LEBESGUE'S DENSITY THEOREM AND DEFINABLE SELECTORS FOR IDEALS

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Abstract. We introduce a notion of density point and prove results analogous to Lebesgue's density theorem for various well-known ideals on Cantor space and Baire space. In fact, we isolate a class of ideals for which our results hold. As a contrasting result of independent interest, we show that there is no reasonably definable selector that chooses representatives for the equivalence relation on the Borel sets of having countable symmetric difference. In other words, there is no notion of density which makes the ideal of countable sets satisfy an analogue to the density theorem.

1. Introduction

1.1. **Lebesgue's Density Theorem.** We write $\mathrm{MEAS}(\mathfrak{X}, \mu)$ for the collection of measurable subsets of a measure space $(\mathfrak{X}, \mu)$ and $A =_\mu B$ to mean that two subsets A and B of $\mathfrak{X}$ are equal up to a null set, i.e., $\mu(A \Delta B) = 0$. Suppose $(\mathfrak{X}, d, \mu)$ is a metric measure space, by which we mean a metric space $(\mathfrak{X}, d)$ equipped with the completion μ of a Borel measure on $\mathfrak{X}$. For $A \in \mathrm{MEAS}(\mathfrak{X}, \mu)$,

$$d^\mu_A(x) = \liminf_{\epsilon \to 0} \frac{\mu(A \cap B_d(\epsilon, x))}{\mu(B_d(\epsilon, x))}$$

is called the density of x in A, and

$$D_\mu(A) = \{x \in \mathfrak{X} \mid d^\mu_A(x) = 1\}$$

is called the set of density 1 points (or just density points) of A.

Now take $\mathfrak{X}$ to be Cantor space ${}^\omega 2$, consisting of infinite binary sequences with the natural ultrametric $d(x, y) = \frac{1}{2^n}$, where n is least such that $x(n) \neq y(n)$, and let μ be the coin-tossing measure on ${}^\omega 2$, i.e., the resulting product measure where 0 and 1 are assigned weight $\frac{1}{2}$. This is also known as the *Bernoulli measure*, *uniform measure*, or *Lebesgue measure on* ${}^\omega 2$. The following result is known as Lebesgue's Density Theorem (for Cantor space).[1]

Theorem 1.1. *For any* $A, B \in \mathrm{MEAS}({}^\omega 2, \mu)$

(A) $A =_\mu B \Rightarrow D_\mu(A) = D_\mu(B)$, *and*
(B) $D_\mu(A) =_\mu A$.

For more on descriptive set theoretical aspects of Theorem 1.1 see [AC13, ACC15, AC18]; for a proof see [AC13] or [Mil08, Proposition 2.10] for ultrametric spaces. We give a different (finitized) proof just after Theorem 3.4 below.

When μ is a measure on a set $\mathfrak{X}$, the collection I_μ of its null sets is a typical example of a *σ-ideal on* $\mathfrak{X}$. By this we mean a collection I of subsets of $\mathfrak{X}$ which is closed under taking subsets and countable unions. We shall also assume that ideals are proper, i.e., that $\mathfrak{X} \notin \mathrm{I}$.

Theorem 1.1 is not merely a result about a particular measure on ${}^\omega 2$ but rather a statement about I_μ, its null-ideal: For it can be shown that $D_\mu(A) = D_{\mu'}(A)$ if $\mathrm{I}_\mu = \mathrm{I}_{\mu'}$ (for

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[1]This theorem is more usually stated for $\mathfrak{X} = \mathbb{R}^n$ and $\mu = \lambda^n$, the n-dimensional Lebesgue measure. We concentrate on Cantor space since it is a more familiar setting for tree ideals (see below).

any two Borel measures μ, μ' on ${}^{\omega}2$).[2] This motivates the main question which we answer in this paper. To state it, let for any Polish space $\mathfrak{X}$ (i.e., every completely metrizable separable space) $\textsc{Borel}(\mathfrak{X})$ denote the collection of its Borel sets (the smallest collection containing the open sets which is closed under countable unions and taking complements).

Question 1.2. *Let* I *be a* σ*-ideal on a Polish space* $\mathfrak{X}$. *Can one define a set of density points* $D_{\mathrm{I}}(A)$ *for each* $A \in \textsc{Borel}(X)$ *(or perhaps even for* A *from a larger collection) so that the analogue of Theorem 1.1 holds?*

Of course we want the assignment $A \mapsto D_{\mathrm{I}}(A)$ be to *definable* to rule out using the Axiom of Choice to select $D_{\mathrm{I}}(A)$. An instance where the answer to Question 1.2 is positive is given by the following fact:

Fact 1.3. Suppose I is the collection of null sets of the completion ν of a Borel probability measure on a Polish space (or even a standard Borel space) $\mathfrak{X}$. Then there is a function $D_{\mathrm{I}}\colon \textsc{Meas}(\mathfrak{X}, \nu) \to \textsc{Borel}(\mathfrak{X})$ so that the analogues of (A) and (B) in Theorem 1.1 hold.

The reason is that there is a Borel isomorphism ϕ of $({}^{\omega}2, \mu)$ with $(\mathfrak{X}, \nu)$ so that images of sets in I_{μ} exactly the sets in I_{ν} [Kec95, 17.39]. Thus the map D_{I} is moreover induced by a Borel function on codes of analytic sets (see [Kec73]).

Question 1.2 also has a positive answer for the *meager ideal*, i.e., the smallest σ-ideal containing all closed nowhere dense sets (closed sets with empty interior): In this case let $D_{\mathrm{I}}(A) = \bigcup\{O \mid O \text{ open}, A \cap O \text{ comeager in } O\}$ for any set A with the Baire property (i.e., any set which has meager symmetric difference with some Borel set).[3]

1.2. **Strongly linked tree ideals.** We answer Question 1.2 by utilizing the connection between σ-ideals and *forcing*.[4] For any ideal I on a Polish space $\mathfrak{X}$, $\textsc{Borel}(\mathfrak{X})\backslash\mathrm{I}$ is a preorder (i.e., a transitive binary relation, a.k.a. a *forcing* or *forcing notion*) under $\supseteq$. For all ideals of practical relevance, the structure of I can be recovered from a particularly nice dense subset[5] of this preorder, consisting of perfect sets (closed sets without isolated points). Conversely, to any preorder $\mathbb{P} = (\mathcal{P}, \supseteq)$ with $\mathcal{P} \subseteq \mathcal{P}(\mathfrak{X})$ there is a standard way to construct a σ-ideal $\mathrm{I}_{\mathbb{P}}$ and a collection of $\mathbb{P}$-measurable sets $\textsc{Meas}(\mathfrak{X}, \mathbb{P})$. We give a detailed review of this construction in Section 4.

We shall mostly restrict our attention to spaces of the form ${}^{\omega}X$, where $X = 2$ or $X = \omega$, with the product topology—i.e., Cantor space and Baire space.[6] Recall that $C \subseteq {}^{\omega}X$ is closed if and only if it is the branch set $[T]$ of a subtree T of ${}^{<\omega}X$ (this is arguably the main reason for working in these spaces).

A *tree forcing* is a preorder $(\mathbb{P}, \supseteq)$ where $\mathbb{P}$ is a collection of perfect subtrees of ${}^{<\omega}X$, corresponding to perfect subsets of ${}^{\omega}X$.[7] A *tree* σ*-ideal* is simply an ideal of the form $\mathrm{I}_{\mathbb{P}}$ where $(\mathbb{P}, \supseteq)$ is a tree forcing.

Tree σ-ideals are ubiquitous and relevant in many areas of mathematics, and to our knowledge every practically relevant ideal on a Polish space is of this form. Examples include: the null ideal of a Borel probability measure, the meager ideal on any non-meager Polish space, the K_{σ} sets, i.e., the ideal generated by the compact sets, on any Polish space which is not itself K_{σ}, or the nowhere Ramsey sets on $[\omega]^{\omega}$, whose associated notion of measurability, the Ramsey property, is of immense importance in areas as diverse as

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[2] On the other hand, Theorem 1.1 depends strongly on the metric. See [Mat95], [KRS16, Example 5.6], [ACC13], and [KRS16, Lemma 2.3].

[3] For a different definition of density point which generalizes the classical notion and at the same time is meaningful for the meager ideal, see [PWBW85] (we discuss this in Section 5.3).

[4] This connection has been explored also in [GRSS95, Zap08, Ike10, BKW18, SS18, KSZ13].

[5] For a given preorder $(\mathbb{P}, \leqslant_{\mathbb{P}})$, a set $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ if for any $p \in \mathbb{P}$ there is $d \in D$ with $d \leqslant_{\mathbb{P}} p$.

[6] Note that if I is a σ-ideal on an arbitrary Polish space $\mathfrak{X}$ then there is $D \in \mathrm{I}$ such that $\mathfrak{X}\backslash D$ is homeomorphic to ${}^{\omega}\omega$ so this is not a serious restriction.

[7] We omit for the sake of this introduction two minor technical requirements; see Section 4.1 for the precise definition.

combinatorics and functional analysis. (The associated forcings are of course Random, Cohen, Miller, and Mathias forcing, respectively.) For further examples see Section 6.

In Section 4.5 we shall show an analogue of Lebesgue's Density Theorem for the class of *strongly linked tree ideals* which we now define. Towards this, let us use the following notation: When T is a subtree of ${}^{<\omega}X$, write stem_T (called the *stem* of T) for the longest $s \in {}^{<\omega}X$ such that $[T] \subseteq N_s$. Recall here that the product topology on ${}^{\omega}X$ has the basic open sets $N_s = \{x \in {}^{\omega}X \mid s \sqsubseteq x\}$ for $s \in {}^{<\omega}X$, where $\sqsubseteq$ means initial segment.

Definition 1.4. A collection $\mathbb{P}$ of subtrees of ${}^{<\omega}X$ is called *strongly linked* if whenever $T, T' \in \mathbb{P}$ and stem_T and $\mathrm{stem}_{T'}$ are comparable under $\sqsubseteq$, then T and T' are compatible, i.e., there is $S \in \mathbb{P}$ with $S \subseteq T \cap T'$. A *strongly linked tree ideal* is an ideal of the form $\mathrm{I}_{\mathbb{P}}$ where $\mathbb{P}$ is a strongly linked tree forcing.

To state our first main result, we write $A =_{\mathrm{I}} B$ to mean that $A \Delta B \in \mathrm{I}$ for $A, B \subseteq \mathcal{P}({}^{\omega}X)$.

Theorem 1.5. *Let* I *be a strongly linked tree ideal, i.e., let* $\mathrm{I} = \mathrm{I}_{\mathbb{P}}$ *where* $\mathbb{P}$ *is a strongly linked tree forcing. For* $A \in \mathrm{MEAS}({}^{\omega}X, \mathbb{P})$ *define the set of* I-shift density points *(or short, just* I-density points*) of* A *as follows:*

$$D_{\mathrm{I}}(A) = \{x \in {}^{\omega}X \mid (\exists n \in \omega)(\forall m \geqslant n)(\forall T \in \mathbb{P})\ \mathrm{stem}_T = x{\upharpoonright}n \Rightarrow [T] \cap A \notin \mathrm{I}\}. \qquad (1)$$

Then the analogue of Theorem 1.1 holds, i.e., for any $A, B \in \mathrm{MEAS}({}^{\omega}X, \mathbb{P})$

(A) $A =_{\mathrm{I}} B \Rightarrow D_{\mathrm{I}}(A) = D_{\mathrm{I}}(B)$, *and*
(B) $D_{\mathrm{I}}(A) =_{\mathrm{I}} A$.

Intuitively, as we shall see, $x \in D_{\mathrm{I}}(A)$ means that for every sufficiently small neighborhood N_s of x, $A \cap N_s$ is "very large" in some sense. We show in Section 4.6 that D_{I} also is reasonable in terms of complexity: When $\mathbb{P}$ is analytic the map $D_{\mathrm{I}}\colon \mathrm{MEAS}({}^{\omega}X, \mathbb{P}) \to \mathrm{BOREL}$ is induced by a map from Borel codes to Borel codes which is absolutely Δ^1_2 and hence universally Baire measurable.

It follows immediately from Theorem 1.5 that D_{I} is a useful notion of density point for the ideals associated to Cohen forcing, Hechler forcing, eventually different forcing, Laver forcing with a filter, and Mathias forcing with a shift invariant filter (cf. Section 6). Although one can define D_{I} as in (1) for arbitrary tree ideals, we verify in Section 5.1 that Theorem 1.5 fails for $\mathbb{P}$ equal to Sacks, Miller, Mathias, Laver, or Silver forcing.

While the null ideal is *not* a strongly linked tree ideal (see Remark 4.16) our methods also yield a variant of density point for the null ideal. Namely, when $\mathrm{I} = \mathrm{I}_\mu$ is the ideal of null sets with respect to the Lebesgue measure μ on ${}^{\omega}2$ and $\mathbb{P}$ is Random forcing (see Definition 6.1(a) or Example 4.1) let

$$D_{\mathrm{I}}(A) = \left\{ x \in {}^{\omega}X \mid (\exists n \in \omega)(\forall m \geqslant n)(\forall T \in \mathbb{P}) \right.$$
$$\left. \left[\frac{\mu([T] \cap N_{x\upharpoonright n})}{\mu(N_{x\upharpoonright n})} > \frac{1}{2}\right] \Rightarrow [T] \cap A \notin \mathrm{I}_\mu \right\}. \qquad (2)$$

We show in Section 3 that $D_{\mathrm{I}}(A) =_\mu A$ for any $A \in \mathrm{MEAS}({}^{\omega}2, \mu)$.

It is also interesting and not hard to characterize the class of strongly linked tree ideals without any reference to forcing. For this, write I^+ for $\mathcal{P}({}^{\omega}X)\backslash\mathrm{I}$, i.e., for the co-ideal associated to an ideal I on ${}^{\omega}X$ and write $A \subseteq_{\mathrm{I}} B$ to mean $A \backslash B \in \mathrm{I}$.

Fact 1.6. A σ-ideal I on ${}^{\omega}X$ is a strongly linked tree ideal if and only if

1. There is a collection $\mathcal{C}$ of non-empty perfect subsets of ${}^{\omega}X$ such that for any $A \in \mathrm{BOREL}({}^{\omega}X)$, $A \in \mathrm{I}^+$ if and only if there is $C \in \mathcal{C}$ such that $C \subseteq_{\mathrm{I}} A$.
2. $\mathcal{C}$ is *shift-invariant* (or *strongly arboreal*) in the following sense: For all $C \in \mathcal{C}$ and all $s \in {}^{<\omega}X$ either $C \cap N_s = \varnothing$ or $C \cap N_s \in \mathcal{C}$.

(3) $\mathcal{C}$ is *strongly linked*: For any $C_0, C_1 \in \mathcal{C}$ either there is $C_2 \in \mathcal{C}$ with $C_2 \subseteq C_0 \cap C_1$ or there is $s \in {}^{\omega}X$ and $i \in \{0,1\}$ such that $C_i \subseteq N_s$ and $N_s \cap C_{1-i} = \varnothing$.

The reason is that $\mathbb{P} = (\mathcal{C}, \supseteq)$ is isomorphic to a tree forcing by (2), and using (1) and (3) one can show that $\mathrm{I} = \mathrm{I}_{\mathbb{P}}$ (this uses only that $\mathbb{P}$ has the ccc; see Section 4).

1.3. **No selector for the ideal of countable sets.** Finally, we show that Question 1.2 does not have a positive answer in general; in fact we show that for the simplest σ-ideal, it is not possible to define a reasonable notion of density point. First, given a tree ideal $\mathrm{I} = \mathrm{I}_{\mathbb{P}}$ consider the equivalence relation $=_{\mathrm{I}}$ on $\textsc{Meas}({}^{\omega}X, \mathbb{P})$, and note that (A) and (B) in Theorem 1.1 mean precisely that D_{I} gives one and the same representative on each equivalence class. Such a map is said to *lift a selector* for $=_{\mathrm{I}}$.

Let I be the ideal of countable sets and consider the equivalence relation on $\textsc{Borel}({}^{\omega}X)$ given by $A =_{\mathrm{I}} B$ (i.e., $A \Delta B$ is countable). We show the following result in Section 5.2:

Theorem 1.7. *There is no map $D\colon \textsc{Borel}({}^{\omega}X) \to \textsc{Borel}({}^{\omega}X)$ which lifts a selector for the equivalence relation $=_{\mathrm{I}}$ on $\textsc{Borel}({}^{\omega}X)$ such that D is induced by a Baire measurable map on the set of Borel codes.*

Thus under the Axiom of (Projective) Determinacy there is no (projective) such map. We also show in Theorem 5.9 that it is consistent with ZF that there is no definable map lifting a selector (if ZF is consistent), and that there is no such map in Solovay's model.

The aim of these results is to find dividing lines between tree ideals with and without a good notion of "density point". They show that *in all analyzed cases* of tree forcings $\mathbb{P}$ the following three conditions (a)-(c) are equivalent. Moreover for any $\mathbb{P}$ which is a strongly linked tree forcing or equal to Random or Sacks forcing, all four conditions (a)-(d) are equivalent.

(a) $\mathbb{P}$ is σ-linked.
(b) $\mathbb{P}$ satisfies the countable chain condition (ccc).
(c) With $D_{\mathrm{I}_{\mathbb{P}}}$ as in Definition 2.3 below, $D_{\mathrm{I}_{\mathbb{P}}}(A) =_{\mathrm{I}_{\mathbb{P}}} A$ for each Borel set $A \subseteq {}^{\omega}X$. In other words $\mathrm{I}_{\mathbb{P}}$ has the density property with respect to $D_{\mathrm{I}_{\mathbb{P}}}$ in the sense of Definition 2.1 below.
(d) There is a simply definable map which lifts to a selector for $=_{\mathrm{I}_{\mathbb{P}}}$.

Whether (d) holds for other well-known forcings is an important question left open in this paper (see Section 7)

1.4. **Structure of the paper.** We introduce density points for ideals in Section 2 and study them for the null ideal in Section 3. We further show in Section 3.1 that one can effectively construct density points of a closed set from weights on the basic open sets. We introduce tree ideals and some of their properties in Sections 4.1 to 4.3. For instance, we show that for any tree forcing $\mathbb{P}$ with the ω_1-covering property, all Borel sets are $\mathbb{P}$-measurable. This improves a result of Ikegami. We then prove the density property for strongly linked tree ideals in Sections 4.4 and 4.5. In Section 4.6, we determine a bound on the complexity of density operators for a large class of ccc forcings and show that they are universally Baire measurable. Section 5.1 contains counterexamples for the remaining tree forcings listed in Section 6. In Section 5.2, we prove that there is no simply definable selector for the ideal of countable sets. This is extended to a version for projective selectors from PD. Moreover, we show that it is consistent with ZF that there is no selector at all for the ideal of countable sets. Section 6 contains a list of the tree forcings we consider in this article. We end with some open questions in Section 7.

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2. Density points for tree ideals

In this section we introduce some notation and define our notion of density point for tree ideals in a way that allows us to treat *both*, D_{I} as defined in (1) for strongly linked tree ideals *and* D_{I_μ}, our variant of density point for the null ideal from (2) simultaneously.

Recall that we write $X = 2$ or $X = \omega$. We consider subtrees T of ${}^{<\omega}X$ and write

$$[T] = \{x \in {}^{\omega}X \mid (\forall n \in \omega)\ x{\upharpoonright}n \in T\}$$

for the set of branches through T. A tree T is *perfect* if it has no end nodes and some splitting node above each node. Let

$$T/s = \{t \in X^{<\omega} \mid s^\frown t \in T\}$$

$$s^\frown T = \{s^\frown t \mid t \in T\}$$

$$T_u = \{t \in T \mid u \subseteq t \vee t \subseteq u\}$$

for $s \in {}^{<\omega}X$ and $u \in T$. Let further

$$\sigma_s \colon {}^{\leqslant\omega}X \to {}^{\leqslant\omega}X,\ \sigma_s(x) = s^\frown x$$

denote the shift by $s \in {}^{<\omega}X$. Thus $\sigma_s^{-1}[T] = [T/s] = \{t \mid s^\frown t \in [T]\}$. For $s, t \in X^{\leqslant\omega}$, we write $s \sqsubseteq t$ if s is an initial segment of t. Recall that the *stem* stem_T of a tree T is the longest $s \in T$ such that $s \sqsubseteq t$ or $t \sqsubseteq s$ for all nodes $t \in T$. The set of *splitting nodes* of T (those with at least two direct successors in T) is denoted split_T. Moreover, let $s \wedge t$ denote the longest common initial segment of s and t.

If I is an ideal on ${}^{\omega}X$ and A and B are subsets of ${}^{\omega}X$, recall that we write $A =_{\mathrm{I}} B$ for $A \triangle B \in I$. We also write $A \subseteq_{\mathrm{I}} B$ for $A \setminus B \in \mathrm{I}$ and $A \perp_{\mathrm{I}} B$ for $A \cap B \in \mathrm{I}$. An ideal on ${}^{\omega}X$ is called *shift invariant* if it is closed under pointwise images and preimages of σ_t for all $t \in {}^{<\omega}X$. For an ideal I, a set is called I-*positive* if it is not in I; recall that the set of I-positive sets (the co-ideal of I) is denoted I^+.

The central notion for this article is the *density property*:

Definition 2.1. If I is an ideal and D is a map from the collection of Borel subsets of ${}^{\omega}X$ to $\mathcal{P}({}^{\omega}X)$, we say that the *density property* holds (with respect to D and I) if $D(A) =_{\mathrm{I}} A$ for all A.

Ideally, we would like to define a notion of density points relative to an arbitrary shift invariant ideal I on ${}^{\omega}X$. For our definition we find it necessary to fix a collection L_{I} of sets which we consider "large"; but for strongly linked ideals and for the null ideal there is a natural choice of L_{I}—namely when $\mathrm{I} = \mathrm{I}_\mu$, L_{I} is defined as the set of perfect sets of measure at least $\frac{1}{2}$ and when $\mathrm{I} = \mathrm{I}_{\mathbb{P}}$ and $\mathbb{P}$ is strongly linked, L_{I} is defined to be the set $\{[T] \in \mathbb{P} \mid \mathrm{stem}_T = \varnothing\}$.

Since we want to speak about arbitrary tree ideals in some of our results below, we make the following convention.

Convention 2.2. *Let* I *be a tree ideal, and fix* $\mathbb{P}$ *such that* $\mathrm{I} = \mathrm{I}_{\mathbb{P}}$*. If* $\mathrm{I} = \mathrm{I}_\mu$ *we shall assume that* $\mathbb{P}$ *is Random forcing, i.e. the collection of trees* $T \subseteq {}^{<\omega}2$ *such that for all* $s \in T$*,* $\mu([T] \cap N_s) > 0$*; further, we let*

$$\mathrm{L}_\mathrm{I} = \{[T] \mid T \in \mathbb{P}, \mu([T]) > \frac{1}{2}\}.$$

If $\mathbb{P}$ *is any other tree forcing then we let*

$$\mathrm{L}_\mathrm{I} = \{[T] \mid T \in \mathbb{P}, \mathrm{stem}_T = \varnothing\}.$$

We say that elements of L_I *are* large *with respect to* I.

Definition 2.3. Suppose that A is a subset of ${}^\omega X$.

(a) An element x of ${}^\omega X$ is an I-*shift density point of* A if there is some n_x such that for all $B \in \mathrm{L}_\mathrm{I}$ and $n \geqslant n_x$
$$\sigma_{x \upharpoonright n}(B) \cap A \notin \mathrm{I}.$$
(b) $D_\mathrm{I}(A)$ denotes the set of I-shift density points of A.

We further say that an ideal I satisfies the I-*shift density property* if $D_\mathrm{I}(A) \triangle A \in \mathrm{I}$ for all Borel sets A.

For simplicity, we sometimes just write I-density point and I-density property. It is clear that by Convention 2.2, Definition 2.3 just repeats the defintion of D_I given in (1) for strongly linked tree ideals as well as the one in (2) for the null ideal. Note that $D_\mathrm{I}(A)$ is $\mathbf{\Sigma}^0_2$ for any subset A of ${}^\omega X$. To see this, let

$$S = \{s \in {}^{<\omega}X \mid \forall B \in \mathrm{L}_\mathrm{I}\ \sigma_s(B) \cap A \notin \mathrm{I}\}$$

and observe that $x \in D_\mathrm{I}(A) \iff \exists m\ \forall n \geqslant m\ x \upharpoonright n \in S$; thus $D_\mathrm{I}(A)$ is $\Sigma^0_2(S)$.

Finally, note that in those cases where we verify the I-shift density property we obtain that $D_\mathrm{I}(A) \triangle A \in \mathrm{I}$ for all $A \in \textsc{Meas}(\mathbb{P}, {}^\omega X)$, not just for Borel A (since in all these cases $\mathbb{P}$ is ccc; cf. Remark 4.19).

Remark 2.4. Definition 2.3 can be rephrased in the following fashion. We call a subset $A^\star$ of ${}^\omega X$ I-*full* if for all $B \in \mathrm{L}_\mathrm{I}$, the set $B \cap A^\star$ is I-positive (this is analogous to the definition of stationary sets from club sets). Then x is a density point of A if and only if $A/(x \upharpoonright n) = \sigma^{-1}_{x \upharpoonright n}(A)$ is eventually I-full as n increases.

Remark 2.5. We notice that definition 2.3 can also be rephrased via the following notion of convergence. We say that a sequence $\vec{f} = \langle f_n \mid n \in \omega\rangle$ of functions $f_n \colon {}^\omega X \to \mathbb{R}$ *converges in* I to a function $f \colon {}^\omega X \to \mathbb{R}$ if the following condition holds:[8] For all $\epsilon > 0$, there is some n_0 such that for all $B \in \mathrm{L}_\mathrm{I}$ and $n \geqslant n_0$,

$$B \setminus \{x \in {}^\omega X \mid |f_n(x) - f(x)| \geqslant \epsilon\} \notin \mathrm{I}.$$

By shift invariance, the condition $\sigma_{x \upharpoonright n}(B) \cap A \notin \mathrm{I}$ in Definition 2.3 (a) is equivalent to $B \cap \sigma^{-1}_{x \upharpoonright n}(A) \notin \mathrm{I}$. Moreover, $B \cap \sigma^{-1}_{x \upharpoonright n}(A) = B \setminus \{x \in {}^\omega X \mid |1_{\sigma^{-1}_{x \upharpoonright n}(A)}(x) - 1| \geqslant \epsilon\}$ for any ϵ with $0 < \epsilon < 1$. Therefore x is an I-shift density point of A if and only if the functions $1_{\sigma^{-1}_{x \upharpoonright n}(A)}$ converge in I to the constant function with value 1.

3. The null ideal

In this section, we outline the situation in the special case of the σ-ideal on ${}^\omega 2$ of Lebesgue null sets to illustrate some ideas used in this paper.

Let μ denote the Lebesgue measure on ${}^\omega 2$, i.e., the measure generated by the assignment $\mu(N_t) = 2^{-n}$ for any $t \in {}^n 2$. Recall that we write I_μ for the σ-ideal of μ-null sets. The next lemma implies that the I_μ-shift density property holds.

Lemma 3.1. *Let* A *be a Borel subset of* ${}^\omega 2$.

[8]Compare this with convergence in measure in Lemma 5.10.

(1) *If* $d^{\mu}_{A}(x) = 1$ *and* $\epsilon > 0$, *then there is some* $n_{x,\epsilon}$ *such that for all* $n \geqslant n_{x,\epsilon}$ *and all Borel sets* B *with* $\mu(B) \geqslant \epsilon$, $\mu(\sigma_{x\upharpoonright n}(B) \cap A) > 0$.

(2) *If* $d^{\mu}_{A}(x) = 0$ *and* $\epsilon > 0$, *then there is a Borel set* B *with*

$$\exists^{\infty} n \ (\sigma_{x\upharpoonright n}(B) \cap A = \varnothing)$$

and $\mu(B) \geqslant 1 - \epsilon$.

Proof. For the first claim, note that there is some $n_{x,\epsilon}$ with $\frac{\mu(A\cap N_{x\upharpoonright n})}{\mu(N_{x\upharpoonright n})} > 1-\epsilon$ for all $n \geqslant n_{x,\epsilon}$, since

$$d^{\mu}_{A}(x) = \liminf_{n} \frac{\mu(A \cap N_{x\upharpoonright n})}{\mu(N_{x\upharpoonright n})} = 1.$$

If B is any I_{μ}-positive Borel set of size at least ϵ, then $\mu(\sigma_{x\upharpoonright n}(B) \cap A) > 0$ for all $n \geqslant n_{x,\epsilon}$.

For the second claim, let $\vec{\epsilon} = \langle \epsilon_i \mid i < \omega \rangle$ be a sequence in $\mathbb{R}^{+}$ with $\sum_i \epsilon_i \leqslant \epsilon$. Since

$$d^{\mu}_{A}(x) = \liminf_{n} \frac{\mu(A \cap N_{x\upharpoonright n})}{\mu(N_{x\upharpoonright n})} = 0,$$

there is a strictly increasing sequence $\vec{n} = \langle n_i \mid i \in \omega \rangle$ with $\frac{\mu(A\cap N_{x\upharpoonright n_i})}{\mu(N_{x\upharpoonright n_i})} < \epsilon_i$ for all $i \in \omega$. Let $B_i = \sigma^{-1}_{x\upharpoonright n_i}(A)$ for $i \in \omega$ and $B = \bigcup_{i\in\omega} B_i$. Then

$$\mu(B) \leqslant \sum_{i} \mu(B_i) \leqslant \sum_{i} \frac{\mu(A \cap N_{x\upharpoonright n_i})}{\mu(N_{x\upharpoonright n_i})} \leqslant \sum_{i} \epsilon_i \leqslant \epsilon.$$

Let C be an I_{μ}-positive set with $\mu(C) \geqslant 1-\epsilon$ that is disjoint from B. Then $C \cap B_i = \varnothing$ for all $i \in \omega$. Since $B_i = \sigma^{-1}_{x\upharpoonright n_i}(A)$, it follows that $\sigma_{x\upharpoonright n_i}(C) \cap A = \sigma_{x\upharpoonright n_i}(C) \cap \sigma_{x\upharpoonright n_i}(B_i) = \sigma_{x\upharpoonright n_i}(C \cap B_i) = \varnothing$. □

For $\epsilon = \frac{1}{2}$, we obtain that $d^{\mu}_{A}(x) = 1$ implies that x is an I_{μ}-shift density point and $d^{\mu}_{A}(x) = 0$ implies that this fails. This yields the following corollary (by Theorem 1.1).

Corollary 3.2. *For every measurable* $A \subseteq {}^{\omega}2$, $D_{\mathrm{I}}(A) =_{\mu} D_{\mu}(A)$. *In particular, the* I_{μ}*-shift density property holds.*

We shall give a proof of Lebesgue's Density Theorem below in Section 3.1 making our proof of the I_{μ}-shift density property completely self-contained (cf. also Section 4.5).

In the next lemma, we give two examples which show that if $d^{\mu}_{A}(x) \in (0,1)$, then x can but does not have to be an I_{μ}-shift density point of A.

Lemma 3.3. *Each of the following statements is satisfied by some Borel subset* A *of* ${}^{\omega}2$ *and some* $x \in {}^{\omega}2$ *with* $d^{\mu}_{A}(x) \in (0,1)$.

(a) x *is an* I_{μ}*-shift density point of* A.

(b) x *is not an* I_{μ}*-shift density point of* A.

Proof. Let $A = \{0^{n\frown}1^{3\frown}x \in {}^{\omega}2 \mid n \in \omega,\ x \in {}^{\omega}2\}$ and B its complement.

For (a) note that $d^{\mu}_{B}(0^{\omega}) \in (0,1)$ since $\frac{\mu(B\cap N_{0^n})}{\mu(N_{0^n})} = \frac{3}{4}$ for all $n \in \omega$. Thus $\mu(\sigma_{0^n}(C) \cap B) > 0$ for any Borel set C with $\mu(C) \geqslant \frac{1}{2}$. So 0^{ω} is an I_{μ}-shift density point of B.

For (b) we have $d^{\mu}_{A}(0^{\omega}) = 1 - d^{\mu}_{B}(0^{\omega}) \in (0,1)$. Since $\mu(B) \geqslant \frac{1}{2}$ and $\sigma_{0^n}(B) \cap A = \varnothing$ for all $n \in \omega$, 0^{ω} is not an I_{μ}-shift density point of A. □

3.1. **An explicit construction of density points.** In this section, we show how to explicitly construct density points of a closed set C of positive measure. In fact, there is an algorithm that takes as input a list of data from C and outputs a perfect tree (level by level) all of whose branches are density points. By choosing e.g. the leftmost branch, we can approximate a single density point with arbitrary precision.

The data consists of a tree T with weights $w_t = \frac{\mu([T]\cap N_t)}{\mu(N_t)} > 0$ for all $t \in T$; we call this a *weighted tree*. The input has the form of a Turing program with an oracle that can decide whether $t \in T$ and approximate w_t with any given precision $\epsilon \in \mathbb{Q} \cap (0,1)$.

Theorem 3.4. *There is an algorithm that computes from any weighted tree T and any $q \in \mathbb{Q} \cap (0,1)$ a subtree S of T such that $[S]$ consists of Lebesgue density points of $[T]$ and $\mu([S]) \geqslant (1-q)\mu([T])$.*

Note that for all strongly linked collections of trees $\mathbb{P}$ (see Definition 4.15) listed in Section 6 and all $T \in \mathbb{P}$, $S = T$ already satisfies the conditions in the previous theorem. For these forcings, any $T \in \mathbb{P}$ has the property that for all $x \in [T]$, there are infinitely many $n \in \omega$ such that there is some $S \leqslant T$ with $x \in [S]$ and $\mathrm{stem}_S = x{\upharpoonright}n$. Hence all elements of $[T]$ are density points of $[T]$ by the proof of Lemma 4.17 below.

Theorem 3.4 will follow from the next lemmas. To state them, we fix the following notation: Let $C = [T]$, $L_{t,i} = \mathrm{Lev}_{|t|+i}(T_t)$ be the level of T_t at height $|t| + i$ and write

$$w_{t,i} = \frac{|L_{t,i}|}{2^i} = \mu(N_t)^{-1} \frac{|L_{t,i}|}{2^{|t|+i}}$$

for all $t \in 2^{<\omega}$ and $i \in \omega$. This is the *relative size* of levels of T above t. The next result shows that these values converge to the relative measure at t.

Lemma 3.5. $\lim_{i\to\infty} w_{t,i} = w_t$ *for all* $t \in 2^{<\omega}$.

Proof. We have $w_t \leqslant \lim_{i\to\infty} w_{t,i}$, since $C \cap N_t \subseteq \bigcup_{u \in L_{t,i}} N_u$ and hence $w_t \leqslant w_{t,i}$ for all $i \in \omega$. To prove that $\lim_{i\to\infty} w_{t,i} \leqslant w_t$, suppose that $\epsilon > 0$ is given. Let U be an open set with $C \cap N_t \subseteq U$ and $\mu(U) < \mu(C \cap N_t) + \epsilon \cdot \mu(N_t)$. By compactness of C, we can assume that U is a finite union of basic open sets. We can thus write $U = \bigcup_{j \leqslant n} N_{s_j}$ for some $\vec{s} = \langle s_j \mid j \leqslant n \rangle$ that consists of pairwise incompatible sequences s_j of the same length $|t| + i$. Since $C \cap N_t \subseteq U$, we have

$$w_{t,i} = \mu(N_t)^{-1} \frac{|L_{t,i}|}{2^{|t|+i}} \leqslant \frac{\mu(\bigcup_{j \leqslant n} N_{s_j})}{\mu(N_t)} = \frac{\mu(U)}{\mu(N_t)}.$$

Hence $w_{t,i} - w_t \leqslant \frac{\mu(U) - \mu(C \cap N_t)}{\mu(N_t)} < \epsilon$ by the previous inequality and the definition of w_t. □

For any $t \in 2^{<\omega}$ and $i \in \omega$, let

$$r_{t,i} = \inf\{c \in (0,1) \mid \frac{|\{u \in L_{t,i} \mid w_u \geqslant c\}|}{|L_{t,i}|} \geqslant c\}$$

denote the ratio of nodes on level $|t| + i$ above t with large weight.

Lemma 3.6. $\liminf_{i\to\infty} r_{t,i} = 1$ *for all* $t \in 2^{<\omega}$.

Proof. Let $b = w_t$ and assume that $c \in (0,1)$ is given. Since $b > bc + b(1-c)c$, there is some $\epsilon > 0$ with $b > (b+\epsilon)c + (b+\epsilon)(1-c)c$. By Lemma 3.5, we can take $i \in \omega$ to be sufficiently large such that $w_{t,i} \leqslant b + \epsilon$. Moreover, let α denote the fraction of nodes $u \in L_{t,i}$ with weight $w_u \geqslant c$. Then

$$b = w_t = 2^{-i} \sum_{u \in L_{t,i}} w_u \leqslant w_{t,i}\alpha + w_{t,i}(1-\alpha)c.$$

We claim that $\alpha \geqslant c$. Otherwise $\alpha < c$ and $b \leqslant w_{t,i}\alpha + w_{t,i}(1-\alpha)c \leqslant w_{t,i}c + w_{t,i}(1-c)c$. Since $w_{t,i} \leqslant b + \epsilon$, we obtain $b \leqslant (b+\epsilon)c + (b+\epsilon)(1-c)c$, contradicting the definition of ϵ. □

We need the following notion to ensure that weights converge to 1 along branches of the tree constructed below. We say that $v \sqsupseteq t$ is (t,a)-*good* if $w_u \geqslant a$ for all u with $t \sqsubseteq u \sqsubseteq v$; otherwise it is called (t,a)-*bad*. Let

$$s_{t,a,i} = \frac{|\{u \in L_{t,i} \mid u \text{ is } (t,a)\text{-good}\}|}{|L_{t,i}|}$$

denote the fraction of (t,a)-good nodes on level $|t| + i$ above t.

We fix a computable function $f\colon \mathbb{Q} \cap (0,1) \to \mathbb{Q} \cap (0,1)$ such that $\frac{1-b}{b(b-a)} < \frac{1-a}{a}$ holds for all $a, b \in \mathbb{Q} \cap (0,1)$ with $b > f(a)$.

Lemma 3.7. *If $a \in \mathbb{Q} \cap (0,1)$ and $w_t = b > f(a)$, then* $\liminf_{i\to\infty} s_{t,a,i} \geqslant a$.

Proof. Since $b > f(a)$, there is some $c \in (0,1)$ with $1-b < \frac{1-a}{a}b(b-a)c$. Let i be sufficiently large such that the fraction of $v \in L_{t,i}$ with $w_v \geqslant c$ is at least b by Lemma 3.6. Then the fraction of nodes $v \in L_{t,i}$ with $w_v < c$ is at most $1-b$ and their number at most $2^i w_{t,i}(1-b)$.

Let A the set of (t,a)-bad nodes in $L_{t,i}$, $U = \bigcup_{v\in A} N_v$ and $\alpha = \frac{|A|}{|L_{t,i}|}$. We aim to show that $\alpha \leqslant 1-a$. The number of (t,a)-bad nodes in $L_{t,i}$ is $2^i w_{t,i}\alpha$. Since all of these except at most $2^i w_{t,i}(1-b)$ have weight at least c, we have

$$2^{|t|+i}\mu(C\cap U) = \sum_{v\in A} w_v \geqslant (2^i w_{t,i}\alpha - 2^i w_{t,i}(1-b))c.$$

Claim. $1-b \geqslant \frac{1-a}{a} w_{t,i}(\alpha-(1-b))c$.

Proof. For each $v \in A$, take some u_v with $t \sqsubseteq u_v \sqsubseteq v$ and $w_{u_v} < a$ by the definition of (t,a)-bad. In particular $\frac{1-w_{u_v}}{w_{u_v}} > \frac{1-a}{a}$.

Let $B = \{u_v \mid v \in A\}$ and $B^\star$ the set of $\sqsubseteq$-minimal elements of B. For each $v \in A$ and $u = u_v$

$$\mu(N_u\backslash C) = 2^{-|u|}(1-w_u) > 2^{-|u|}w_u\frac{1-a}{a} = \frac{1-a}{a}\mu(C\cap N_u).$$

Since the sets N_u for $u \in B^\star$ are pairwise disjoint, the previous inequality implies

$$\mu(U\backslash C) = \sum_{u\in B^\star}\mu(N_u\backslash C) > \frac{1-a}{a}\sum_{u\in B^\star}\mu(C\cap N_u) = \frac{1-a}{a}\mu(C\cap U).$$

By this inequality and the one before the claim, we have $\mu(N_t\backslash C) \geqslant \mu(U\backslash C) > \frac{1-a}{a}\mu(C\cap U) \geqslant 2^{-|t|}\frac{1-a}{a}(w_{t,i}\alpha - w_{t,i}(1-b))c$. Since $\frac{\mu(N_t\backslash C)}{2^{-|t|}} = \frac{\mu(N_t\backslash C)}{\mu(N_t)} = 1-w_t = 1-b$ and $\mu(N_t) = 2^{-|t|}$, the claim follows. ☐

It is sufficient to show that $\alpha \leqslant 1-a$. Otherwise by the previous claim and since $b = w_t \leqslant w_{t,i}$

$$\begin{aligned} 1-b &\geqslant \frac{1-a}{a}w_{t,i}(\alpha-(1-b))c \\ &> \frac{1-a}{a}w_{t,i}((1-a)-(1-b))c \\ &= \frac{1-a}{a}w_{t,i}(b-a)c \\ &\geqslant \frac{1-a}{a}b(b-a)c. \end{aligned}$$

But this contradicts the choice of c. ☐

Proof of Theorem 3.4. Let $\vec{a} = \langle a_i \mid i \in \omega\rangle$ be a computable sequence in $\mathbb{Q}\cap(0,1)$ with $\prod_{i\in\omega} a_i^2 > 1-q$. Using Lemmas 3.6 and 3.7, we will inductively construct a strictly increasing sequence $\vec{n} = \langle n_i \mid i \in \omega\rangle$ and sets $S_i \subseteq \mathrm{Lev}_{n_i}(T)$ with $n_0 = 0$ and $S_0 = \{\varnothing\}$ by induction on $i \in \omega$. The sets S_i are compatible levels of a tree in the sense that each $t \in S_i$ has an extension $u \in S_{i+1}$ and conversely, each $u \in S_{i+1}$ extends some $t \in S_i$. We further let $T^{(i)} = \{t \in T \mid \exists u \in S_i\ (t \sqsubseteq u \vee u \sqsubseteq t)\}$ denote the subtree of T induced by S_i.

We will maintain during the induction that (a) u is (t,a_i)-good for all $t \in S_{i+1}$ and $u \in S_{i+2}$ and (b) $\frac{\mu([T^{(i+1)}])}{\mu([T^{(i)}])} \geqslant a_i^2$.

Description. This describes the construction. We simultaneously construct auxiliary numbers $a_i', a_i'', b_i \in \mathbb{Q}$ and $j_i \in \omega$ with $a_i < a_i' < a_i'' < b_i < 1$ and $b_i > f(a_i'')$ for all $i \geqslant 1$. It is not hard to see that all steps are effective.

Let $n_0 = 0$ and $S_0 = \mathrm{Lev}_{n_0}(T) = \mathrm{Lev}_0(T)$.

For $i = 1$, we first choose some $a_1' \in \mathbb{Q}$ with $a_1 < a_1' < 1$ and $b_1 \in \mathbb{Q}\cap(0,1)$ with $b_1 > f(a_1')$. By Lemma 3.6 applied to $t = \varnothing$, there is some j_1 with $r_{\varnothing,j_1} > b_1$. Let $n_1 = j_1$

and S_1 a subset of $\mathrm{Lev}_{n_1}(T)$ with $w_u > b_1$ for all $u \in S_1$ and $\frac{|S_1|}{|\mathrm{Lev}_{n_1}(T)|} > b_1$. We can further take b_1 to be sufficiently large such that $\frac{\mu([T^{(1)}])}{\mu([T^{(0)}])} > a_0^2$ by $r_{\varnothing,j_1} > b_1$ and the definition of $r_{\varnothing,j_1}$.

Fix $i \geqslant 1$ and assume that step i is completed. First take some $a'_{i+1}, a''_{i+1} \in \mathbb{Q}$ with $a_{i+1} < a'_{i+1} < a''_{i+1} < 1$ and $1 - a'_{i+1} < a'_i - a_i$. Let $b_{i+1} \in \mathbb{Q} \cap (0,1)$ with $b_{i+1} > f(a''_{i+1}), a''_{i+1}, a_i$. By Lemma 3.6, there is some j_{i+1} such that $r_{t,j_{i+1}} > b_{i+1}$ for all $t \in S_i$. Since $b_i > f(a''_i)$ by the inductive hypothesis and $a'_i < a''_i$, we can take j_{i+1} to satisfy $s_{t,a''_i,j_{i+1}} > a'_i$ for all $t \in S_i$ by Lemma 3.7. Let $n_{i+1} = n_i + j_{i+1}$. By definition of j_{i+1}, there is a subset S_{i+1} of $\mathrm{Lev}_{n_{i+1}}(T)$ such that for all $u \in S_{i+1}$, we have $w_u > b_{i+1}$, there is a (unique) $t \in S_i$ with $t \sqsubseteq u$, u is (t, a'_i)-good and $\frac{|L_{t,j_{i+1}} \setminus S_{i+1}|}{|L_{t,j_{i+1}}|} < (1 - a'_i) + (1 - b_{i+1})$.

Verification. We show that the algorithm computes the required tree. Clearly condition (a) is maintained in the construction. The next claim shows (b).

Claim. $\frac{\mu([T^{(i+1)}])}{\mu([T^{(i)}])} \geqslant a_i^2$ for all $i \in \omega$.

Proof. This is clear for $i = 0$. Let $i \geqslant 1$ and fix any $t \in S_i$. Since $a'_{i+1} < b_{i+1}$ and by the definition of S_{i+1}, we have $\frac{|L_{t,j_{i+1}} \setminus S_{i+1}|}{|L_{t,j_{i+1}}|} \leqslant (1 - a'_i) + (1 - b_{i+1}) \leqslant (1 - a'_i) + (1 - a'_{i+1}) \leqslant 1 - a_i$. Hence $\frac{|L_{t,j_{i+1}} \cap S_{i+1}|}{|L_{t,j_{i+1}}|} \geqslant a_i$.

Since each $u \in S_{i+1}$ has weight at least b_{i+1}, we have $c := \frac{\mu([T^{(i+1)}] \cap N_t)}{\mu([T^{(i)}] \cap N_t)} \geqslant c' := a_i b_{i+1}$. By the definition of $T^{(i+1)}$ from S_{i+1}, $d := \frac{\mu(([T^{(i)}] \cap N_t) \setminus [T^{(i+1)}])}{\mu([T^{(i)}] \cap N_t)} \leqslant d' := 1 - a_i$. Moreover $c + d = 1$. Since $c \geqslant c'$, $\frac{c+d'}{c} = 1 + \frac{d'}{c} \leqslant 1 + \frac{d'}{c'} = \frac{c'+d'}{c'}$. Since $d \leqslant d'$ and by the last inequality $\frac{c}{c+d} \geqslant \frac{c}{c+d'} \geqslant \frac{c'}{c'+d'}$. Therefore $c = \frac{c}{c+d} \geqslant \frac{c'}{c'+d'} = \frac{a_i b_{i+1}}{a_i b_{i+1} + (1 - a_i)} \geqslant a_i b_{i+1} \geqslant a_i^2$. Since this inequality holds for all $t \in S_i$, we have $\frac{\mu([T^{(i+1)}])}{\mu([T^{(i)}])} \geqslant a_i^2$. □

To see that conditions (a) and (b) are sufficient, let S be the unique perfect subtree of ${}^{<\omega}2$ with $\mathrm{Lev}_{n_i}(S) = S_i$ for all $i \in \omega$. This tree can be computed level by level via the algorithm above. We have $\lim_{i\to\infty} a_i = 1$ by the definition of $\vec{a}$. Thus (a) implies that all elements of $[S]$ are density points of $[T]$. Moreover, $\mu([S]) = \inf_{i\in\omega} \mu([T^{(i)}]) \geqslant \prod_{i\in\omega} a_i^2 \mu([T]) < (1 - q)\mu([T])$ by (b) as required. □

The previous result also provides a different (finitized) proof of Lebesgue's density theorem for the uniform measure on Cantor space, since any Borel set can be approximated in measure by closed subsets. To see this, note that trivially $D(A) \cap D({}^{\omega}X \setminus A) = \varnothing$ for any subset A of ${}^{\omega}X$. Thus it is sufficient to show that for any Borel set A and any $\epsilon > 0$, there is a closed subset C of A with $\mu(A \setminus C) < \epsilon$ consisting of density points of A; the density property for A follows by applying this property to both A and its complement. To see that this property holds, take a closed subset B of A with $\mu(A \setminus B) < \frac{\epsilon}{2}$. By Theorem 3.4, there is a closed subset C of B with $\mu(B \setminus C) < \frac{\epsilon}{2}$ that consists of density points of B and therefore also of A. Since $\mu(A \setminus C) < \epsilon$, C is as required.

Note that the algorithm also produces lower bounds for weights along branches of S.

4. Tree ideals

In this section, we study ideals induced by collections of trees. We introduce the class of strongly linked tree ideals which includes various well-known ones and show that the density property holds for this class.

4.1. **What is a tree ideal?** A tree ideal on ${}^{\omega}X$ is induced by a collection $\mathbb{P}$ of perfect subtrees of ${}^{<\omega}X$ that contains ${}^{<\omega}X$ and T_s for all $T \in \mathbb{P}$ and $s \in T$ (we say that $\mathbb{P}$ is *shift invariant*, or following [Ike10], *strongly arboreal*). We will always assume this condition for any collection of trees.

Any such collection of trees carries the partial order $S \leqslant T :\Longleftrightarrow S \subseteq T \Longleftrightarrow [S] \subseteq [T]$. Such partial orders are also called *tree forcings*;[9] some well-known examples are listed in Section 6. For instance, the null ideal is induced by the collection of Random trees, given as follows:

Example 4.1. A subtree T of ${}^{<\omega}2$ is *Random* if $\mu([T_s]) > 0$ for all $s \in T$ with $\mathrm{stem}_T \sqsubseteq s$.

We next associate an ideal to any collection of trees as in [Ike10, Definition 2.6]. The underlying idea is based on the special case that the sets $[T]$ for $T \in \mathbb{P}$ form a base for a topology. In this case, $\mathbb{P}$ is called *topological* and its topology is denoted $\tau_{\mathbb{P}}$. For instance, the collection of Hechler trees (see Definition 6.1 (e)) is topological. In fact, all tree forcings of the kind studied in Section 4.5 below have this property.[10]

Usually, one defines nowhere dense sets relative to a given topology, or equivalently, to a base of that topology. Moreover, meager sets are defined as countable unions of these sets. In the next definition, these notions are generalized by replacing a base by an arbitrary collection of trees.

Definition 4.2. [Ike10, Definition 2.6] Let $\mathbb{P}$ be a collection of trees.

(a) A set A is $\mathbb{P}$-*nowhere dense* if for all $T \in \mathbb{P}$ there is some $S \leqslant T$ with $[S] \cap A = \varnothing$. Moreover, $\mathrm{N}_{\mathbb{P}}$ is the ideal of $\mathbb{P}$-nowhere dense sets.
(b) $\mathrm{I}_{\mathbb{P}}$ is the σ-ideal of $\mathbb{P}$-*meager sets* generated by $\mathrm{N}_{\mathbb{P}}$.

Tree ideals are those of the form $\mathrm{I}_{\mathbb{P}}$ for a collection $\mathbb{P}$ of trees.[11] This presentation allows for uniform proofs of results for various ideals. Many well-known ideals are of this form; for instance, for Cohen forcing[12] $\tau_{\mathbb{P}}$ is the standard topology, so $\mathrm{N}_{\mathbb{P}}$ is the collection of nowhere dense sets and $\mathrm{I}_{\mathbb{P}}$ that of meager sets. For Random forcing, $\mathrm{N}_{\mathbb{P}}$ and $\mathrm{I}_{\mathbb{P}}$ equal the σ-ideal of null sets. Sacks forcing is the collection of all perfect trees; here both ideals equal the *Marczewski ideal* (see [Szp35, 3.1]). Its restriction to Borel sets equals the ideal of countable sets by the perfect set property for Borel sets. For Mathias forcing, $\tau_{\mathbb{P}}$ is the Ellentuck topology, and $\mathrm{N}_{\mathbb{P}}$ and $\mathrm{I}_{\mathbb{P}}$ are equal to the ideal of nowhere Ramsey sets (see [BKW18]). The ideal associated to Silver forcing consists of the completely doughnut null sets (see [Hal03]).

4.2. **Measurability for tree ideals.** Let $\mathbb{P}$ be a collection of trees. The next definition introduces a form of indivisibility[13] of $\mathbb{P}$ with respect to A: If $T \in \mathbb{P}$ and $[T]$ is split into two pieces by forming intersections with A and its complement, then at least one of these pieces contains a set of the form $[S]$ for some $S \in \mathbb{P}$, up to some $\mathbb{P}$-meager set.

Definition 4.3. [Ike10, Definition 2.8][14] A subset A of ${}^{\omega}X$ is called $\mathbb{P}$-*measurable* if for every $T \in \mathbb{P}$, there is some $S \leqslant T$ with at least one of the properties (a) $[S] \subseteq_{\mathrm{I}_{\mathbb{P}}} A$ and (b) $[S] \perp_{\mathrm{I}_{\mathbb{P}}} A$.

Note that the properties (a) and (b) are mutually exclusive (see Lemma 4.10 below).

The main motivation for introducing this notion is the fact that it formalizes various well-known properties. For instance, we will see in Lemma 4.8 that $\mathbb{P}$-measurability for Random forcing means Lebesgue measurability. For Sacks forcing it is equivalent to the Bernstein property for collections of sets closed under continuous preimages and intersections with closed sets [BL99, Lemma 2.1]. Moreover, for Mathias forcing it is equivalent to being completely Ramsey.

[9] These forcings, but without the condition that ${}^{\omega}X$ is in $\mathbb{P}$, are called *strongly arboreal* in [Ike10, Definition 2.4].

[10] Topological does not imply the density property. For instance, assuming CH one can construct a dense topological (shift-invariant) subforcing of Sacks forcing, while we show in Propostion 5.1 and Theorem 5.5 that the density property fails for the ideal associated to Sacks forcing.

[11] In [BL99, Section 2] and various other papers, tree ideals mean the ideals $\mathrm{N}_{\mathbb{P}}$ instead of $\mathrm{I}_{\mathbb{P}}$.

[12] See Section 6 for this and the following forcings.

[13] See e.g. [LNVTPS09] for the concept of indivisibility in combinatorics.

[14] This is a modification of a definiton in [BHL05, Section 0].

Our next goal is to show that for a very large class of forcings, all Borel sets are $\mathbb{P}$-measurable. This will be important in the proofs of the following sections.

[Ike10, Lemma 3.5] shows that for proper tree forcings $\mathbb{P}$, all Borel sets are $\mathbb{P}$-measurable.[15] We will show a slightly more general version of this result. To state this, recall that a forcing $\mathbb{P}$ has the *ω_1-covering property* if for any $\mathbb{P}$-generic filter G over V, any countable set X of ordinals in $V[G]$ is covered by a set in $Y \in V$ that is countable in V. For instance, this statement holds for all proper and thus for all Axiom A forcings.[16] In particular, it holds for all forcings considered in this paper.

We will need the following characterization of the ω_1-covering property. Here we will write $D^{\parallel p} = \{q \in D \mid p \parallel q\}$ if $D \subseteq \mathbb{P}$ and $p \in \mathbb{P}$, where $p \parallel q$ denotes that there is an $r \in \mathbb{P}$ such that $r \leqslant p$ and $r \leqslant q$.

Lemma 4.4. *The following conditions are equivalent for any forcing $\mathbb{P}$:*

(a) *$\mathbb{P}$ has the ω_1-covering property.*
(b) *For any condition $p \in \mathbb{P}$ and any sequence $\vec{D} = \langle D_n \mid n \in \omega\rangle$ of antichains in $\mathbb{P}$, there is some $q \leqslant p$ such that for any $n \in \omega$, the set $D_n^{\parallel q}$ is countable.*

Proof. We first assume (a). Let $p \in \mathbb{P}$ and let $\vec{D} = \langle D_n \mid n \in \omega\rangle$ be as in (b). Take a $\mathbb{P}$-generic filter G over V. Moreover, let $f(n)$ be an element of $D_n \cap G$ for each $n \in \omega$. By the ω_1-covering property, there is a countable subset $C \in V$ of $\mathbb{P}$ such that $f(n) \in C$ for all $n \in \omega$. Let $\dot{f}$ be a name for f such that $q \leqslant p$ forces $\dot{f}(n) \in \check{C}$ for all $n \in \omega$. It follows that $D_n^{\parallel q} \subseteq C$, since for any $\mathbb{P}$-generic filter H over V that contains both q and r we have $r = \dot{f}^H(n) \in C$.

For the converse implication, assume (b) and suppose that G is $\mathbb{P}$-generic over V and C is a countable set of ordinals in $V[G]$. Moreover, let f be an enumeration of C and $\dot{f}$ a name with $\dot{f}^G = f$. Then there is a condition $p \in G$ which forces that $\dot{f}\colon \omega \to \mathrm{Ord}$ is a function. For each $n \in \omega$, let D_n be a maximal antichain of conditions deciding $\dot{f}(n)$. By our assumption, there are densely many conditions $q \leqslant p$ as in (b). Hence there is some $q \in G$ as in (b). Since $D_n^{\parallel q}$ is countable for all $n \in \omega$, $C_n = \{\alpha \mid r \Vdash \dot{f}(n) = \alpha \text{ for some } r \leqslant q, q' \text{ for a } q' \in D_n^{\parallel q}\}$ is countable and hence $C = \bigcup_{n\in\omega} C_n$ is a countable cover of $\mathrm{ran}(f)$. □

To show that all Borel sets are $\mathbb{P}$-measurable if $\mathbb{P}$ has the ω_1-covering property, we need the next two easy lemmas. We will use the following notation. If A is a subset of ${}^{\omega}X$, let $\mathbb{P}_A = \{T \in \mathbb{P} \mid [T] \subseteq_{\mathrm{I}_{\mathbb{P}}} A\}$. We further say that a subset of $\mathbb{P}$ is *A-good* if it is contained in $\mathbb{P}_{(A)} = \mathbb{P}_A \cup \mathbb{P}_{{}^{\omega}X\setminus A}$.

Lemma 4.5. *A subset A of ${}^{\omega}X$ is $\mathbb{P}$-measurable if and only if there is an A-good maximal antichain in $\mathbb{P}$.*

Proof. If A is $\mathbb{P}$-measurable, then $\mathbb{P}_{(A)}$ is a dense subset of $\mathbb{P}$. Then there is a maximal antichain in $\mathbb{P}$ contained in $\mathbb{P}_{(A)}$ and hence an A-good maximal antichain. Conversely, if D is a maximal A-good antichain in $\mathbb{P}$ and $S \in \mathbb{P}$, let $T \in D^{\parallel S}$ and $R \leqslant S, T$. Then $[R] \subseteq_{\mathrm{I}_{\mathbb{P}}} A$ or $[R] \subseteq_{\mathrm{I}_{\mathbb{P}}} {}^{\omega}X\setminus A$. □

If $D \subseteq \mathbb{P}$, write $\bigcup^{\square} D = \bigcup_{T\in D}[T]$.

Lemma 4.6. *If D is a maximal antichain in $\mathbb{P}$ and $T \in \mathbb{P}$, then $[T]\setminus\bigcup^{\square} D^{\parallel T} \in \mathrm{N}_{\mathbb{P}}$.*

Proof. Let $S \in \mathbb{P}$, $R \in D^{\parallel S}$ and $Q \leqslant R, S$. If $Q \perp T$, then there is some $P \leqslant Q$ with $[P] \cap [T] = \varnothing$ by the closure property of $\mathbb{P}$ defined in the beginning of Section 4.1. If $Q \parallel T$, let $P \leqslant Q, T$. Since $P \in D^{\parallel T}$, $[P]$ is disjoint from $[T]\setminus\bigcup^{\square} D^{\parallel T}$, as required. □

The next result shows that Borel sets are $\mathbb{P}$-measurable in all relevant cases.

[15]The proof of this and several other results in [Ike10] can also be done from the weaker assumption that $\mathbb{P}$ has the ω_1-covering property. We give a more direct proof.

[16]See [Jec03, Definition 31.10].

Lemma 4.7. *If $\mathbb{P}$ has the ω_1-covering property, then the $\mathbb{P}$-measurable sets form a σ-algebra. In particular, all Borel sets are $\mathbb{P}$-measurable.*

Proof. For the first claim, it suffices to show that the class of $\mathbb{P}$-measurable sets is closed under forming countable unions. To this end, let $\vec{A} = \langle A_n \mid n \in \omega\rangle$ be a sequence of $\mathbb{P}$-measurable subsets of ${}^\omega X$. Furthermore, let D_n be an A_n-good maximal antichain for each $n \in \omega$ by Lemma 4.5. We will show that $A = \bigcup_{n\in\omega} A_n$ is $\mathbb{P}$-measurable.

Fix any $T \in \mathbb{P}$. Since $\mathbb{P}$ has the ω_1-covering property, there is some $S \leqslant T$ such that the sets $E_n = D_n^{\parallel S}$ are countable for all $n \in \omega$ by Lemma 4.4. First assume that for some $n \in \omega$, there is a tree $R \in E_n$ with $[R] \subseteq_{\mathrm{I}_\mathbb{P}} A_n$. Then there is some $Q \leqslant S$ with $[Q] \subseteq_{\mathrm{I}_\mathbb{P}} A_n \subseteq A$ as required. So we can assume that the previous assumption fails. We claim that then $[S] \perp_{\mathrm{I}_\mathbb{P}} A$. It suffices to show that $[S] \cap A_n \in \mathrm{I}_\mathbb{P}$ for each $n \in \omega$, since $\mathrm{I}_\mathbb{P}$ is a σ-ideal. To see this, note that $[R] \cap A_n \in \mathrm{I}_\mathbb{P}$ for all $R \in E_n$ by our case assumption. Hence $\bigcup^{\square} E_n \cap A_n \in \mathrm{I}_\mathbb{P}$. Moreover, $[S]\setminus \bigcup^{\square} E_n \in \mathrm{N}_\mathbb{P}$ by Lemma 4.6 and therefore $[S] \cap A_n \in \mathrm{I}_\mathbb{P}$.

Since it easy to see that closed sets are $\mathbb{P}$-measurable, it follows that all Borel sets are $\mathbb{P}$-measurable. □

Next is the observation that $\mathbb{P}$-measurability for Random forcing means Lebesgue measurability. We recall the argument from [Ike10] for the benefit of the reader.

Lemma 4.8. [Ike10, Proposition 2.9] *If $\mathbb{P}$ is a ccc tree forcing and A is any subset of ${}^\omega X$, then the following conditions are equivalent.*

(a) *A is $\mathbb{P}$-measurable.*
(b) *There is a Borel set B with $A \triangle B \in \mathrm{I}_\mathbb{P}$.*

Proof. If A is $\mathbb{P}$-measurable, then $\mathbb{P}_{(A)}$ is dense in $\mathbb{P}$. Let $D \subseteq \mathbb{P}_{(A)}$ be a maximal antichain in $\mathbb{P}$. Since D is countable, the sets $B_0 = \bigcup^{\square}(D \cap \mathbb{P}_A) \subseteq_{\mathrm{I}_\mathbb{P}} A$ and $B_1 = \bigcup^{\square}(D \setminus \mathbb{P}_A) \subseteq_{\mathrm{I}_\mathbb{P}} {}^\omega X \setminus A$ are Borel. Since D is maximal, ${}^\omega X \setminus \bigcup^{\square} D \in \mathrm{N}_\mathbb{P}$ by Lemma 4.6. Thus $A \triangle B_0 \in \mathrm{I}_\mathbb{P}$.

The reverse implication follows from the fact that all Borel sets are $\mathbb{P}$-measurable by Lemma 4.7. □

4.3. **Positive Borel sets.** The following characterization of positive Borel sets via trees will be important below. It uses an auxiliary ideal from [Ike10].

Definition 4.9. [Ike10, Definitions 2.11] $A \in \mathrm{I}_\mathbb{P}^\star$ if for all $T \in \mathbb{P}$, there is some $S \leqslant T$ with $[S] \cap A \in \mathrm{I}_\mathbb{P}$.

It is clear that $\mathrm{I}_\mathbb{P} \subseteq \mathrm{I}_\mathbb{P}^\star$, but it is open whether equality holds for all proper tree forcings.[17] Note that equality holds for ccc forcings. To see this, assume that $A \in \mathrm{I}_\mathbb{P}^\star$. Then $\mathbb{P}_{{}^\omega X \setminus A}$ (as defined before Lemma 4.5) is dense in $\mathbb{P}$ and therefore contains a (countable) maximal antichain D. We have $A \cap \bigcup^{\square} D \in \mathrm{I}_\mathbb{P}$. Since ${}^\omega X \setminus \bigcup^{\square} D \in \mathrm{N}_\mathbb{P}$ by Lemma 4.6, we have $A \in \mathrm{I}_\mathbb{P}$. Moreover, equality holds for fusion forcings (then in fact $\mathrm{N}_\mathbb{P} = \mathrm{I}_\mathbb{P}$) [18] and for topological tree forcings as in the proof of [FKK16, Lemma 3.8].

The next lemma characterizes $\mathrm{I}_\mathbb{P}^\star$-positive sets.

Lemma 4.10. *Suppose that $\mathbb{P}$ is a tree forcing.*

(1) *For any $T \in \mathbb{P}$, $[T] \notin \mathrm{I}_\mathbb{P}^\star$.*
(2) *A $\mathbb{P}$-measurable subset A of ${}^\omega X$ is $\mathrm{I}_\mathbb{P}^\star$-positive if and only if there is some $T \in \mathbb{P}$ with $[T] \subseteq_{\mathrm{I}_\mathbb{P}^\star} A$.*

[17] To our knowledge, every known proper tree forcing satisfies either the ccc or fusion and thus equality holds.

[18] For a tree forcing $\mathbb{P}$, we define *fusion* as the existence of a sequence $\vec{\leqslant} = \langle \leqslant_n \mid n \in \omega\rangle$ of partial orders on $\mathbb{P}$ with $\leqslant_0 = \leqslant$ which satisfy the following conditions:

(a) (*decreasing*) If $S \leqslant_n T$ and $m \leqslant n$, then $S \leqslant_m T$.
(b) (*limit*) If $\vec{T} = \langle T_n \mid n \in \omega\rangle$ is a sequence in $\mathbb{P}$ with $T_{n+1} \leqslant_n T_n$ for all $n \in \omega$, then there is some $S \in \mathbb{P}$ with $S \leqslant_n T_n$ for all $n \in \omega$.
(c) (*covering*) If $T \in \mathbb{P}$, $n \in \omega$ and D is dense below T, then there is some $S \leqslant_n T$ with $[S] \subseteq \bigcup^{\square} D$.

Proof. We show the first claim. If $[T] \in \mathrm{I}^\star_{\mathbb{P}}$, then $[S] \in \mathrm{I}_{\mathbb{P}}$ for some $S \leqslant T$. Let $\vec{A} = \langle A_n \mid n \in \omega \rangle$ be a sequence of sets in $\mathrm{N}_{\mathbb{P}}$ with $[S] \subseteq \bigcup_{n\in\omega} A_n$. We can then recursively construct a sequence $\vec{S} = \langle S_n \mid n \in \omega \rangle$ in $\mathbb{P}$ such that $S_0 = S$, $S_{n+1} \subseteq S_n$, $[S_n] \cap A_n = \varnothing$ for all $n \in \omega$ and the sequence $\vec{s} = \langle \mathrm{stem}_{S_n} \mid n \in \omega \rangle$ of stems is strictly increasing. Then $x = \bigcup_{n\in\omega} \mathrm{stem}_{S_n} \in \bigcap_{n\in\omega} [S_n]$. Since $[S_n] \cap A_n = \varnothing$ for all $n \in \omega$ and $[S] \subseteq \bigcup_{n\in\omega} A_n$, we have $x \notin [S]$. But this contradicts the fact that $x \in [S]$.

We now show the second claim. By the first part, it is sufficient to take any $\mathbb{P}$-measurable $\mathrm{I}^\star_{\mathbb{P}}$-positive set A and find some $S \in \mathbb{P}$ with $[S] \subseteq_{\mathrm{I}^\star_{\mathbb{P}}} A$. Assume that there is no such tree. Since A is $\mathbb{P}$-measurable, we have that for any $T \in \mathbb{P}$ there is some $S \leqslant T$ with $[S] \perp_{\mathrm{I}_{\mathbb{P}}} A$. Hence $A \in \mathrm{I}^\star_{\mathbb{P}}$ by the definition of $\mathrm{I}^\star_{\mathbb{P}}$. □

For ideals I of the form $\mathrm{I}^\star_{\mathbb{P}}$ such that $\mathbb{P}$ has the ω_1-covering property, the definition of I-shift density points (Definition 2.3) of a Borel set A can now be formulated in the following way: An element x of ${}^\omega X$ is an I-*shift density point of* A if there is some n_x such that for all $B \in \mathrm{L}_{\mathrm{I}}$ and $n \geqslant n_x$, there is some $T \in \mathbb{P}$ with

$$[T] \subseteq_I \sigma_{x\restriction n}(B) \cap A.$$

Note that it is easy to see that $\mathbb{P}$-measurability remains equivalent if $\mathrm{I}_{\mathbb{P}}$ is replaced with $\mathrm{I}^\star_{\mathbb{P}}$. Moreover, the ideals $\mathrm{N}_{\mathbb{P}}$, $\mathrm{I}_{\mathbb{P}}$ and $\mathrm{I}^\star_{\mathbb{P}}$ remain the same if $\mathbb{P}$ is replaced by a dense subset by the next remark.

Remark 4.11. $\mathbb{P}$ is dense in $\mathbb{Q}$ if for every $T \in \mathbb{P}$, there is some $S \leqslant T$ in $\mathbb{Q}$. We define $\mathbb{P}$ and $\mathbb{Q}$ to be *mutually dense* if $\mathbb{P}$ is dense in $\mathbb{Q}$ and conversely.

We claim that $\mathrm{N}_{\mathbb{P}} = \mathrm{N}_{\mathbb{Q}}$, $\mathrm{I}_{\mathbb{P}} = \mathrm{I}_{\mathbb{Q}}$ and $\mathrm{I}^\star_{\mathbb{P}} = \mathrm{I}^\star_{\mathbb{Q}}$ if $\mathbb{P}$ and $\mathbb{Q}$ are mutually dense. To see that $\mathrm{N}_{\mathbb{P}} \subseteq \mathrm{N}_{\mathbb{Q}}$, take any $A \in \mathrm{N}_{\mathbb{P}}$ and $T \in \mathbb{Q}$. As $\mathbb{P}$ is dense in $\mathbb{Q}$, there is some $T' \leqslant T$ in $\mathbb{P}$. Since $A \in \mathrm{N}_{\mathbb{P}}$, there is some $S \leqslant T'$ in $\mathbb{P}$ with $[S] \cap A = \varnothing$. As $\mathbb{Q}$ is dense in $\mathbb{P}$, there is some $S' \leqslant S$ with $S' \in \mathbb{Q}$ such that $[S'] \cap A \subseteq [S] \cap A = \varnothing$. Thus $\mathrm{N}_{\mathbb{P}} = \mathrm{N}_{\mathbb{Q}}$ and $\mathrm{I}_{\mathbb{P}} = \mathrm{I}_{\mathbb{Q}}$. A similar argument shows $\mathrm{I}^\star_{\mathbb{P}} = \mathrm{I}^\star_{\mathbb{Q}}$.

Conversely, any two collections of trees $\mathbb{P}$ and $\mathbb{Q}$ with the ω_1-covering property and $\mathrm{I}^\star_{\mathbb{P}} = \mathrm{I}^\star_{\mathbb{Q}}$ are mutually dense by Lemmas 4.7 and 4.10.

4.4. **An equivalence to the density property.** Let I be an ideal on ${}^\omega X$. We say that a function $D\colon \mathrm{Borel}({}^\omega X) \to \mathrm{Borel}({}^\omega X)$ is *lifted from* $\mathrm{Borel}({}^\omega X)/\mathrm{I}$ if for any $A, B \in \mathrm{Borel}({}^\omega X)$, $A =_{\mathrm{I}} B \Longrightarrow D(A) = D(B)$.

For instance, D_{I} as introduced in Definition 2.3 is *lifted from* $\mathrm{Borel}({}^\omega X)/\mathrm{I}$ if I is shift invariant (see p. 5). Then in fact $A \subseteq_{\mathrm{I}} B \Longrightarrow D_{\mathrm{I}}(A) \subseteq D_{\mathrm{I}}(B)$ by the definition of D_{I}.

The I-*density property* for D means that $D(A) =_{\mathrm{I}} A$ for all Borel sets A. We now give a useful condition for proving this for specific ideals. To state this condition, we say that D is I-*compatible* if $A \subseteq_{\mathrm{I}} B \Longrightarrow D(A) \subseteq_{\mathrm{I}} D(B)$ and $A \perp_{\mathrm{I}} B \Longrightarrow D(A) \perp_{\mathrm{I}} D(B)$ for all Borel sets A and B. We further say that D is I-*positive* if $D(A) \cap A$ is I-positive for all I-positive Borel sets A.

Proposition 4.12. *If $D\colon \mathrm{Borel}({}^\omega X) \to \mathrm{Borel}({}^\omega X)$ is a function that is lifted from $\mathrm{Borel}({}^\omega X)/\mathrm{I}$, then the following statements are equivalent.*

(a) *D is* I-*compatible and* I-*positive.*
(b) *D has the* I-*density property.*

Proof. It is clear that the I-density property implies that D is I-positive and I-compatible. To see that these conditions imply the density property, take any Borel set A. We aim to show that $D(A) =_{\mathrm{I}} A$.

We first show that $B_0 = A \setminus D(A)$ is in I. Towards a contradiction, assume that B_0 is I-positive. Then $D(B_0) \setminus D(A)$ is also I-positive, since it contains $D(B_0) \cap B_0$ as a subset, and the latter is I-positive since D is I-positive. On the other hand, we have $D(B_0) \setminus D(A) \in \mathrm{I}$ since $B_0 \subseteq A$ and D is I-compatible, contradiction.

It remains to show that $B_1 = D(A) \setminus A$ is in I. Assume that B_1 is I-positive, so that in particular $D(A)$ is I-positive. The set $C = D(B_1) \cap B_1$ is I-positive, since D is I-positive.

We thus obtain $C \perp_{\mathrm{I}} D(A)$, as $C \subseteq D(B_1)$ and we have $D(B_1) \perp_{\mathrm{I}} D(A)$ since $B_1 \perp_{\mathrm{I}} A$ (in fact B_1 and A are disjoint) and D is I-compatible. However, this contradicts the fact that C is I-positive and $C \subseteq B_1 \subseteq D(A)$. □

4.5. **Strongly linked collections of trees have the density property.** To obtain the density property for $D_{\mathrm{I}_{\mathbb{P}}}$, we will make two modest assumptions on $\mathbb{P}$. Let K_{I} denote a fixed subset of $\mathbb{P}$ coding L_{I} (from Convention 2.2) in the sense that $\mathrm{L}_{\mathrm{I}} = \{[T] \mid T \in \mathrm{K}_{\mathrm{I}}\}$.

Definition 4.13. A collection $\mathbb{P}$ of trees has the *stem property* if for all $T \in \mathbb{P}$ and $\mathrm{I}^\star_{\mathbb{P}}$-almost all $x \in [T]$, there are infinitely many $n \in \omega$ such that there is some $S \leqslant T$ with $x \in [S]$ and $S/(x{\restriction}n) \in \mathrm{K}_{\mathrm{I}}$.

The condition is trivially true for all $x \in [T]$ when $\mathrm{K}_{\mathrm{I}} = \{T \in \mathbb{P} \mid \mathrm{stem}_T = \varnothing\}$ and we only introduce it to deal with the case of Random forcing. For Random forcing, we let $\mathrm{K}_{\mathrm{I}} = \{T \in \mathbb{P} \mid \mu([T]) > \frac{1}{2}\}$. Then the stem property follows from Theorem 3.4 or Lebesgue's density theorem.

The next lemma shows that D_{I} is I-compatible for $\mathrm{I} = \mathrm{I}^\star_{\mathbb{P}}$ provided that D_{I} is I-positive and $\mathbb{P}$ has the stem property.

Lemma 4.14. *Assume that $\mathbb{P}$ is a collection of trees with the stem property, all Borel sets are $\mathbb{P}$-measurable, $\mathrm{I} = \mathrm{I}^\star_{\mathbb{P}}$ and $D_{\mathrm{I}}([T]) \notin \mathrm{I}$ for all $T \in \mathbb{P}$. Then $D_{\mathrm{I}}(A) \cap D_{\mathrm{I}}(B) =_{\mathrm{I}} D_{\mathrm{I}}(A \cap B)$ for all Borel sets A, B.*

Proof. It is easy to see that $D_{\mathrm{I}}(A \cap B) \subseteq D_{\mathrm{I}}(A) \cap D_{\mathrm{I}}(B)$, so suppose towards a contradiction that there are Borel sets A, B with $C = (D_{\mathrm{I}}(A) \cap D_{\mathrm{I}}(B)) \setminus D_{\mathrm{I}}(A \cap B) \notin \mathrm{I}$. Since C is Borel (it is a Boolean combination of $\mathbf{\Sigma}^0_2$ sets) and hence $\mathbb{P}$-measurable, by Lemma 4.10 there is some $S_0 \in \mathbb{P}$ with $[S_0] \subseteq_{\mathrm{I}} C$.

Since $[S_0] \subseteq_{\mathrm{I}} D_{\mathrm{I}}(A)$ we can pick $x \in [S_0] \cap D_{\mathrm{I}}(A)$ witnessing the stem property for S_0. Let $n_x \in \omega$ witness that $x \in D_{\mathrm{I}}(A)$. By the choice of x, there is some $m \geqslant n_x$ and $S_1 \leqslant S_0$ with $S_1/(x{\restriction}m) \in \mathrm{K}_{\mathrm{I}}$. Since $m \geqslant n_x$, $[S_1] \cap A \notin \mathrm{I}$. By Lemma 4.10, there is some $S_2 \in \mathbb{P}$ with $[S_2] \subseteq_{\mathrm{I}} [S_1] \cap A$.

Thus $[S_2] \subseteq_{\mathrm{I}} C \subseteq D_{\mathrm{I}}(B)$. Repeating the previous argument with A replaced by B yields some $S_3 \in \mathbb{P}$ with $[S_3] \subseteq_{\mathrm{I}} [S_2] \cap B$.

Since $[S_3] \subseteq_{\mathrm{I}} A \cap B$, we have $D_{\mathrm{I}}([S_3]) \subseteq D_{\mathrm{I}}(A \cap B)$, Note that $D_{\mathrm{I}}([S_3]) \subseteq [S_3] \subseteq [S_0]$. As $D_{\mathrm{I}}([S_3]) \notin \mathrm{I}$ by our assumption, this contradicts the fact that S_0 was chosen so that $[S_0] \cap D_{\mathrm{I}}(A \cap B) \in \mathrm{I}$. □

To obtain I-positivity we assume the following property.

Definition 4.15. A collection of trees $\mathbb{P}$ is called *strongly linked* if any $S, T \in \mathbb{P}$ with $\mathrm{stem}_S \sqsubseteq \mathrm{stem}_T$ and $\mathrm{stem}_T \in S$ are compatible in $\mathbb{P}$.

This condition holds for all ccc tree forcings that we study except Random forcing. For instance for Hechler forcing, eventually different forcing, Laver forcing $\mathbb{L}_F$ with a filter and Mathias forcing $\mathbb{R}_F$ with a shift invariant filter. Clearly the condition implies that $\mathbb{P}$ is σ-linked and ccc.

Remark 4.16. Random forcing doesn't have a strongly linked dense subset. To see this, assume that $\mathbb{B}'$ is strongly linked and dense in $\mathbb{B}$. Let $S \in \mathbb{B}'$ and $t \in S$ with $\mathrm{stem}_S \sqsubseteq t$ and $\frac{\mu([S] \cap N_t)}{\mu(N_t)} < 1$. For example, such a $t \in S$ exists if $[S]$ is nowhere dense. Let further $A = N_t \setminus [S]$. Since $\frac{\mu(A \cap N_t)}{\mu(N_t)} > 0$, there is some $T \in \mathbb{B}$ with $[T] \subseteq A$ and $t \sqsubseteq \mathrm{stem}_T$. Since $\mathbb{B}'$ is dense in $\mathbb{B}$, we can assume that $T \in \mathbb{B}'$. Then $\mathrm{stem}_S \sqsubseteq \mathrm{stem}_T$, but S and T are incompatible.

Lemma 4.17. *Assume that $\mathbb{P}$ is a strongly linked tree forcing—whence by Convention 2.2 $\mathrm{L}_{\mathrm{I}} = \{[T] \in \mathbb{P} \mid \mathrm{stem}_T = \varnothing\}$. Moreover assume that $\mathrm{I} = \mathrm{I}^\star_{\mathbb{P}}$. Then D_{I} is I-positive. In fact for any $T \in \mathbb{P}$, all $x \in [T]$ are I-density points of $[T]$.*

Proof. Let $x \in [T]$ be given. Then there is some $m \in \omega$ such that letting $T' = T_{x \restriction m}$, $\mathrm{stem}_{T'} = x \restriction m$. It is sufficient to show that x is an I-density point of $[T]$. To see this, suppose that $S \in \mathbb{P}$ is such that $\mathrm{stem}_S = x \restriction n$ for some $n \geqslant m$. Since the stems of S and T' are compatible, $\mathrm{stem}_S \in T'$, and $\mathbb{P}$ is strongly linked, S and T' are compatible. Then $[S] \cap [T']$ and thus also $[S] \cap [T]$ are I-positive sets as required. □

This implies a version of Lebesgue's density theorem.

Corollary 4.18. $\mathrm{I}^{\star}_{\mathbb{P}}$ *has the density property for any strongly linked tree forcing with the stem property. In particular,* $\mathrm{I}_{\mathbb{P}}$ *has the density property for Cohen forcing* $\mathbb{C}$*, Hechler forcing* $\mathbb{H}$*, eventually different forcing* $\mathbb{E}$*, Laver forcing* $\mathbb{L}_F$ *with a filter and Mathias forcing* $\mathbb{R}_F$ *with a shift invariant filter.*

For Random forcing, the I_μ-density property follows from Lemma 3.1 and Lebesgue's density theorem. An alternative proof (which does not appeal to Lebesgue's density theorem) is now obtained via Lemma 3.1, Theorem 3.4 and Proposition 4.12.

Note that if $\mathbb{P}$ is topological and $\mathrm{I}_{\mathbb{P}}$ has the density property, then one can describe $D_{\mathrm{I}_{\mathbb{P}}}(A)$ as follows using $\tau_{\mathbb{P}}$. First note that $\mathrm{I}_{\mathbb{P}} = \mathrm{I}^{\star}_{\mathbb{P}}$ by the proof of [FKK16, Lemma 3.8]. Moreover, for any Borel set A there is some $\tau_{\mathbb{P}}$-open set U with $A \triangle U$ $\tau_{\mathbb{P}}$-meager by the $\tau_{\mathbb{P}}$-Baire property. Since $\mathrm{I}_{\mathbb{P}}$ equals the set of $\tau_{\mathbb{P}}$-meager sets, $D_{\mathrm{I}_{\mathbb{P}}}(A)$ is almost equal to a $\tau_{\mathbb{P}}$-open set. However, note that even for ccc collections it is not clear how to find such a set in a simply definable way.

We conclude this section with some observations about other variants of the Definition 2.3 of density points and the density property.

Remark 4.19. If $\mathbb{P}$ is ccc and the density property holds for all Borel sets, then it already holds for all $\mathbb{P}$-measurable sets. This is the case because any $\mathbb{P}$-measurable set equals $A \triangle B$ for some Borel set A and some $B \in \mathrm{I}_{\mathbb{P}}$ by the ccc and Lemma 4.8.

Remark 4.20. (a) If we let $\mathrm{L}_\mathrm{I} = \mathrm{I}^+$ be the I-positive sets for Hechler forcing $\mathbb{H}$ and $\mathrm{I} = \mathrm{I}_{\mathbb{H}}$, then the density property fails. To see this, let T be the Hechler tree with empty stem given by the constant function with value 1. With this variant $[T]$ does not satisfy the density property, since no $x \in [T]$ is an I-shift density point of $[T]$, as witnessed by $N_{\langle 0 \rangle} \in \mathrm{L}_\mathrm{I}$.

(b) If n_0 depends on $B \in \mathrm{L}_\mathrm{I}$ in Definition 2.3, then the density property fails for $\mathrm{I}_{\mathbb{H}}$ as well. To see this, let x be the function with $x(n) = n+1$ for all $n \in \omega$ and let $T_{\varnothing,x}$ be the tree with empty stem given by x. Let further $A = \bigcup_{t \in {}^{<\omega}\omega} [T_{t^\frown \langle |t| \rangle, t^\frown \langle |t| \rangle^\frown 0^\infty}]$; this contains all $y \in {}^{\omega}\omega$ with $y(n) = n$ for some $n \in \omega$. It is sufficient to show $[T_{\varnothing,x}] \subseteq D(A) \setminus A$ by Lemma 4.10.

It is easy to see that $[T_{\varnothing,x}] \cap A = \varnothing$, since any $y \in [T_{\varnothing,x}]$ satisfies $y(n) \geqslant n+1$ for all $n \in \omega$.

We now claim that $y \in D(A)$ for all $y \in \omega^\omega$. So take any $\mathrm{I}_{\mathbb{H}}$-positive Borel set B. It is sufficient to assume that $B = [T_{s,u}]$ for some $s \in {}^{<\omega}\omega$ and $u \in {}^{\omega}\omega$ by Lemma 4.10. Then for any $n \geqslant u(|s|)$, $\sigma^{-1}_{y \restriction n}([T_{s,u}/s)]) \cap A = [(y \restriction n)^\frown (T_{s,u}/s)] \cap A$ contains $[T_{t,t^\frown v}]$ for $t = y \restriction n^\frown \langle n \rangle$ and $v \in {}^{\omega}\omega$ with $v(i) = u(|s|+i+1)$ for all $i \in \omega$. Hence it is $\mathrm{I}_{\mathbb{H}}$-positive and thus y is a density point of A.

4.6. **Complexity of the density operator.** In this section, we give an upper bound for the complexity of the operator D_I for relevant cases of tree ideals I of the form $\mathrm{I}_{\mathbb{P}}$. K_I is fixed as in the previous section.

A definable forcing $\mathbb{P}$ is called *absolutely ccc* if the ccc holds in all generic extensions. A Δ^1_2 predicate is *absolutely* Δ^1_2 if its Σ^1_2 and Π^1_2 definitions are equivalent in all generic extensions.

Lemma 4.21. *Suppose that* $\mathrm{I} = \mathrm{I}_{\mathbb{P}}$*, where* $\mathbb{P}$, $\leqslant_{\mathbb{P}}$ *and* $\perp_{\mathbb{P}}$ *are* Σ^1_1, K_I *is a* Σ^1_1 *subset of* $\mathbb{P}$ *and* $\mathbb{P}$ *is absolutely ccc. Then*

$$D_\mathrm{I} \colon \textsc{Borel}({}^{\omega}X) \to \textsc{Borel}({}^{\omega}X)$$

is induced by an absolutely Δ^1_2 function from Borel codes to Borel codes.

Proof. Let B denote the Π^1_1 set of Borel codes and B_x the set coded by a Borel code x. Let $\varphi(x)$ denote the Π^1_2 formula $x \in \mathrm{B}$ & $\forall T \in \mathrm{K_I}\ B_x \cap [T] \notin \mathrm{I}$. Let $\psi(x)$ denote the Σ^1_2 statement: $x \in \mathrm{B}$ and there is some $m \leqslant \omega$ and a sequence $\vec{T} = \langle T_i \mid i < m \rangle$ from $\mathbb{P}$ with $\forall i < m\ [T_i] \subseteq_{\mathrm{I}} B_x$ and $\forall T \in L_{\mathrm{I}}\ \exists i < m\ T \parallel_{\mathbb{P}} T_i$.

Claim. $\forall x\ \varphi(x) \Longleftrightarrow \psi(x)$.

Proof. If $\varphi(x)$ holds, inductively define an antichain $\vec{T} = \langle T_\xi \mid \xi < \theta \rangle$ from $\mathbb{P}$ with $[T_\xi] \subseteq_{I_{\mathbb{P}}} B_x$ for all $\xi < \theta$. Suppose that $\vec{T}^{(\alpha)} = \langle T_\xi \mid \xi < \alpha \rangle$ is already defined. If there is $T \in \mathrm{L_I}$ which is incompatible with each element of $\vec{T}^{(\alpha)}$, we may find $T_\alpha \in \mathbb{P}$ with $T_\alpha \subseteq T$ and $[T_\alpha] \subseteq B_x$ since $\Phi(x)$ holds. We can thus extend the antichain by adding T_α. By the ccc we must reach some $\theta < \omega_1$ such that each $T \in \mathrm{L_I}$ is compatible to an element of $\vec{T} = \langle T_\xi \mid \xi < \theta \rangle$. Enumerate $\vec{T}$ in order type $m \leqslant \omega$ to obtain a witness to $\psi(x)$.

If conversely $\psi(x)$ holds, then for any $T \in \mathrm{K_I}$ we may find some $i \in \omega$ with $T_i \parallel T$ and $[T_i] \subseteq_{\mathrm{I}} B_x$, so one can infer $B_x \cap [T] \notin \mathrm{I}$ from Lemma 4.10. Thus $\varphi(x)$ holds. □

We now check that $\varphi(x)$ is a Π^1_2 and $\psi(x)$ a Σ^1_2 formula. First note that the statement $B_x \in \mathrm{N}_{\mathbb{P}}$ is Σ^1_2, since this holds if and only if there is a (countable) maximal antichain $\vec{S} = \langle S_i \mid i < m \rangle$ in $\mathbb{P}$ with $B_x \cap \bigcup_{i<m}[S_i] = \emptyset$. Since $\mathrm{I} = \mathrm{I}_{\mathbb{P}}$ is the σ-ideal generated by $\mathrm{N}_{\mathbb{P}}$, the statements $B_x \in \mathrm{I}$ and $[T] \subseteq_{\mathrm{I}} B_x$ are Σ^1_2 as well. The claim follows.

Since $\mathbb{P}$ is absolutely ccc, the argument above shows that $\forall x\ \Phi(x) \Longleftrightarrow \Psi(x)$ is absolute to generic extensions. Given a Borel code x, we can thus determine $S_x = \{s \in {}^{<\omega}X \mid \forall T \in \mathrm{K_I}\ \sigma_s([T]) \cap B_x \notin \mathrm{I}\}$ in an absolutely Δ^1_2 fashion. From this, we can compute a Borel code for $D_{\mathrm{I}}(B_x)$. □

We want to point out that the previous lemma remains true with virtually the same proof if we replace *absolutely ccc* and *absolutely* Δ^1_2 by *provably ccc* and *provably* Δ^1_2.

We now show that for all strongly linked tree forcings listed in Section 6, the density operator $D_{\mathrm{I}_{\mathbb{P}}}$ is induced by a universally Baire measurable function.

Recall that a forcing is called *Suslin* if $\mathbb{P}$, $\leqslant_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are Σ^1_1. Also recall that a subset A of ${}^\omega\omega$ is called *universally Baire* if for any topological space Y and any continuous function $f\colon Y \to {}^\omega\omega$, $f^{-1}(A)$ has the property of Baire. A function $f\colon A \to B$ between subsets of ${}^\omega\omega$ is called *universally Baire measurable* if $f^{-1}(U)$ is universally Baire for every relatively open subset U of B. Note that all universally Baire sets have the Baire property, are Lebesgue measurable and Ramsey [FMW92, Theorem 2.2].

Proposition 4.22. *For any strongly linked Suslin tree forcing $\mathbb{P}$, $D_{\mathrm{I}_{\mathbb{P}}}$ is induced by a universally Baire function.*

Proof. If $\mathbb{P}$ is strongly linked, then $S \parallel T$ if and only if stem_S and stem_T are compatible, so $\perp_{\mathbb{P}}$ is arithmetical and hence Σ^1_1. Moreover, the fact that a Suslin tree forcing is strongly linked is Π^1_2 and hence absolute. Thus $D_{\mathrm{I}_{\mathbb{P}}}$ is induced by an absolutely Δ^1_2 function on the Borel codes by Lemma 4.21. Any such function is universally Baire measurable by [FMW92, Theorem 2.1]. □

5. Ideals without density

In this section, we study various counterexamples to density properties.

5.1. Counterexamples.

We first give counterexamples to the density property in Definition 2.3 for several non-ccc tree forcings.

Proposition 5.1. *Let $\mathbb{R}$, $\mathbb{V}$, $\mathbb{S}$ denote Mathias, Silver and Sacks forcing. Then $\mathrm{I}_{\mathbb{R}}$, $\mathrm{I}_{\mathbb{V}}$ and $\mathrm{I}_{\mathbb{S}}$ do not have the shift density property.*

Proof. To see that $\mathrm{I}_{\mathbb{R}}$ does not have the $\mathrm{I}_{\mathbb{R}}$-shift density property, let $A = \{f \in {}^{\omega}2 \mid f(2n+1) = 1 \text{ for all } n \in \omega\}$. Note that $A = [S]$ for some $S \in \mathbb{R}$ and hence $A \notin \mathrm{I}_{\mathbb{R}}$. We aim to show that no $x \in A$ is an $\mathrm{I}_{\mathbb{R}}$-density point of A, i.e. $A \cap D_{\mathbb{R}}(A) = \varnothing$. Then in particular $A \triangle D_{\mathbb{R}}(A) = A \cup D_{\mathbb{R}}(A) \notin \mathrm{I}_{\mathbb{R}}$.

Let $x \in A$ be arbitrary and let $T \in \mathbb{R}$ be a perfect tree such that $\mathrm{split}(T) = 2\mathbb{N}$ and $t^\frown i^\frown j \in T$ iff $j = 0$ for all $t \in \mathrm{split}(T)$ and $i, j \in \{0,1\}$. In particular $\mathrm{stem}_T = \varnothing$. Let $n_0 \in \omega$ be arbitrary and let $n \geqslant n_0$ be even. Then $f_{x\restriction n}[T] \cap A = \varnothing$ and thus x is not an $\mathrm{I}_{\mathbb{R}}$-density point of A.

As $\mathbb{R} \subseteq \mathbb{V} \subseteq \mathbb{S}$, the claim also holds for $\mathrm{I}_{\mathbb{V}}$ and $\mathrm{I}_{\mathbb{S}}$. □

The following is a similar counterexample for Laver and Miller forcing.

Proposition 5.2. *Let $\mathbb{L}$, $\mathbb{M}$ denote Laver and Miller forcing. Then $\mathrm{I}_{\mathbb{L}}$ and $\mathrm{I}_{\mathbb{M}}$ do not have the shift density property.*

Proof. Let $A = \{f \in \omega^\omega \mid f(n) \text{ is even for all } n \in \omega\}$. Then $A = [S]$ for some $S \in \mathbb{L}$ so in particular $A \notin \mathrm{I}_{\mathbb{L}}$. We aim to show that no $x \in A$ is an $\mathrm{I}_{\mathbb{L}}$-density point of A, i.e. $A \cap D_{\mathbb{L}}(A) = \varnothing$. As above this implies $A \triangle D_{\mathbb{L}}(A) \notin \mathrm{I}_{\mathbb{L}}$.

Let $x \in A$ be arbitrary and let $T \in \mathbb{L}$ be a perfect tree such that $\mathrm{stem}_T = \varnothing$ and $[T] = \{g \in \omega^\omega \mid g(n) \text{ is odd for all } n \in \omega\}$. Let $n_0 \in \omega$ be arbitrary and let $n \geqslant n_0$. Then $f_{x\restriction n}[T] \cap A = \varnothing$ and thus x is not an $\mathrm{I}_{\mathbb{L}}$-density point of A.

Since $\mathbb{L} \subseteq \mathbb{M}$, the claim for $\mathrm{I}_{\mathbb{M}}$ follows. □

5.2. **Selectors modulo countable.** We now study the ideal I of countable sets. In contrast to the previous results, we will show that there is no Baire measurable selector with Borel values for the equivalence relation of having countable symmetric difference on the set of Borel subsets. This implies that the density property fails for I for any reasonable notion of density points.

To state the result formally, we need the following notions. A *selector* for an equivalence relation $E^\star$ is a function that chooses an element from each equivalence class. We generalize this notion by replacing equality with a subequivalence relation E of $E^\star$.

Definition 5.3. Suppose that $E \subseteq E^\star$ are equivalence relations on a set Y and $X \subseteq Y$. A *selector for $E^\star/E$ on X* is an $(E^\star, E)$-homomorphism $X \to Y$ that uniformizes $E^\star$.

Equivalently, the induced map on Y/E is a selector for the equivalence relation on Y/E induced by $E^\star$.

In the following, E will be equality of decoded sets, $E^\star$ the equivalence relation of having countable symmetric difference, X the set of F_σ-codes and Y the set of Borel codes.

More precisely, Λ_{F_σ} denotes the set of F_σ-codes (sequences $\vec{T} = \langle T_n \mid n \in \omega\rangle$ where T_n is a subtree of $2^{<\omega}$ for each n), Λ the set of Borel codes and B_x the Borel set coded by $x \in \Lambda$. We can assume that $\Lambda_{F_\sigma} \subseteq \Lambda$. Let $E_=$ denote the equivalence relation on Λ of equality of decoded sets, i.e. $(x,y) \in E_= \iff B_x = B_y$. Let further $(x,y) \in E_{\mathrm{I}} \iff B_x \triangle B_y \in \mathrm{I}$ for $x, y \in \Lambda$.

Definition 5.4. A *selector for* I *with Borel values* is a selector for $E_{\mathrm{I}}/E_=$ on Λ_{F_σ}.

The motivation for this definition is as follows. Consider any notion of density points for I with Borel values, i.e. such that for any Borel set A the set of density points of A is Borel. If the density property holds for this notion, then it induces a selector for I on the set of Borel codes.

The restriction to F_σ-codes is purely for the technical reason that it is already sufficient to show that no such selectors exist.

Theorem 5.5. *There is no Baire measurable selector for* I *with Borel values.*

Note that this result is analogous to the fact that E_0 does no have a Baire measurable selector. The proof for E_0 is a short argument, see for example [Hjo10, Example 1.6.2].

The idea of the proof is to add an F_σ set $B_C = \bigcup_{n\in\omega}[T_n]$ by forcing over a countable elementary submodel M of H_{ω_1}. One then shows that the properties of a selector are not satisfied on the E_{I}-equivalence class of B_C. The trees T_n will be added by the following forcing $\mathbb{T}$. The conditions in $\mathbb{T}$ are finite subtrees t of $2^{<\omega}$, ordered by end extension. Moreover, we let $\dot{T}$ denote a name for the generic tree $\bigcup G$ that is added by a $\mathbb{T}$-generic filter G. Note that $\mathbb{T}$ is equivalent to Cohen forcing, since it is countable and non-atomic. We include a proof of the following well-known fact for the reader's convenience.

Lemma 5.6. *Assume that G is $\mathbb{T}$-generic over V. Then in any further generic extension, any two branches in $\dot{T}^G$ are mutually Cohen generic over V.*

Proof. Assume that $\dot{\mathbb{Q}}$ is a $\mathbb{T}$-name for a forcing and $1_{\mathbb{T}\star\dot{\mathbb{Q}}}$ forces that σ and τ are distinct branches in $\dot{T}$. Let further D be a dense subset of $\mathrm{Add}(\omega,1)^2$ and

$$E = \{(p,\dot{q}) \in \mathbb{T}\star\dot{\mathbb{Q}} \mid \exists (u,v)\in D\ (p,\dot{q}) \Vdash_{\mathbb{T}\star\dot{\mathbb{Q}}} u \subseteq \sigma \ \&\ v\subseteq \tau\}.$$

It is sufficient to show that E is a dense subset of $\mathbb{T}\star\dot{\mathbb{Q}}$. To this end, assume that a condition $(p,\dot{q})\in\mathbb{T}\star\dot{\mathbb{Q}}$ is given. By extending it, we can assume that $(p,\dot{q}) \Vdash_{\mathbb{T}\star\dot{\mathbb{Q}}} u\subseteq\sigma \ \&\ v\subseteq\tau$ for two incompatible $u,v\in 2^{<\omega}$. We first add all direct successors to end nodes of p to add another splitting level. It is easy to see that one can successively extend each pair of new end nodes to an element of D to obtain a condition $r\leqslant_{\mathbb{T}} p$ with $(r,\dot{q})$ in E. □

Proof of Theorem 5.5. Suppose that there is a selector F on Λ_{F_σ} as in the statement of the theorem. Since F is Baire measurable, there is a comeager G_δ subset A of Λ_{F_σ} such that $F{\upharpoonright}A$ is continuous. Let x_A be a real that codes this restriction. Moreover, let $M \prec H_{\omega_1}$ be countable with $x_A\in M$.

The following items are defined in M. Let $\mathbb{T}^\omega$ denote the finite support product of ω copies of $\mathbb{T}$, $\vec{T} = \langle \dot{T}_n \mid n\in\omega\rangle$ a sequence of $\mathbb{T}^\omega$-names for the generic trees and $\dot{C}$ a $\mathbb{T}^\omega$-name for the canonical F_σ-code for $\bigcup_{n\in\omega}[\dot{T}_n]$. Let further $\dot{F}$ be a $\mathbb{T}^\omega$-name for the continuous function coded by x_A. Then $\dot{F}^g(x) = F(x)$ for all $x\in A\cap M[g]$ and $\mathbb{T}^\omega$-generic filters g over M in V.

Claim. $1 \Vdash^M_{\mathbb{T}^\omega} [\dot{T}_n]\cap B_{\dot{F}(\dot{C})} \neq \varnothing$ for all $n\in\omega$.

Proof. Assume towards a contradiction that $p \Vdash^M_{\mathbb{T}^\omega} [\dot{T}_n]\cap B_{\dot{F}(\dot{C})} = \varnothing$ for some $p\in\mathbb{T}^\omega$ and some $n\in\omega$. Since A is comeager, there is a $\mathbb{T}^\omega$-generic filter g over M with $p\in g$ and $\dot{C}^g\in A$. Then $\dot{F}^g(\dot{C}^g) = F(\dot{C}^g)$ and $B_{\dot{F}^g(\dot{C}^g)} = B_{F(\dot{C}^g)}$. We have $M[g]\models [\dot{T}^g_n]\cap B_{\dot{F}^g(\dot{C}^g)} = \varnothing$ and hence $[\dot{T}^g_n]\cap B_{F(\dot{C}^g)} = [\dot{T}^g_n]\cap B_{\dot{F}^g(\dot{C}^g)} = \varnothing$ by Π^1_1-absoluteness between $M[g]$ and V. But this contradicts the assumption that F is a selector. □

By the maximum principle, there is a $\mathbb{T}^\omega$-name $\sigma\in M$ for an element of $[\dot{T}_0]\cap B_{\dot{F}(\dot{C})}$.

We now proceed as follows. We first rearrange a given $\mathbb{T}^\omega\times\mathrm{Add}(\omega,1)$-generic extension $M[g\times h]$ of M as a $\mathbb{T}^\omega\times\mathrm{Add}(\omega,1)$-generic extension $M[G\times H]$ such that σ^G is equal to the Cohen real given by h and $\dot{C}^G$ and $\dot{C}^g$ are in the same $E_=$-equivalence class. We then use this to show that $B_{F(\dot{C}^g)}\setminus B_{\dot{C}^g}$ contains all Cohen reals in V over $M[g]$ and hence is uncountable. But this contradicts the fact that F is a selector for E_{I}.

To this end, we will use the following forcings. Let $\mathbb{T}_\sigma$ be the forcing that consists of all conditions (p,s), where $p\in\mathbb{T}$ and s is a maximal branch in p such that $\exists q\leqslant p\ q\Vdash s\sqsubseteq\sigma$. It is ordered by pointwise extension. Moreover, let $\mathbb{P}_\sigma$ be the finite support product of $\mathbb{T}_\sigma$ and ω copies of $\mathbb{T}$. Since $\mathbb{T}^\omega$ is separative, its Boolean completion exists and $\mathbb{P}_\sigma$ is a dense subset. We will also need the following forcing $\mathbb{T}_\star$, which adds a $\mathbb{T}$-generic tree and simultaneously shoots a branch through it. The conditions in $\mathbb{T}_\star$ are pairs (p,s), where $p\in\mathbb{T}$ and s is a maximal branch in p.

Now let $\mathbb{P}_\star$ denote the finite support product of $\mathbb{T}_\star$ and ω copies of $\mathbb{T}$. Let τ be a name for the branch added by $\mathbb{T}_\star$. In the following claims, we will translate values of σ (when forcing with $\mathbb{P}_\sigma$) to those of τ (when forcing with $\mathbb{T}_\star$) and conversely. Note that the sets of values of σ and of τ are not necessarily equal, since σ could for instance be the leftmost branch of $\dot{T}_0$ and is then different from τ. However, we can translate values of τ to those of σ if they are compatible with the information forced about σ. Therefore, we need to introduce the following notation. For any $p \in \mathbb{T}^\omega$, let σ_p denote the unique maximal sequence $w \in 2^{<\omega}$ with $p \Vdash_{\mathbb{T}^\omega} w \sqsubseteq \sigma$ and $T_\sigma = \{\sigma_p \mid p \in \mathbb{T}^\omega\}$ the set of possible initial segments of σ, as decided by conditions in $\mathbb{T}^\omega$. Moreover, let U_σ denote the set of $w \in 2^{<\omega}$ such that T_σ is dense above w.

Claim. U_σ is a nonempty open subset of $2^{<\omega}$.

Proof. It is clear that U_σ is open. To see that it is nonempty, assume towards a contradiction that T_σ is nowhere dense. For every $u \in 2^{<\omega}$, let $v = v_u$ be an extension of u such that $C_v = \{w \in 2^{<\omega} \mid v \sqsubseteq w\}$ is disjoint from T_σ. Then $D = \bigcup_{u \in 2^{<\omega}} C_{v_u}$ is an open dense subset of Cohen forcing. Since σ is a name for a Cohen real, it follows that for any Cohen generic g over M, there is some u with $v_u \sqsubseteq \sigma^g$. But this contradicts the fact that $\sigma^g \in [T_\sigma]$. □

Claim. (a) Every $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$-generic extension $M[G \times H]$ is equal to a $\mathbb{P}_\star \times \mathrm{Add}(\omega, 1)$-generic extension $M[g \times h]$ with $\sigma^G = \tau^g$ and $B_{\dot{C}^G} = B_{\dot{C}^g}$.

(b) Every $\mathbb{P}_\star \times \mathrm{Add}(\omega, 1)$-generic extension $M[g \times h]$ with $\tau^g \restriction n \in U_\sigma$ for some $n \in \omega$ is equal to a $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$-generic extension $M[G \times H]$ with $\sigma^G = \tau^g$ and $B_{\dot{C}^G} = B_{\dot{C}^g}$.

Proof. To prove the first claim, we will define a projection from a dense subset D_σ of $\mathbb{P}_\sigma \times \mathrm{Add}(\omega, 1)$ to $\mathbb{P}_\star \times \mathrm{Add}(\omega, 1)$. We refer to [Cum10, Definition 5.2] for the definition of projections. To define D_σ, we associate to each $(\vec{p}, q) \in \mathbb{P}_\sigma \times \mathrm{Add}(\omega, 1)$ with $\vec{p} = ((p_0, s_\star), p_1, \ldots, p_k)$ a strictly increasing sequence of indices which single out certain trees p_i for $1 \leqslant i < k$. Let $s_i = \mathrm{stem}_{p_i}$ for all $i < k$. If u, v are incompatible elements of $2^{<\omega}$, we will also write $\mathrm{dir}(u, v)$ for the unique node below v that is a direct successor of $u \wedge v$.

A sequence $\vec{n} = \langle n_i \mid i < l \rangle$ is called a *gluing sequence* for $(\vec{p}, q)$ if the following conditions hold. The first value n_0 is least such that $q(2n_0) = 1$, s_{n_0} is incompatible with $s_\star$ and $\mathrm{dir}(s_\star, s_{n_0}) \notin p_0$. Moreover, for all $i < l$ the value n_{i+1} is least with $n_{i+1} > n_i$, $q(2n_{i+1}) = 1$, $s_{n_{i+1}}$ is incompatible with $s_\star$, $\mathrm{dir}(s_\star, s_{n_{i+1}}) \notin p_0$ and $s_{n_{i+1}} \wedge s_\star$ is a proper end extension of $s_{n_i} \wedge s_\star$. For every condition $(\vec{p}, q) \in \mathbb{P}_\sigma \times \mathrm{Add}(\omega, 1)$, there is a unique maximal gluing sequence we denote by $\vec{n} = \vec{n}_{\vec{p},q} = \langle n_i \mid i < n_{\vec{p},q} \rangle$. We define $(\vec{p}, q) \in D_\sigma$ to hold if $n_l = k = \max(\mathrm{dom}(q))$ for $l = n_{\vec{p},q}$.

We now define a projection $\pi \colon D_\sigma \to \mathbb{P}_\star \times \mathrm{Add}(\omega, 1)$ that glues these trees to p_0. We define $\pi(\vec{p}, q)$ for $\vec{p} = ((p_0, s_\star), p_1, \ldots, p_k)$ as follows. Let $J = J_{\vec{p},q} = \{0\} \cup \{n_i \mid i < n_{\vec{p},q}\}$ and let $\vec{m} = \vec{m}_{\vec{p},q} = \langle m_i \mid i < m_{\vec{p},q} \rangle$ be the order preserving enumeration of $(k+1) \setminus J$. Now let $\pi(\vec{p}, q) = (\vec{t}, u)$ where $\vec{t} = ((t_0, t_\star), t_1, \ldots, t_m)$, $t_0 = \bigcup_{i \in J} p_{n_i}$, $t_{i+1} = p_{m_i}$ for $i < m_{\vec{p},q}$ and u the sequence of odd digits of q.

Subclaim. π is a projection.

Proof. It is clear that π is a homomorphism. To see that π is a projection, suppose that $\pi(\vec{p}, q) = (\vec{t}, u)$, $\vec{v} = ((v_0, v_\star), v_1, \ldots, v_l) \leqslant \vec{t} = ((t_0, t_\star), t_1, \ldots, t_k)$ and $w \leqslant u$. We need to find some $(\vec{a}, b) \leqslant (\vec{p}, q)$ with $\pi(\vec{a}, b) \leqslant (\vec{v}, w)$. Let $\vec{m} = \vec{m}_{\vec{p},q}$, $\vec{n} = \vec{n}_{\vec{p},q}$ and $J = J_{\vec{p},q}$ be as above. Let $\mathrm{end}(p_i)$ denote the set of all elements of $2^{<\omega}$ that are compatible with some maximal node in p_i. In other words, $\mathrm{end}(p_i)$ is the union of all end extensions of p_i.

We now define a_i for $i \leqslant l$. Let $a_0 = \{r \in v_0 \mid t_\star \sqsubseteq r \Rightarrow r \sqsubseteq v_\star\}$ and note that $(a_0, v_\star)$ is a condition in $\mathbb{T}_\sigma$. If $1 \leqslant i \leqslant k$ and $i \in J$, we let $a_i = v_0 \cap \mathrm{end}(p_i)$. If $1 \leqslant i \leqslant k$ and $i \notin J$, then $i = m_j$ for some $j < m_{\vec{p},q}$ and we let $a_i = v_{m_j}$. Let further $a_i = v_{i-|J|+1}$if $k < i \leqslant l$. Moreover, let j be the number of direct successors in v_0 of nodes strictly below $v_\star$ at or above $t_\star$ that are incompatible with $v_\star$. The subtrees of v_0 through each of these stems are missing in t_0 and we add them as a_i for $l < i \leqslant l + j$. Let $\vec{a} = ((a_0, s_\star), a_1, \ldots, a_{l+j})$. We can further assume that $j \geqslant |w| - |q|$ by extending $(v_0, v_\star)$. We finally extend q to b by adding j even digits with value 1, $|w| - |q|$ odd digits of w above q and $j - (|w| - |q|)$ arbitrary odd digits.

Since $n_i = k = \max(\mathrm{dom}(q))$ for $i = n_{\vec{p},q}$ by the definition of D_σ, the choice of a_0, a_i for $l < i \leqslant l + j$ and b implies that the first coordinate of $\pi(\vec{a}, b)$ is equal to $(v_0, v_\star)$. Hence $\pi(\vec{a}, b) \leqslant (\vec{v}, w)$ as required. □

Since π is a projection, there is a $\mathbb{P}_\star$-generic filter g over M in $M[G \times H]$ with $\sigma^G = \tau^g$ and $B_{\dot{C}^G} = B_{\dot{C}^g}$. It is clear that $M[g] \subsetneq M[G \times H]$ because of the second coordinate of the projection. Hence the quotient forcing for $M[g]$ in $M[G \times H]$ is non-atomic. Since $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$ is countable, the quotient forcing has a dense subset that is isomorphic to Cohen forcing. This shows (a).

We now complete the proof of the claim by showing (b). Note that for any $\mathbb{T}_\star \times \mathrm{Add}(\omega, 1)$-generic filter $g \times h$ over M, the quotient forcing for τ^g in $\mathbb{T}_\star \times \mathrm{Add}(\omega, 1)$ is countable and non-atomic, therefore it has a dense subset that is isomorphic to Cohen forcing. Assuming that the claim fails, there is some $r \in U_\sigma$ that forces as a Cohen condition that in any further Cohen extension, the required rearrangement doesn't exist. In other words, for every $\mathbb{T}_\star \times \mathrm{Add}(\omega, 1)$-generic extension $M[g \times h]$ with $\tau^h \restriction n = r$ for $n = |r|$, there is no such rearrangement. Since $r \in U_\sigma$, T_σ is dense above r and hence there is some $p \in T_\sigma$ with $\sigma_p \leqslant r$. Now let $G \times H$ be $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$-generic over M with $p \in G$. Then $M[G \times H]$ can be rearranged as $M[g \times h]$ with $\sigma^G = \tau^h$ and $B_{\dot{C}^G} = B_{\dot{C}^g}$ by the first claim. Hence $\tau^h \restriction n = \sigma^G \restriction n = r$, but this contradicts the choice of r. □

Let θ be a $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$-name for the Cohen real in the second coordinate.

Claim. (a) Every $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$-generic extension $M[G \times H]$ is equal to a $\mathbb{P}_\star \times \mathrm{Add}(\omega, 1)$-generic extension $M[g \times h]$ with $\theta^H = \tau^g$ and $B_{\dot{C}^g} = B_{\dot{C}^G} \cup \{\theta^H\}$.

(b) Every $\mathbb{P}_\star \times \mathrm{Add}(\omega, 1)$-generic extension $M[g \times h]$ is equal to a $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$-generic extension $M[G \times H]$ with $\theta^H = \tau^g$ and $B_{\dot{C}^g} = B_{\dot{C}^G} \cup \{\theta^H\}$.

Proof. To prove the claim, we will define a projection $\rho\colon D \to \mathbb{P}_\star \times \mathrm{Add}(\omega, 1)$, where D is a dense subset of $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$. We first define D as follows. Suppose that $(\vec{p}, q) \in \mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$ and $\mathrm{supp}(\vec{p}) = \mathrm{dom}(q) = [0, k]$ for some $k \geqslant 1$. Let $\vec{p} = \langle p_i \mid i \leqslant k \rangle$ and $s_i = \mathrm{stem}_{p_i}$ for $i \leqslant k$. Note that if all nodes in p_i are compatible, then $s_i = p_i$. We associate a *splitting sequence* $\vec{n} = \vec{n}_{\vec{p},q} = \langle n_i \mid i \leqslant n \rangle$ to $\vec{p}, q$ as follows. Let n_0 be least such that s_{n_0} is incompatible with q and $s_{n_0} \wedge q \neq \varnothing$. If n_i is already defined, let n_{i+1} be least such that $s_{n_{i+1}}$ is incompatible with q, $n_{i+1} > n_i$ and $s_{n_{i+1}} \wedge q$ is a proper end extension of $s_{n_i} \wedge q$. There is a unique maximal splitting sequence $\vec{n} = \vec{n}_{\vec{p},q} = \langle n_i \mid i \leqslant n_{\vec{p},q} \rangle$. We now let $(\vec{p}, q) \in D$ if $n_{\vec{p},q} = k$, i.e. p_k appears in the splitting sequence.

The projection constructs a tree from the splitting sequence. Given any $(\vec{p}, q) \in D$, we now define $\rho(\vec{p}, q) = (\vec{t}, u)$ with $\vec{t} = ((t_0, t_\star), t_1, \ldots, t_l)$. Let $t_0 = \bigcup_{i \leqslant k} p_{n_i} \cup \{q \restriction j \mid j \leqslant |q|\}$ and $t_\star = q$. If $\vec{m} = \vec{m}_{\vec{p},q} = \langle m_i \mid i < m_{\vec{p},q} \rangle$ is the order preserving enumeration of $[0, k] \setminus \{n_i \mid i \leqslant n_{\vec{p},q}\}$, let $t_{j+1} = p_{m_j}$ for $j < m_{\vec{p},q}$. Since s_{n_k} is incompatible with q, $(t_0, t_\star)$ is a condition in $\mathbb{T}_\star$ and hence $\rho(p, q) \in \mathbb{T}_\star \times \mathrm{Add}(\omega, 1)$. Moreover, let $u(i) = 0$ if m_i is even and $u(i) = 1$ if it is odd for all $i < m_{\vec{p},q}$.

Subclaim. ρ is a projection.

Proof. It is clear that ρ is a homomorphism. To see that ρ is a projection, suppose that $\rho(\vec{p}, q) = (\vec{t}, u)$ with $\vec{t} = ((t_0, t_\star), t_1, \ldots, t_l)$, $\vec{v} = ((v_0, v_\star), v_1, \ldots, v_m) \leqslant \vec{t}$ and $w \leqslant u$. We will find some $(\vec{a}, b) \leqslant (\vec{p}, q)$ with $\rho(\vec{a}, b) \leqslant (\vec{v}, w)$ and start with defining $\vec{a}$. Let $a_{m_i} = v_i$ for $1 \leqslant i < m_{\vec{p},q}$ and $a_{n_i} = v_0 \cap \mathrm{end}(p_{n_i})$ for $i \leqslant k$. It remains to define a_i for $i > k$. To this end, consider the set K of all nodes in v_0 that are direct successors of nodes in $[t_\star, v_\star) = \{r \in 2^{<\omega} \mid t_\star \sqsubseteq r \subsetneq v_\star\}$, but incompatible with $v_\star$. Note that K contains at most one node of each length. Let $\vec{r} = \langle r_i \mid i < j \rangle$ enumerate the elements of K in the order of increasing length. We define $a_{k+i+1} = v_0 \cap \mathrm{end}(r_i)$ for $i < j$. We can assume that $|\mathrm{dom}(w) \setminus \mathrm{dom}(u)| \leqslant j$ by extending $(v_0, v_\star)$. Using this, we add the trees v_i for $k < i \leqslant k+j$ to $\vec{a}$ in this order and for each of them, we first add a tree to the splitting sequence to adjust the parity, if necessary. We thus obtain $b \leqslant v_\star$ and an extension $\vec{a}$ of the previously defined values with $(\vec{a}, b) \leqslant (\vec{p}, q)$ and $\rho(\vec{a}, b) \leqslant (\vec{v}, w)$. □

The (a) and (b) in the last claim about rearranging $M[G \times H]$ as $M[g \times h]$ and conversely now follow similarly as in the proof of the previous claim. □

Finally, there is a $\mathbb{T}^\omega$-generic filter g over M in V with $C = \dot{C}^g \in A$, since A is comeager. We pick some $p \in U_\sigma$ and let h be any Cohen generic filter over $M[g]$ in V with $p \in h$ and let $x = \bigcup h$. Since x is a Cohen real over $M[g]$, it is easy to see by a density argument that $x \notin B_C$. Since $p \in U_\sigma$, the previous two claims show that we can rearrange $M[g \times h]$ as a $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$-generic extension $M[G \times H]$ with $\sigma^G = x$. We have $\sigma^G \in B_{F(C)}$ by the choice of σ and Borel absoluteness between $M[G]$ and V. Thus $B_{F(C)} \setminus B_C$ contains every Cohen real over $M[g]$ in V and hence it is uncountable by Lemma 5.6. But this contradicts the assumption that F is a selector. □

The next results use the following principle. Let *internal projective Cohen absoluteness* ($\mathsf{IA}^{\mathbb{C}}_{\mathrm{proj}}$) denote the statement that $H^{M[g]}_{\omega_1} \prec H_{\omega_1}$ holds for all sufficiently large regular cardinals θ, countable elementary submodels $M \prec H_\theta$ and Cohen generic filters g over M in V.

We will only need the first part of the next lemma; the second part is an easy observation. Recall that PD denotes the axiom of projective determinacy.[19]

Lemma 5.7. (1) PD *implies* $\mathsf{IA}^{\mathbb{C}}_{\mathrm{proj}}$.
(2) *If* $\mathsf{IA}^{\mathbb{C}}_{\mathrm{proj}}$ *holds, then all projective set have the property of Baire.*

Proof. (1) Take a $\boldsymbol{\Sigma}^1_{2n}$-universal Σ^1_{2n} subset A_n of $({}^\omega 2)^2$ for each $n \geqslant 1$. If the Σ^1_{2n}-formula $\varphi_n(x, y)$ defining A_n is chosen in a reasonable way, then for any Σ^1_{2n}-formula $\psi(x)$ with (hidden) real parameters, there is some $y_\psi \in {}^\omega 2$ with $\forall x\ (\psi(x) \iff \varphi_n(x, y_\psi))$. Moreover, this holds in all transitive models of ZFC^- and the map $\psi \mapsto y_\psi$ is absolute between such models. We assume that in any transitive model of ZFC^-, A_n denotes the set defined by $\varphi_n(x, y)$.

Let $M \prec H_\theta$ be a countable elementary submodel for some large enough regular cardinal θ. By [Mos09, Corollary 6C.4] and PD, there is a Σ^1_{2n}-scale $\sigma_n = \langle \leqslant^m_n \mid m \in \omega \rangle$ on A_n for each $n \geqslant 1$. Let $r^m_n(x)$ denote the rank of $x \in A_n$ with respect to $\leqslant^m_n$. Moreover, recall that the *tree of the scale* σ_n is defined as[20]

$$T_{\sigma_n} = \{(x \restriction m, (r^0_n(x), ..., r^{m-1}_n(x))) \mid x \in A_n,\ m \in \omega\}$$

and $p[T_{\sigma_n}] = A_n$. Since $\mathsf{ZFC}^- + \mathsf{PD}$ is sufficient to prove the existence of such scales, let T_{σ_n} denote the tree defined via $\varphi_n(x, y)$ in any transitive model of this theory containing $\mathcal{P}(\omega)$ as an element. Let $g \in V$ be Cohen generic over M.

Claim. $M[g] \models A_n = p[T_{\sigma_n}]$.

Proof. $p[T_{\sigma_n}]$ has a projective definition via the above definitions of T_{σ_n} and A_n. Thus the claim holds by 1-step projective Cohen absoluteness in M [Woo82, Lemma 2]. □

Claim. $T^M_{\sigma_n} = T^{M[g]}_{\sigma_n}$.

Proof. By PD any projective pre-wellorder E is *thin*, i.e. there is no perfect set of reals that are pairwise inequivalent with respect to E. By 1-step projective Cohen absoluteness [Woo82, Lemma 2], [Sch14, Lemma 3.18] and PD, Cohen forcing does not add new equivalence classes to E. Equality now follows from [Sch14, Theorem 5.15]. The inclusion $T^M_\sigma \subseteq T^{M[g]}_\sigma$ holds since the rank function r_n is upwards absolute from M to $M[g]$ by the previous statement for $E = \leqslant_n$. The converse inclusion is proved in [Sch14, Claim 5.16] from PD. □

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[19] See [Jec03, Chapter 33].

[20] The tree of a scale is defined in the discussion before [Mos09, Theorem 8G.10].

We now show that $\mathsf{IA}^{\mathbb{C}}_{\mathrm{proj}}$ holds. Assume that $\psi(x)$ is a Σ^1_{2n}-formula and $x \in M[g]$. Note that the equivalences $\psi(x) \iff \varphi_n(x, y_\psi) \iff (x, y_\psi) \in A_n \iff (x, y_\psi) \in p[T_{\sigma_n}]$ hold in V and in $M[g]$ by the first claim. It remains to show that $M[g] \models (x, y_\psi) \in p[T_{\sigma_n}]$ if and only if $(x, y_\psi) \in p[T_{\sigma_n}]$ holds in V.

We claim that $T^{M[g]}_{\sigma_n} = T_{\sigma_n} \cap M[g]$. To see this, note that $T^M_{\sigma_n} = T_{\sigma_n} \cap M$ since $M \prec H_\theta$. Using the second claim and the fact that $\mathrm{Ord}^M = \mathrm{Ord}^{M[g]}$, we obtain $T^{M[g]}_{\sigma_n} = T^M_{\sigma_n} = T_{\sigma_n} \cap M = T_{\sigma_n} \cap M[g]$. By absoluteness of wellfoundedness, we have $M[g] \models (x, y_\psi) \in p[T_{\sigma_n}] \iff (x, y_\psi) \in p[T^{M[g]}_{\sigma_n}] \iff (x, y_\psi) \in p[T_{\sigma_n}]$.

(2) Since M is countable, the set of Cohen reals over M in V is comeager. Therefore, this claim holds by the argument for the Baire property in Solovay's model. ☐

The above argument for the non-existence of a selector with Borel values can now be used for the following results. We fix codes for $\mathbf{\Sigma}^1_n$ sets via $\mathbf{\Sigma}^1_n$-universal Σ^1_n sets.

Theorem 5.8. *Assuming $\mathsf{IA}^{\mathbb{C}}_{\mathrm{proj}}$, there is no Baire measurable selector for* I *with projective values.*

Proof. This is proved similarly to Theorem 5.5. We only indicate the two necessary changes. In the proof of the first claim, $\mathsf{IA}^{\mathbb{C}}_{\mathrm{proj}}$ implies that the (projective) statement $[\dot{T}^g_n] \cap B_{F(\dot{C}^g)} = [\dot{T}^g_n] \cap B_{\dot{F}^g(\dot{C}^g)} = \varnothing$ is absolute between $M[g]$ and V. The second change is at the very end of the proof. Here projective absoluteness between $M[g \times h]$ and V by $\mathsf{IA}^{\mathbb{C}}_{\mathrm{proj}}$ guarantees that $x = \sigma^G \in B_{F(\dot{C}^g)}$ and thus $B_{F(\dot{C}^g)} \setminus B_{\dot{C}^g}$ contains every Cohen real over $M[g]$ in V. As before, this contradicts Lemma 5.6 and the fact that F is a selector for I. ☐

Note that by Lemma 5.7, PD is sufficient for the previous result. Thus PD implies that there is no projective selector for I. Note that some assumption beyond ZFC is necessary to prove this statement. For instance, in L there are projective selectors for all projectively coded ideals I, since there is a projective, in fact a Σ^1_2, wellorder of the reals.

The previous arguments can also be used to show that it is consistent with ZF that there is no selector at all for I.

Suppose that κ is an uncountable cardinal and G is a $\mathbb{P}$-generic filter over V for $\mathbb{P} = \mathrm{Add}(\kappa, 1)$ or $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$. We call $V^\star = \bigcup_{\alpha<\kappa} V[G{\restriction}\alpha]$ a *symmetric extension* for $\mathbb{P}$. Note that $V^\star = HOD^{V[G]}_{V^\star}$ by homogeneity and for cardinals κ of uncountable cofinality, $\mathbb{R}^{V^\star} = \mathbb{R}^{V[G]}$.

Theorem 5.9. *There are no selectors for* I *in the following models of* ZF.

(a) *Symmetric extensions $V^\star$ for $\mathrm{Add}(\omega, \kappa)$ and $\mathrm{Col}(\omega, {<}\kappa)$, where κ is any uncountable cardinal.*
(b) *Solovay's model.*
(c) *$L(\mathbb{R})$ assuming $\mathsf{AD}^{L(\mathbb{R})}$.*

Proof. The proof of the first claim is similar to that of to Theorem 5.5.

Suppose that F is a selector for I in $V^\star$. It is definable from an element x_0 of $V[G{\restriction}\alpha]$ for some $\alpha < \kappa$. We can further assume that $x_0 \in V$. Hence F is continuous on the set of Cohen reals over V in $V^\star$; let $\dot{F}$ be a name for this function.

We follow the proof of Theorem 5.5 but work with V instead of M. The first claim in the proof of Theorem 5.5 is replaced by the following claim.

We will write $V^\star$ for a $\mathbb{P}$-name for $V^\star$ to keep the notation simple.

Claim. $1 \Vdash^V_{\mathbb{T}^\omega \times \mathbb{P}} V^\star \models [\dot{T}_n] \cap B_{\dot{F}(\dot{C})} \neq \varnothing$ for all $n \in \omega$.

Proof. Assume towards a contradiction that $(p, q) \Vdash^V_{\mathbb{T}^\omega \times \mathbb{P}} V^\star \models [\dot{T}_n] \cap B_{\dot{F}(\dot{C})} = \varnothing$ for some $p \in \mathbb{T}^\omega$, $q \in \mathbb{P}$ and $n \in \omega$. We can assume $q = 1$ by homogeneity. Let $g \times h$ be a $\mathbb{T}^\omega \times \mathbb{P}$-generic filter over V with $p \in g$ whose symmetric model is $V^\star$. Then $[\dot{T}^g_n] \cap B_{F(\dot{C}^g)} =$

$[\dot{T}^g_n] \cap B_{\dot{F}^g(\dot{C}^g)} = \varnothing$. But this contradicts the assumption that F is a selector with respect to I. $\square$

There is a $\mathbb{T}^\omega$-name σ in V that is forced to be an element of $[\dot{T}_n] \cap B_{\dot{F}(\dot{C})} \cap V^\star$ by the previous claim. The next steps of the proof are as before.

In the end of the proof, we rearrange $V[g \times h]$ as a $\mathbb{T}^\omega \times \mathrm{Add}(\omega, 1)$-generic extension $V[G \times H]$ with $\sigma^G = x$ as before. Then $\sigma^G \in B_{F(\dot{C}^g)}$ by the choice of σ. Thus $B_{F(\dot{C}^g)} \setminus B_{\dot{C}^g}$ contains every Cohen real over $V[g]$ in $V^\star$. Since there are uncountably many Cohen reals over $V[g]$ in $V^\star$, this contradicts the assumption that F is a selector.

The second claim holds since Solovay's model is obtained via a symmetric extension for $\mathrm{Col}(\omega, {<}\kappa)$. Note that this model is also a $\mathrm{Add}(\omega, \omega_1)$-generic extension of an intermediate model.

The last claim follows from [SS06, Theorem 0.1] and the first claim. By this theorem and $\mathsf{AD}^{L(\mathbb{R})}$, $L(\mathbb{R})^V = L(\mathbb{R})^{V^\star}$ for some symmetric extension $V^\star$ for $\mathrm{Col}(\omega, {<}\kappa)$ for some κ over some ground model N which is an inner model of some generic extension of V. $\square$

5.3. **Density points via convergence.** In this section we discuss the notion of density point introduced in [PWBW85]. We show that this notion does not satisfy the analogue of Lebesgue's Density Theorem for any of the tree forcings listed in the next section except Cohen and Random forcing.

Lebesgue's density theorem was generalized to the σ-ideals of meager sets on Polish metric spaces in [PWBW85, Theorem 2]. To this end, a notion of density points for ideals was introduced. This notion is based on the following measure theoretic lemma.

Lemma 5.10. *Suppose that (X, d, μ) is a metric measure space, f is a Borel-measurable function and $\vec{f} = \langle f_n \mid n \in \omega\rangle$ is a sequence of Borel-measurable functions from (X, d) to $\mathbb{R}$. The following statements are equivalent.*

(a) *$f_n \to f$ converges in measure, i.e. for all $\epsilon > 0$ we have*
$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geqslant \epsilon\}) = 0.$$
(b) *Every subsequence of $\vec{f}$ has a further subsequence that converges pointwise μ-almost everywhere.*

Condition (b) is suitable for generalizations to other ideals, since it only mentions the ideal of null sets, but not the measure itself.

Definition 5.11. Let I be a σ-ideal on ${}^\omega X$ and $A \subseteq {}^\omega X$.

(1) For each $n \in \omega$ let f_n be the characteristic function of $\sigma^{-1}_{x\restriction n}(A \cap N_{x\restriction n})$. Define x to be an I-*convergence density point* of A if every subsequence of $(f_n)_{n\in\omega}$ has a further subsequence which converges I-almost everywhere to the constant function ${}^\omega X \to \{1\}$ (i.e., converges everywhere on a set A such that ${}^\omega X \setminus A \in \mathrm{I}$).
(2) By the I-*convergence density property* we mean the statement that for every $B \in \textsc{Borel}({}^\omega X)$ it holds that $B \Delta B' \in \mathrm{I}$ where B' is the set of I-convergence density points of B.

We first consider tree forcings on ${}^\omega\omega$. For $s \in {}^{<\omega}\omega$ and $f_0, \ldots, f_m \in {}^\omega\omega$, let
$$T_{s,f_0,\ldots,f_m} = \{t \in C_s \mid \forall i \leqslant m\ \forall n \geqslant |s|\ t(n) \neq f_i(n)\}$$
(cf. Definition 6.1 (f) below) and let $f \in {}^\omega\omega$ denote the constant function with value 0.

Lemma 5.12. *If $N_{\langle 0\rangle}$ and $[T_{\varnothing,f}]$ are* I*-positive, then the* I*-convergence density property fails. In particular, this holds for $\mathrm{I}_\mathbb{H}$, $\mathrm{I}_\mathbb{E}$, $\mathrm{I}_\mathbb{M}$ and $\mathrm{I}_{\mathbb{L}_A}$ if A contains all cofinite sets.*

Proof. We claim that no $x \in {}^\omega\omega$ is an I-convergence density point of $[T_{\varnothing,f}]$. We have $\sigma_t^{-1}([T_{\varnothing,f}]) = [T_{\varnothing,f}]$ if $t \in T_{\varnothing,f}$ and $\sigma_t^{-1}([T_{\varnothing,f}]) = \varnothing$ otherwise. Hence $\sigma_t^{-1}([T_{\varnothing,f}]) \cap N_{\langle 0\rangle} = \varnothing$ for all $t \in {}^{<\omega}\omega$. Thus x is not an I-convergence density point of $[T_{\varnothing,f}]$. $\square$

We now turn to tree forcings on ${}^{\omega}2$. For $s \in 2^{<\omega}$ and $N \in [\omega]^{\omega}$ let

$$T_{s,N} = \{t \in {}^{<\omega}2 \mid t \in C_s \ \&\ \forall n\ t(n) = 1 \Rightarrow n \in N\}$$

(cf. Definition 6.1 (h) below).

Lemma 5.13. *Suppose that $N_{\langle 1\rangle}$ and $[T_{\varnothing,N}]$ are* I*-positive for some infinite set N. Then the* I*-convergence density property fails. In particular, this holds for $\mathbb{R}_A$ and $\mathbb{V}_A$ for any subset A of $\mathcal{P}(\omega)$ that contains at least one infinite set.*

Proof. It is sufficient to show that no $x \in [T_{\varnothing,N}]$ is an I-convergence density point of $[T_{\varnothing,N}]$. We have $\sigma^{-1}_{x\upharpoonright n}([T_{\varnothing,N}]) \cap N_{\langle 1\rangle} = \varnothing$ for all $n \notin N$. Since $N_{\langle 1\rangle}$ is I-positive, any infinite strictly increasing sequence in N witnesses that x is not an I-convergence density point. □

The next lemma takes care of the remaining tree forcings.

Lemma 5.14. *Suppose that* I *is an ideal on ${}^{\omega}2$ with $\mathrm{I}_{\mathbb{S}} \subseteq \mathrm{I} \subseteq \mathrm{I}_{\mathbb{R}}$. Then the* I*-convergence density property fails.*

Proof. Let

$$T = \{t \in 2^{<\omega} \mid \forall i \in \omega\ (t(i) = 1 \Rightarrow \exists j \in \omega\ (i = 2^j + 1))\}.$$

It is sufficient to show that no $x \in {}^{\omega}2$ is an I-convergence density point of $[T]$. Otherwise there is a strictly increasing sequence $\vec{n} = \langle n_i \mid i < \omega\rangle$ such that

$$A = \{y \in {}^{\omega}2 \mid \exists i\ \forall j \geqslant i\ y \in \sigma^{-1}_{x\upharpoonright n_j}([T])\}$$

is co-countable. Let

$$A_{i,j} = \sigma^{-1}_{x\upharpoonright n_i}([T]) \cap \sigma^{-1}_{x\upharpoonright n_j}([T])$$

for $i < j$ in ω. Since $A \subseteq \bigcup_{i,j\in\omega,\ i<j} A_{i,j}$ is uncountable, $A_{i,j}$ is uncountable for some $i < j$. Let $a <_{\mathrm{lex}} b <_{\mathrm{lex}} c$ be elements of $A_{i,j}$. We denote the longest common initial segment of $d, e \in 2^{\leqslant\omega}$ by $d \wedge e$. We can assume without loss of generality that $a \wedge b = s$, $a \wedge c = b \wedge c = t$ and $s \subsetneq t$. Then s and t are splitting nodes in $\sigma^{-1}_{x\upharpoonright n_i}(T)$ and $\sigma^{-1}_{x\upharpoonright n_j}(T)$. Hence $s + n_i$, $s + n_j$, $t + n_i$, $t + n_j$ are of the form $2^k + 1$. Since $s \neq t$, $n_j - n_i$ can be written in the form $2^k - 2^l$ in two different ways. But this contradicts the easy fact that k, l are uniquely determined by $2^k - 2^l$. □

Let $\mathbb{P}_{E_0}$ denote *E_0-forcing* [Zap08, Section 4.7.1] and $\mathbb{W}$ an appropriate representation of *Willowtree forcing* [Bre95, Section 1.1]. Since $\mathbb{R} \subseteq \mathbb{V} \subseteq \mathbb{W} \subseteq \mathbb{P}_{E_0} \subseteq \mathbb{S}$, the previous result holds for the ideals $\mathrm{I}_{\mathbb{P}}$ associated to these forcings as well.

6. A list of tree forcings

We review definitions of some tree forcings for the reader's convenience. If $N \subseteq \omega$ and $m \in \omega$, we write $m + N = \{m + n \mid n \in N\}$. Moreover, a subset A of $\mathcal{P}(\omega)$ is called *shift invariant* if $N \in A \Longleftrightarrow m + N \in A$ for all $m \in \omega$.

Definition 6.1. Assume that A is a subset of $\mathcal{P}(\omega)$.

(a) *Random forcing* is the collection of perfect subtrees T of ${}^{<\omega}2$ with $\mu([T_s]) > 0$ for all $s \in T$ with $\mathrm{stem}_T \sqsubseteq s$.
(b) *Cohen forcing* $\mathbb{C}$ is collection of *cones* $C_s = \{t \in {}^{<\omega}\omega \mid s \sqsubseteq t \text{ or } t \sqsubseteq s\}$ for $s \in {}^{<\omega}\omega$.
(c) *Sacks forcing* $\mathbb{S}$ is the collection of all perfect subtrees of $2^{<\omega}$.
(d) *Miller forcing* $\mathbb{M}$ is the collection of *superperfect* subtrees T of ${}^{<\omega}\omega$. This means that above every node in T there is some infinitely splitting node t in T, i.e. t has infinitely many direct successors.
(e) *Hechler forcing* $\mathbb{H}$ is the collection of trees

$$T_{s,f} = \{t \in C_s \mid \forall n \geqslant |s|\ t(n) \geqslant f(n)\}$$

for $s \in {}^{<\omega}\omega$ and $f \in {}^{\omega}\omega$.

(f) *Eventually different forcing* $\mathbb{E}$ is the collection of trees

$$T_{s,f_0,\ldots,f_m} = \{t \in C_s \mid \forall i \leqslant m \ \forall n \geqslant |s| \ t(n) \neq f_i(n)\}$$

for $s \in {}^{<\omega}\omega$ and $f_0, \ldots, f_m \in {}^{\omega}\omega$.

(g) *A-Laver forcing* $\mathbb{L}_A$ is the collection of subtrees T of ${}^{<\omega}\omega$ such that for every $t \in T$ with $\mathrm{stem}_T \sqsubseteq t$, the set $\mathrm{succ}_T(t)$ of direct successors of t in T is an element of A. *Laver forcing* is $\mathbb{L}_F$ for the Fréchet filter F of cofinite sets.

(h) *A-Mathias forcing* $\mathbb{R}_A$ is the collection of trees

$$T_{s,N} = \{t \in {}^{<\omega}2 \mid t \in C_s \ \& \ \forall n \ t(n) = 1 \Rightarrow n \in N\}$$

for $s \in 2^{<\omega}$ and $N \in A$. *Mathias forcing* $\mathbb{R}$ is $\mathbb{R}_A$ for $A = [\omega]^\omega$.

(i) *A-Silver forcing* $\mathbb{V}_A$ is the collection of trees

$$T_f = \{t \in {}^{<\omega}2 \mid \forall n \in \mathrm{dom}(t) \cap \mathrm{dom}(f) \ f(n) \leqslant t(n)\},$$

where $\mathrm{dom}(f) = \omega \backslash N$ for some $N \in A$. Silver forcing $\mathbb{V}$ is $\mathbb{V}_A$ for $A = [\omega]^\omega$.

All of these satisfy the above condition for collections of trees $\mathbb{P}$ that $T_s \in \mathbb{P}$ for all $T \in \mathbb{P}$ and $s \in T$. Moreover, Random forcing, Cohen forcing, Sacks forcing, Hechler forcing, eventually different forcing, and A-Laver forcing are *shift invariant* in the sense that for all T and $s \in 2^{<\omega}$, $T \in \mathbb{P} \iff \sigma_s(T) \in \mathbb{P}$. If A is shift invariant, then $\mathbb{R}_A$ and $\mathbb{V}_A$ are also shift invariant.

Note that Cohen forcing, Hechler forcing, eventually different forcing, Laver forcing, and Silver forcing are topological, while Random forcing, Sacks forcing, and Miller forcing are not.

7. Open problems

We end with some open questions. The main one asks about equivalence of the properties discussed in the introduction.

Question 7.1. *Are the shift density property, the existence of a simply definable selector and the ccc equivalent for all simply definable σ-ideals?*

For more examples of ideals, see [Zap08]. We further suggest to study the relationship between the shift density property and the condition that the collection of Borel sets modulo I carries a natural Polish metric. For the previous question, we suggest to generalize the proof idea of Theorem 5.5 to other non-ccc ideals such as the K_σ-ideal.

Question 7.2. *Is there a Baire measurable selector with Borel values for the K_σ-ideal?*

Lemma 4.7 showed that the $\mathbb{P}$-measurable sets form a σ-ideal if $\mathbb{P}$ has the ω_1-covering property. If this assumption can be omitted, this would in particular imply a positive answer to the next question.

Question 7.3. *Are all Borel sets $\mathbb{P}$-measurable for all tree forcings $\mathbb{P}$?*

Note that any counterexample $\mathbb{P}$ collapses ω_1 if we assume CH; then $\mathbb{P}$ preserves ω_1 if and only if it has the ω_1-covering property by a similar argument as in the proof of Lemma 4.4.

The last question is about the notion of generic absoluteness introduced in Section 5.2. Lemma 5.7 shows that this follows from PD. Our results leave the consistency strength of this statement open.

Question 7.4. *Does $\mathrm{IA}^{\mathbb{C}}_{\mathrm{proj}}$ imply that $0^\#$ exists?*

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