)(i+1)(2i+1)} \text{ for } k \in g(i)2(i+1) \setminus g(i),\\ 0 \text{ else.}\end{cases}
\end{aligned}$$

By assumption, there is an index set $I \in \mathcal{I}$ such that $\sum_{j \in I} t_j$ is diverging. Let $k \in \omega \setminus 30$ be such that $f_I(\ell) \leqslant g(\ell)$ for all $\ell \in \omega \setminus k$. Since

$$\begin{aligned}
\sum_{j \in I} t_j = \sum_{j \nearrow \infty} u_j \text{ with}\\
u_j := \sum_{\ell \in I \cap g(j)2(j+1) \setminus g(j)} t_\ell,
\end{aligned}$$

and $u_j < 0$ if and only if j is odd, the Leibniz criterion implies that there has to be a $j \in \omega \setminus k$ such that $|u_j| \leqslant |u_{j+1}|$. But then

$$\begin{aligned}
&\sum_{\ell \in I \cap g(j)2(j+1) \setminus g(j)} |t_\ell| = |u_j| \leqslant |u_{j+1}| = \sum_{\ell \in I \cap g(j+1)2(j+2) \setminus g(j+1)} |t_\ell|\\
\iff &\sum_{\ell \in I \cap g(j)2(j+1) \setminus g(j)} \frac{1}{g(j)(j+1)(2j+1)} = |u_j| \leqslant |u_{j+1}| = \sum_{\ell \in I \cap g(j+1)2(j+2) \setminus g(j+1)} \frac{1}{g(j+1)(j+2)(2j+3)}\\
\iff &\frac{|I \cap g(j)2(j+1) \setminus g(j)|}{g(j)(j+1)(2j+1)} = |u_j| \leqslant |u_{j+1}| = \frac{|I \cap g(j+1)2(j+2) \setminus g(j+1)|}{g(j+1)(j+2)(2j+3)}
\end{aligned}$$

We have $k \leqslant j$ and therefore $f_I(j) \leqslant g(j)$, so by definition of f_I, the absolute values of u_j and u_{j+1} can be estimated as follows:

$$\begin{aligned}
|u_j| &\geqslant \frac{(1/2 - 1/2j)\, g(j)2(j+1) - (1/2 + 1/2j)\, g(j)}{g(j)(j+1)(2j+1)} = \frac{(j-1)2(j+1) - (j+1)}{2j(j+1)(2j+1)} = \frac{2j^2 - j - 3}{j(4j^2 + 6j + 2)}\\
|u_{j+1}| &\leqslant \frac{(1/2 + 1/2j)\, g(j+1)2(j+2) - (1/2 - 1/2j)\, g(j+1)}{g(j+1)(j+2)(2j+3)} = \frac{2(j+1)(j+2) - (j-1)}{2j(j+2)(2j+3)} = \frac{2j^2 + 5j + 5}{j(4j^2 + 14j + 12)}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& j^2(4j^2+6j+2)(4j^2+14j+12)|u_j| \leqslant j^2(4j^2+6j+2)(4j^2+14j+12)|u_{j+1}| \\
\Longrightarrow & (2j^2-j-3)(4j^2+14j+12) \leqslant (2j^2+5j+5)(4j^2+6j+2) \\
\Longleftrightarrow & 8j^4+24j^3-2j^2-54j-36 \leqslant 8j^4+22j^3+54j^2+40j+10 \\
\Longleftrightarrow & 2j^3-56j^2-94j-46 \leqslant 0 \\
\Longleftrightarrow & 27j^3-756j^2-1269j-621 \leqslant 0 \\
\Longrightarrow & 27j^3-756j^2-1269j-1958 \leqslant 0 \\
\Longleftrightarrow & (3j-89)(9j^2+15j+22) \leqslant 0 \\
\Longleftrightarrow & 9j^2+15j+22 \leqslant 0 \\
\Longleftrightarrow & 36j^2+60j+88 \leqslant 0 \\
\Longleftrightarrow & (6j+5)^2 \leqslant -63, \text{ a contradiction.}
\end{aligned}
$$

□

2.2. **Various Inequalities.**

2.2.1. *Splitting of Partitions and the Uniformity of the Meagre Ideal.*

Theorem 5. $\mathfrak{s}(\mathfrak{pr}) \leqslant \mathrm{non}(\mathcal{M})$.

The following proof uses Bartoszyński's characterisation of $\mathrm{non}(\mathcal{M})$ via sequences of natural numbers equalling each other infinitely often. Such sequences are used in this proof to code sets of natural numbers thus guessing larger and larger parts of an infinite set of natural numbers thus enabling the definition of a splitting partition.

Proof. Assume towards a contradiction that $\mathrm{non}(\mathcal{M}) < \mathfrak{s}(\mathfrak{pr})$. Recall that $\mathrm{non}(\mathcal{M})$ is the minimal cardinality of an infinitely often equal family, cf. [8, Thm. 5.9]. So let $\mathcal{E}$ be an infinitely often equal family of size $\mathrm{non}(\mathcal{M})$. Let a bijection $e : \omega \longrightarrow [\omega]^{<\omega}$ encode all finite sets of natural numbers by a natural number. For every sequence $s \in {}^{\omega}\omega$ we inductively define an infinite partition P_s of ω. If $i < j$ or $|e(s(i))| \neq (2i)!(i+1)$ let $Q_s(j,i) := 0$, otherwise let $\langle f_s(i,j) \mid j < (2i)!(i+1)\rangle$ be the ascending enumeration of $e(s(i))$ and

$$Q_s(j,i) := \{f_s(i,m) \mid m \in (2i)!(j+1) \setminus (2i)!j\} \setminus \bigcup_{k,\ell<i} Q_s(\ell,k).$$

Note that $|Q_s(j,i)| \leqslant (2i)!$ for all natural numbers i and j. Having so defined pairwise disjoint $Q_s(j,i)$ for all $i,j < \omega$, let

$$P_s(j) := \bigcup_{i<\omega} Q_s(j,i) \text{ for all positive natural numbers } j \text{ and } P_s(0) := \omega \setminus \bigcup_{j\in\omega\setminus 1} P_s(j),$$

we have $P_s(0) \supset \bigcup\limits_{i<\omega} Q_s(0,i)$.

We now define a family of partitions $\mathcal{P} := \{P_s \mid s \in \mathcal{E}\}$. As $|\mathcal{P}| \leqslant |\mathcal{E}| = \mathrm{non}(\mathcal{M}) < \mathfrak{s}(\mathfrak{pr})$ there is an infinite set A of natural numbers which is not split by any member of $\mathcal{P}$. Let $\langle a_i \mid i < \omega \rangle$ be the inreasing enumeration of A. Now consider the sequence $s := \langle e^{-1}(\{a_j \mid j < (2i)!(i+1)\}) \mid i < \omega \rangle$. As $\mathcal{E}$ is an infinitely often equal family, there is a $t \in \mathcal{E}$ infinitely often equalling s. As P_t does not split A there is a natural number n such that $P_t(n) \cap A$ is finite. Let k be a natural number such that $A \cap P_t(n) \subset k$ and ℓ be one such that $B := \{i \in \ell \setminus (n+1) \mid s(i) = t(i)\}$ has more than k elements. We have

$$A \cap \bigcup_{i \in B} Q_s(n,i) = A \cap \bigcup_{i \in B} Q_t(n,i) \subset A \cap \bigcup_{i < \ell} Q_t(n,i) \subset A \cap \bigcup_{i < \omega} Q_t(n,i) = A \cap P_t(n) \subset k$$

so by the pigeonhole principle there has to be an $i \in B$ such that $Q_s(n,i) \subset \omega \setminus A$. Remember that $a_j = f_s(i,j)$ for all $j \leqslant i < \omega$. So we have

$$\{a_j \mid j \in (2i)!(n+1) \setminus (2i)!n\} \setminus \bigcup_{m,j<i} Q_s(m,j) = Q_s(n,i) \subset \omega \setminus A \text{ and hence}$$

$$\{a_j \mid j \in (2i)!(n+1) \setminus (2i)!n\} \subset \bigcup_{m,j<i} Q_s(m,j), \text{ implying } i \geqslant 1 \text{ and}$$

$$(2i)! \leqslant \sum_{j,m<i} |Q_s(m,j)| \leqslant i \sum_{j<i} (2j)! \leqslant i^2(2i-2)!.$$

$$\text{Therefore } (2i)(2i-1) \leqslant i^2,$$

$$\text{so } i(3i-2) \leqslant 0 \text{ contradicting both factors of this product being no smaller than 1.}$$

□

2.2.2. *Pair-Splitting.*

Theorem 6. $\mathfrak{e} \leqslant \mathfrak{s}_{pair}$.

The following proof uses an interval partition with intervals of sufficiently quickly growing lengths such that, regardless of the behaviour of a set S of integers on previous intervals, a predictor will still narrow down the possibilities of that set on the next interval sufficiently to point to two elements in the current interval which are either both in S or both outside of S.

Proof. Assume towards a contradiction that $\mathfrak{s}_{pair} < \mathfrak{e}$ and let $\mathcal{S}$ be a pair-splitting family of cardinality $\mathfrak{s}_{pair}$. We inductively define a function $f : \omega \longrightarrow \omega$ by $f(0) := 3$ and $f(i+1) := 2f(i)\left(1 + 2^{2^{f(i)}}\right)$ and use it to define an interval partition $\langle J_i \mid i < \omega \rangle$ by setting $J_i := f(i) \setminus \bigcup_{k<i} J_k$ for all natural numbers i. Note the all intervals in this partition have odd length. Let $\langle F_i \mid i < \omega \rangle$ be an enumeration of all finite sets of natural numbers. We define a family

$$\mathcal{E} := \{g_S \mid S \in \mathcal{S}\}, \text{ where}$$

$$g_S := \langle m_k^S \mid k < \omega \rangle \text{ for } S \in \mathcal{S}, \text{ and}$$

$$m_k^S \text{ is such that } F_{m_k^S} = S \cap J_k.$$

As $|\mathcal{E}| \leqslant |\mathcal{S}| = \mathfrak{s}_{pair} < \mathfrak{c}$, there is a predictor $\pi = (B, \langle \pi_i \,|\, i \in B\rangle)$ predicting every member of $\mathcal{E}$.

Claim. *For every natural number ℓ there is a pair $a_\ell = \{b_{2\ell}, b_{2\ell+1}\} \subset J_\ell$ (where $b_{2\ell} < b_{2\ell+1}$) such that for every $x \in \mathcal{E}$, we have $a_\ell \cap F_{\pi_k(x \restriction \ell)} \in \{0, a_\ell\}$.*

Proof of Claim. For $\ell = 0$ the claim holds true by the pigeonhole principle. In the following we show the claim to holds for $\ell = k+1$. For any natural number k there are $2^{f(k)}$ subsets of $f(k)$ and hence just as many initial segments $x \restriction k$ for $x \in \mathcal{E}$. Let $\langle t_i \,|\, i < 2^{f(k)}\rangle$ be an enumeration of these initial segments. Set $K_{2^{f(k)}} := J_{k+1}$ and for every $i \leqslant 2^{f(k)}$, let K_i be the bigger one of the two sets $K_{i+1} \cap F_{\pi_k(t_i)}$ and $K_{i+1} \setminus F_{\pi_k(t_i)}$. Inductively we have $|K_i| \geqslant 2^i + 1$ for every $i \leqslant 2^{f(k)}$. Now we let $b_{2k} := \min(K_0)$ and $b_{2k+1} := \max(K_0)$. Then $a_k := \{b_{2k}, b_{2k+1}\}$ provides what was demanded. *(End of Proof of Claim.)*

Now we let $A := \{a_\ell \,|\, \ell < \omega\}$. The set A is an unbounded set of pairs, therefore there is an $S \in \mathcal{S}$ pair-splitting it. Let m be a natural number such that $\pi_i(g_S \restriction i) = g_S(i)$ for all $i \in B \setminus m$ and let $n \in B \setminus m$ be such that $a_n \in A \setminus [g(m)]^2$ is split by S, i.e. exactly one of b_{2n}, b_{2n+1} is in S, in other words, $S \cap a_n \notin \{0, a_n\}$. Then,

$$\{0, a_n\} \ni a_n \cap F_{\pi_n(g_S \restriction n)} = a_n \cap F_{g_S(n)} = a_n \cap F_{m_n^S} = a_n \cap J_n \cap S = a_n \cap S \notin \{0, a_n\}, \text{ a contradiction.}$$

□

Theorem 7. $\mathfrak{s}_{pair} \leqslant \text{ß}$.

The following proof depends on the observation that for any unbounded set A of pairs of natural numbers one can define an alternating harmonic series padded with zeros(a technique employed in [6] and [13]) such that any index set rendering the respective subseries divergent will also pair-split A.

Proof. Assume towards a contradiction that $\text{ß} < \mathfrak{s}_{pair}$ and let $\mathcal{I}$ be an ß-sized family of infinite index sets of natural numbers such that for every conditionally convergent series $\sum\limits_{i \nearrow \infty} r_i$ there is an index set $I \in \mathcal{I}$ such that $\sum\limits_{i \in I} r_i$ is diverging. As $|\mathcal{I}| = \text{ß} < \mathfrak{s}_{pair}$, there is an unbounded collection A of pairs of natural numbers which is not pair-split by any member of the family $\mathcal{S}$. Let $\langle a_i \,|\, i < \omega\rangle$ be an enumeration of A and inductively define a $B \in [A]^\omega$ by setting

$$B_0 := 0;$$
$$B_{i+1} := \begin{cases} B \cup \{a_i\} \text{ if } \max(b) < \min(a_i) \text{ for all } b \in B_i, \\ B_{i+1} := B_i \text{ otherwise.} \end{cases}$$

and finally $B := \bigcup_{i<\omega} B_i$. Clearly, B is an infinite unbounded subcollection of A, moreover we have $a_1 < a_2$ or $a_3 < a_0$ for $\big\{\{a_0, a_1\}_<, \{a_2, a_3\}_<\big\} \in [B]^2$. Let $\langle b_i \,|\, i < \omega\rangle$ be the increasing enumeration of $\bigcup B$. Clearly, $B = \big\{\{b_{2i}, b_{2i+1}\} \,\big|\, i < \omega\big\}$. Let $n(i) := \min(\{j < \omega \,|\, b_j \geqslant i\})$ for all natural numbers i. We now consider the series

$$\sum_{i<\nearrow\infty} r_i,$$

$$\text{where } r_i := \begin{cases} \frac{(-1)^{n(i)}}{n(i)+1} & \text{if } i \in \bigcup A, \\ 0 & \text{else.} \end{cases}$$

This is just the alternating harmonic series padded with zeroes, and therefore it is conditionally convergent. Now let $I \in \mathcal{I}$ be such that $\sum_{i\in I} r_i$ diverges. Let k be a natural number such that $A \subset [k]^2 \cup [\omega \setminus k]^2$ and $a \subset I$ or $a \subset \omega \setminus I$ for all $a \in A \setminus [k]^2$. As $\sum_{i\in I} r_i$ diverges, so does $\sum_{i\in I\setminus k} r_i$. But $\sum_{i\in I\setminus k} r_i = \sum_{i\in I\cap\bigcup A\setminus k} r_i$ which, being divergent, by the Leibniz criterion cannot be an alternating series. Then $a \not\subset I$ or $a \not\subset \omega \setminus I$ for some $a \in A \setminus [k]^2$, a contradiction. $\square$

Since together with Theorem 6 we now established the inequality $\mathfrak{e} \leqslant \text{ß}$ between characteristics originating in, respectively, algebra and analysis, let us point out the following immediate Corollary.

Corollary. *For any cardinal $\kappa < 2^{\aleph_0}$ at least one of the the following holds:*

- *For all families $\mathcal{F} \in \left[[\omega]^\omega\right]^\kappa$ there is a conditionally convergent series $\sum_{i\nearrow\infty} r_i$ such that for all index sets $I \in \mathcal{F}$ the series $\sum_{i\in I} r_i$ converges.*
- *There is a subgroup of the Baer-Specker group no larger than κ exhibiting the Specker phenomenon.*

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