Idiosynchromatic Poetry

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June 14, 2014

Abstract

We prove equivalences of asymmetric partition relations involving natural numbers and products of weakly compact cardinals κ, infinite cardinals $\lambda < \kappa$ and natural numbers to certain classes of finitary problems in the theory of edge-coloured digraphs. We are able to determine three classes of Ramsey numbers exactly, typical examples in these classes are $r(\omega^2 2, 3)$, $r(\kappa\lambda 2, 3)$ and $r(\kappa\lambda 3, 3)$. We moreover provide general upper bounds for $r(\omega^2 m, 3)$ and $r(\kappa\lambda m, n)$.

Keywords: partition, digraph, ordinal, weakly compact, Ramsey, edge-coloured.

Mathematical Subject Classification of the AMS: 03E02, 05C20, 05C63, 05D10.

1 Introduction

The partition calculus as introduced by Erdős and Rado in [56ER] is of interest both to set theorists and combinatorialists. While the former are usually more interested in statements with consistency strength which are hence independent over ZFC the latter are normally interested in finitary problems. In between falls a class of problems the æsthetic appeal of which is in the author's opinion usually underestimated—partition relations between countable ordinals.

[0] This title is a tribute both to my PhD advisor who regarded this work at an early stage as poetry—the author concedes but hopes it to be good poetry—and to David Schrittesser who came up with the word "idiosynchromatic".

Notation 1.1. *In this paper we use the terminology which is nowadays pretty common. Ordinals are von-Neumann-ordinals so in particular an ordinal is identical to the set of its ordinal predecessors. Therefore* $\alpha \setminus \beta$ *denotes the half-open interval from* β *to* α *containing the former but not the latter.* Ω *denotes the class of all ordinals.* $[\alpha]^\beta$ *denotes the set of subsets of* α *with order type* β*.* $c``X$ *for* $X \subset \mathrm{dom}(c)$ *is defined to be* $\{y \in \mathrm{ran}(c) | \exists x \in X : c(x) = y\}$ *i.e. the pointwise image of* X *under the function* c*. Recall that the asymmetric partition relation* $\alpha \to (\beta, n)$ *means* $\forall c : [\alpha]^2 \longrightarrow 2 \exists X \in [\alpha]^\beta : c``X = \{0\} \vee \exists X \in [\alpha]^n : c``X = \{1\}$, *i.e. for every colouring of the unordered pairs of ordinals less than* α *with two colours we find a subset of* α *of order type* β *homogeneous in colour* 0 *or a subset of* α *of order type* n *homogeneous in colour* 1.

Moreover we employ the common notation for Ramsey numbers also in this more general case. So $r(\beta, n) = \alpha$ *means that* α *is the least ordinal such that* $\alpha \to (\beta, n)$ *holds true.*

When v, w *are vertices of a directed graph* $v \mapsto_b w$ *means that there is a solid black arrow from* v *to* w *while* $v \mapsto_g w$ *says that there is a solid grey arrow from* v *to* w, $v \mapsto_d w$ *claims the existence of a dashed black arrow from* v *to* w *and finally* $v \mapsto w$ *means that there is an arrow of some kind from* v *to* w*. Sometimes we want to talk about there being an arrow of a certain colour between* v *and* w *without claiming anything about its direction. For this we write* $v -_b w$*—there is a solid black arrow between* v *and* w, $v -_g w$*—there is a solid grey arrow between* v *and* w *or* $v -_d w$*—there is a dashed black arrow between* v *and* w*. The solid black in-neighborhoods of a vertex* v *will be denoted by* $N_B^-(v)$, *the solid black out-neighborhoods by* $N_B^+(v)$ *and* $N_G^-(v)$, $N_G^+(v)$, $N_D^-(v)$ *and* $N_D^+(v)$ *have their expected denotations. The set of vertices independent from* v *will be denoted by* $I(v)$.

In this paper, a digraph has at most one arrow between any two vertices and never any from a vertex to itself.

For the graph-theoretic terminology we follow [09BG]. Some of the notation was inspired by [09GS].

People working in this area seem to have been mainly interested in evaluating partition relations of the form $\alpha \to (\alpha, n)$ for countable α and natural n. There are three cases in which partition relations of the form $\alpha \to (\beta, n)$ for $\beta < \alpha$ both countable and n natural have been analysed in some generality. This was the case for α, β finite powers of ω, for α, β finite multiples of ω and for α, β natural—the last case of course being by far the most popular. In this last case people went to great lengths to calculate the $r(m, n)$'s, see for example [95MR]! There are a few results which do not fall into one of these categories, for example in [69Mi], [69HSc], [69HSa] and [69HSb].

Definition 1.2. *A* coloured digraph *is a triple* $\langle V, A, c \rangle$ *such that* V *is a set(of vertices),* A *is a binary irreflexive asymmetric relation on* V*—a set of arcs—and* $c : A \longrightarrow \mathrm{ran}(c)$ *is a colouring of* A.

We are in the following going to be concerned with arc-coloured digraphs, i.e. directed graphs with coloured arcs. We will be especially interested in three-coloured digraphs, i.e. in the case where $\mathrm{ran}(c) = 3$. Remember that a complete digraph is called a *tournament*.

For brevity in the following we will speak of *triples* instead of three-person-three-coloured-tournaments.

In [56ER] Erdős and Rado prove the following theorem[1]

Theorem 1.3. *Let $m, n \in \omega \setminus 2$ and denote by $l_0 = l_0(m,n)$ the least finite number l possessing the following property.*

Property P_{mn}. Whenever $\rho(\lambda,\mu) < 2$ for $\{\lambda,\mu\} \in [l]^2$, then there is either $\{\lambda_0,\ldots,\lambda_{m-1}\} \in [l]^m$ such that $\rho(\lambda_\alpha,\lambda_\beta) = 0$ for $\alpha < \beta < m$ or there is $\{\lambda_0,\ldots,\lambda_{n-1}\} \in [l]^n$ such that $\rho(\lambda_\alpha,\lambda_\beta) = 1$ for $\{\alpha,\beta\} \in [n]^2$. Then $\omega l_0 \to (m,\omega n)^2$ but $\gamma \not\to (m,\omega n)^2$ for $\gamma < \omega l$. Moreover, if $l_1 \to (m,m,n)^2$, then $l_0 \leqslant l_1$.

Note, however, that a possible alternative formulation of all but the last sentence of this theorem would nowadays be the following.

Theorem 1.4. *The partition relation $\omega l \to (\omega m, n)$ holds true if and only if for every digraph C on l vertices contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are transitive.*

These two theorems are equivalent by a well-known fact.

Fact 1.5. *If a tournament has a cycle then it has one of length* 3.

Later theorem 1.3 was generalized by Baumgartner, see [74Ba], in the following way:

Theorem 1.6. *Let κ be any infinite cardinal. The partition relation $\kappa l \to (\kappa m, n)$ holds true if and only if for every digraph C on l vertices contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are transitive.*

1.1 Added in Proof

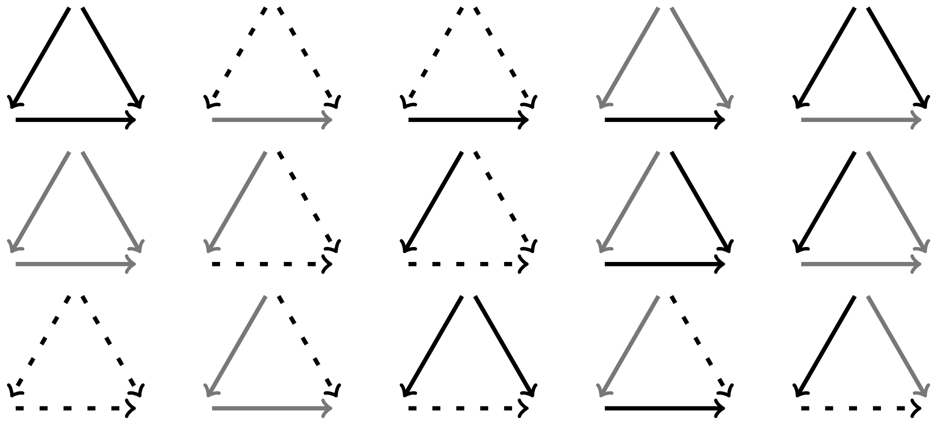
The author believed for quite some time that the discovery that $r(\omega^2 2, 3) = \omega^2 10$ was due to him. Only later he discovered that it already appeared in [69HSb]. No proof was given though and there was also neither a formulation of a finitary problem equivalent to the calculation of the $r(I_m, A_n)$'s nor an upper bound for the $r(I_m, A_3)$'s given.

2 Determining a Ramsey number

In this section we mainly build on work of Pál Erdős and Richard Rado in the aforementioned [56ER] and Ernst Specker in [57Sp]. The following eclectic definition is justified by theorem 2.6 the proof of which explains the background of it.

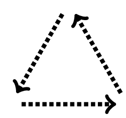
[1]The author changed the notation partially to his own. This Theorem was first published as Theorem 25 in "A Partition Calculus in Set Theory" in the Bulletin of the American Mathematical Society in No. 62, pp. 427–489, published by the American Mathematical Society, ©Pál Erdős and Richard Rado, 1956.

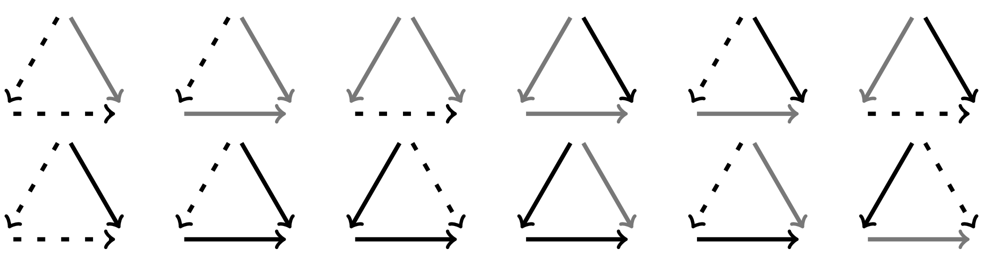
Definition 2.1. *We identify* 0 *with solid black,* 1 *with solid grey and* 2 *with dashed black arrows.*[2] *A triple is called* agreeable *if and only if it is one of the following.*



If a triple is not agreeable we call it *disagreeable*. Note that all agreeable triples are *transitive* rather than *cyclic*, i.e. there is both a vertex emitting two arrows and a vertex absorbing two arrows.

Fact 2.2. *A triple is disagreeable if and only if it is either...*

- ♢ *... a cyclic triple, regardless of the colouring, i.e.* 
- ♢ *... or one of the following:*



We remind the reader of the definition of weakly compacts.

Definition 2.3. $\kappa \in \Omega \setminus 3$ *is called weakly compact iff* $\kappa \to (\kappa)^2_\omega$.

Fact 2.4. *For all* $\kappa \in \Omega$ *we have* $\kappa \to (\kappa)^2_2$ *iff* $\kappa \in \{0, 1, 2, \omega\}$ *or* κ *is weakly compact.*

Fact 2.5. $\kappa \to (\kappa)^2_2$ *iff* $\forall n < \omega, \alpha < \kappa : \kappa \to (\kappa)^n_\alpha$

Both facts are corollaries of theorem 7.6 of [03Ka] which can be found on page 76.

[2] It is planned to make a coloured postprint version of this paper available on the web.

Theorem 2.6. *Let $\kappa \in \Omega \setminus 3$ satisfy $\kappa \to (\kappa)^2_2$.*

The partition relation $\kappa^2 l \to (\kappa^2 m, n)$ holds true if and only if every coloured digraph $C = \langle l, A, c\rangle$ with $\operatorname{ran}(c) = 3$ contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are agreeable.

Proof.

◇ Let us first assume that the finite combinatorial characterization above holds true. Given any colouring $\chi : [\kappa^2 l]^2 \longrightarrow 2$ we define a new colouring χ' as follows:

$$\begin{aligned}
\chi' : [\kappa]^4 &\longrightarrow 2^{4l^2}\\
\{h, j, i, k\}_< &\longmapsto \sum_{f,g<l} \big(\chi(\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\})2^{4(lf+g)}\\
&\qquad +\chi(\{\kappa^2 f + \kappa h + i, \kappa^2 g + \kappa j + k\})2^{4(lf+g)+1}\\
&\qquad +\chi(\{\kappa^2 f + \kappa h + k, \kappa^2 g + \kappa j + i\})2^{4(lf+g)+2}\\
&\qquad +\chi(\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa h + i\})2^{4(lf+g)+3}\big)
\end{aligned}$$

Now we use fact 2.5 in the form $\kappa \to (\kappa)^4_{2^{4l^2}}$ thereby finding an $X \in [\kappa]^\kappa$ homogeneous for χ'.

Now fix a surjection $b : X \twoheadrightarrow \{\langle \nu, \gamma\rangle | \nu < l \wedge \gamma < \kappa + 1\}$ and monotonic enumeration functions $e_\nu : \kappa \longleftrightarrow b^{-1}``(\{\langle \nu, \kappa\rangle\})$ such that

- $\forall \nu < l \forall \gamma < \kappa + 1 : \overline{\overline{b^{-1}``(\{\langle \nu, \gamma\rangle\})}} = \kappa$.
- $\forall \nu < l \forall \gamma < \kappa \forall \delta \in b^{-1}``(\{\langle \nu, \gamma\rangle\}) : e_\nu(\gamma) < \delta$.

Now let $Y := \{\kappa^2 f + \kappa h + j | f < l \wedge h \in b^{-1}(\langle f, \kappa\rangle) \wedge j \in b^{-1}(\langle f, e_f^{-1}(h)\rangle)\}$.

In prose what's happening is this. We distribute the elements of X to signify multiples of 1 or κ in the l blocks of size κ^2. No two ordinals may appear in two roles and within one κ^2-block the 1-coordinate is always larger than the κ-coordinate. Note that $Y \in [\kappa^2 l]^{\kappa^2 l}$.

Now observe that the colour of an element $\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\} \in [Y]^2$—where we suppose that $h < i$—is completely determined by f, g and whether $h < j < i < k$, $h < i < j < k$, $h < i < k < j$ or $h = i < j < k$.

Now we define a coloured digraph $C = \langle l, A, c\rangle$ as follows. If for $\{f, g\} \in [l]^2$ we have

- $\forall \{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\} \in [Y]^2 : \chi(\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\}) = 0$ then there is no arc between f and g in C.
- $\exists \{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\} \in [Y]^2 : \chi(\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\}) = 1$ where we again assume that $h < i$ and
  - ◇ $h < i < k < j$ then we let $f \mapsto_b g$.
  - ◇ $h < i < j < k$ then we let $f \mapsto_g g$.

◇ $h < j < i < k$ then we let $f \mapsto_d g$.

In this case there may very well be more than one way to put an arc between f and g and to colour it—clearly there could be six possible ways to do this yet it suffices just to do it in one way.

Our remark above shows us that this is well-defined. Since l is natural we do not even have to use the axiom of choice.

Now we may use our combinatorial statement. If $I \in [l]^m$ is independent in C then obviously $Z := \{\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\} \in [Y]^2 | \{f, g\} \subset I \wedge h, i, j, k < \kappa\}$ is a homogeneous set of size $\kappa^2 m$ in colour 0. So let us assume that there is a subtournament S of C of size n such that all triples in S are agreeable.

Let $\langle m_0, m_2, \ldots, m_{n-1}\rangle$ be the sequence of natural numbers defined by the following backwards recursion:

- $m_n := 0$,
- $m_{i-1} := 3m_i + 2$.

We have $m_i := 3^{n-i} - 1$. Furthermore let $\langle v_0, v_1, \ldots, v_{n-1}\rangle$ be a sequence of vertices such that $S := \{v_i | i < n\}$.

Now we are going to inductively construct the 1-homogeneous set $\{\kappa a_i + b_i | i < n\}$ where a_i and b_i are ordinals less than κ for $i < n$.

First we are going to inductively define a sequence $\langle s_i | i < m_0\rangle$ of ordinals less than κ. Let $s_0 := 0$. Given $\langle s_j | j \leqslant i\rangle$ choose $\alpha_i < \kappa$ such that $e_{v_j}(\alpha_i) \geqslant s_i$ for all $j < n$ and then choose $s_{i+1} < \omega$ such that $\forall j < n, k \leqslant i : b^{-1}``(\{\langle v_j, \alpha_k\rangle\}) \cap s_{i+1} \setminus s_i \supsetneq \emptyset$.

In step $i < n$ suppose we have already defined a 1-homogeneous set of size i.

The following will be our induction hypothesis.

1. $\min\{j \in \omega \setminus 1 | \exists k < \omega : \{a_h, b_h | h < i\} \cap s_{j+k+1} \setminus s_{j+k} \supsetneq \emptyset \wedge \{a_h, b_h | h < i\} \cap s_{k+1} \setminus s_k \supsetneq \emptyset\} > m_{i+1}$,
2. $\forall j, k < i : v_j \mapsto_b v_k \leftrightarrow a_j < a_k < b_k < b_j$,
3. $\forall j, k < i : v_j \mapsto_g v_k \leftrightarrow a_j < a_k < b_j < b_k$,
4. $\forall j, k < i : v_j \mapsto_d v_k \leftrightarrow a_j < b_j < a_k < b_k$.

This we do as follows, let $L_\kappa := \max(\{a_h | h < i \wedge v_h \mapsto v_i\} \cup \{b_h | h < i \wedge v_h \mapsto_d v_i\})$ and $U_\kappa := \min(\{a_h | h < i \wedge v_i \mapsto v_h\} \cup \{b_h | h < i \wedge v_h \mapsto_b v_i\} \cup \{b_h | h < i \wedge v_h \mapsto_g v_i\})$.

Claim 2.7. $L_\kappa < U_\kappa$.

Proof of Claim:

Suppose not. The claim could fail in the following ways:

- $\max\{a_h|h < i \wedge v_h \mapsto v_i\} \geqslant \min\{a_j|j < i \wedge v_i \mapsto v_j\}$. Fix h and j witnessing this. The induction hypothesis implies that $v_j \mapsto v_h$. But then v_h, v_i and v_j form a cyclic and hence disagreeable triple. Contradiction!
- $\max\{a_h|h < i \wedge v_h \mapsto v_i\} \geqslant \min\{b_j|j < i \wedge v_j \mapsto_b v_i\}$. As before fix witnesses h and j. By induction hypothesis we have $v_j \mapsto_d v_h$ and fact 2.2 implies that $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- $\max\{a_h|h < i \wedge v_h \mapsto v_i\} \geqslant \min\{b_j|j < i \wedge v_j \mapsto_g v_i\}$. Fix h and j witnessing this. The induction hypothesis implies that $v_j \mapsto_d v_h$. A look at fact 2.2 shows that $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- $\max\{b_h|h < i \wedge v_h \mapsto_b v_i\} \geqslant \min\{a_j|j < i \wedge v_i \mapsto v_j\}$. Fix h, j witnessing this. By induction hypothesis we now either have $v_j \mapsto v_h$, $v_h \mapsto_b v_j$ or $v_h \mapsto_g v_j$. In every case $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- $\max\{b_h|h < i \wedge v_h \mapsto_d v_i\} \geqslant \min\{b_j|h < i \wedge v_j \mapsto_b v_i\}$. Let h, j witness this. As before the induction hypothesis implies that either $v_h \mapsto_b v_j$, $v_j \mapsto_g v_h$ or $v_j \mapsto_d v_h$. Again by fact 2.2 $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- $\max\{b_h|h < i \wedge v_h \mapsto_d v_i\} \geqslant \min\{b_j|h < i \wedge v_j \mapsto_g v_i\}$. Fix witnesses h and j to this fact. By induction hypothesis we now either have $v_h \mapsto_b v_j$, $v_j \mapsto_g v_h$ or $v_j \mapsto_d v_h$. Fact 2.2 again shows $\{v_h, v_i, v_j\}$ to be disagreeable. Contradiction!

■(Claim)

Let $h < m_0$ be minimal such that $L_\kappa < s_h$. Then take

$$a_i \in e_{v_i}``(\kappa) \cap s_{h+m_i+1} \setminus s_{h+m_i}.$$

Now we are going to define b_i. Similarly, let $L_1 := \max(\{a_i\} \cup \{a_h|h < i \wedge v_i \mapsto_g v_h\} \cup \{b_h|h < i \wedge v_i \mapsto_b v_h\} \cup \{b_h|h < i \wedge v_h \mapsto_g v_i\})$ and $U_1 := \min(\{a_h|h < i \wedge v_i \mapsto_d v_h\} \cup \{b_h|h < i \wedge v_h \mapsto_b v_i\} \cup \{b_h|h < i \wedge v_i \mapsto_g v_h\})$.

Claim 2.8. $L_1 < U_1$.

Proof of Claim: As above we distinguish cases and reach contradictions.

- Suppose $a_i \geqslant \min\{a_h|h < i \wedge v_i \mapsto_d v_h\}$. Since $\{a_h|h < i \wedge v_i \mapsto_d v_h\} \subset \{a_h|h < i \wedge v_i \mapsto v_h\}$ we have $\min\{a_h|h < i \wedge v_i \mapsto_d v_h\} \geqslant \min\{a_h|h < i \wedge v_i \mapsto v_h\}$ and since $\min\{a_h|h < i \wedge v_i \mapsto v_h\} \geqslant U_\kappa$ it follows that $a_i \geqslant U_\kappa$. Contradiction!
- If $a_i \geqslant \min\{b_h|h < i \wedge v_h \mapsto_b v_i\}$ then analogously we get $\min\{b_i|h < i \wedge v_h \mapsto_b v_i\} \geqslant U_\kappa$ implying $a_i \geqslant U_\kappa$. Contradiction!

- Suppose $a_i \geqslant \min\{b_h | h < i \wedge v_i \mapsto_g v_h\}$. Similarly we have $\min\{b_h | h < i \wedge v_h \mapsto_g v_i\} \geqslant U_\kappa$ and thus $a_i \geqslant U_\kappa$. Contradiction!
- Suppose $\max\{a_h | h < i \wedge v_i \mapsto_g v_h\} \geqslant \min\{a_j | j < i \wedge v_i \mapsto_d v_j\}$. Fix witnesses h and j to this fact. The induction hypothesis implies $v_j \mapsto v_h$. Fact 2.2 shows that in all three cases $\{v_h, v_i, v_j\}$ is disagreeable.
- Suppose $\max\{a_h | h < i \wedge v_i \mapsto_g v_h\} \geqslant \min\{b_j | j < i \wedge v_j \mapsto_b v_i\}$. Once more fix witnesses h, j. Again the induction hypothesis implies that $v_j \mapsto_d v_h$. Fact 2.2 implies that $\{v_h, v_i, v_j\}$ is a disagreeable triple. Contradiction!
- In case $\max\{a_h | h < i \wedge v_i \mapsto_g v_h\} \geqslant \min\{b_j | j < i \wedge v_i \mapsto_g v_j\}$ we can again fix h and j witnessing this and use the induction hypothesis to see that $v_j \mapsto_d v_h$. A backcheck with fact 2.2 assures us of the disagreeability of $\{v_h, v_i, v_j\}$. Contradiction!
- In case $\max\{b_h | h < i \wedge v_i \mapsto_b v_h\} \geqslant \min\{a_j | j < i \wedge v_i \mapsto_d v_j\}$ let us once more fix witnesses h and j to this fact and use the induction hypothesis. We must either have $v_j \mapsto v_h$ or $v_h \mapsto_b v_j$ or $v_h \mapsto_g v_j$. Fact 2.2 shows that in all five cases $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- Suppose that $\max\{b_h | h < i \wedge v_i \mapsto_b v_h\} \geqslant \min\{b_j | j < i \wedge v_j \mapsto_b v_i\}$. Fix witnesses h and j and use the induction hypothesis in order to see that we either have $v_h \mapsto_b v_j$ and hence a cyclic triple $\{v_h, v_i, v_j\}$ or we have either $v_j \mapsto_g v_h$ or $v_j \mapsto_d v_h$. In both of the last two cases $\{v_h, v_i, v_j\}$ is disagreeable by fact 2.2. Contradiction!
- If $\max\{b_h | h < i \wedge v_i \mapsto_b v_h\} \geqslant \min\{b_j | j < i \wedge v_i \mapsto_g v_j\}$ we again fix witnesses h and j. By induction hypothesis we have $v_h \mapsto_b v_j$ or $v_j \mapsto_g v_h$ or $v_j \mapsto_d v_h$. Fact 2.2 assures us once again of $\{v_h, v_i, v_j\}$'s disagreeability in any case. Contradiction!
- Suppose $\max\{b_h | h < i \wedge v_h \mapsto_g v_i\} \geqslant \min\{b_j | j < i \wedge v_j \mapsto_b v_i\}$. Let h and j be witnesses to this fact. The induction hypothesis implies that either $v_h \mapsto_b v_j$ or $v_j \mapsto_g v_h$ or $v_j \mapsto_d v_h$. By fact 2.2 in any case $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- If $\max\{b_h | h < i \wedge v_h \mapsto_g v_i\} \geqslant \min\{b_j | j < i \wedge v_i \mapsto_g v_j\}$ we fix witnesses h and j. By induction hypothesis we either have $v_h \mapsto_b v_j$ or $v_j \mapsto_g v_h$ or $v_j \mapsto_d v_h$. In each case by fact 2.2 $\{v_h, v_i, v_j\}$ would be disagreeable. Contradiction!
- If $\max\{b_h | h < i \wedge v_h \mapsto_g v_i\} \geqslant \min\{a_j | j < i \wedge v_i \mapsto_d v_j\}$ we can again fix witnesses h, j to this fact. By induction hypothesis we can conclude that either $v_j \mapsto v_h$ in which case $\{v_h, v_i, v_j\}$ is cyclic and in particular disagreeable or that either $v_h \mapsto_b v_j$ or $v_h \mapsto_g v_j$. But fact 2.2 tells us that in these cases $\{v_h, v_i, v_j\}$ is disagreeable too. Contradiction!

■(Claim)

Now let $h < m_0$ be minimal such that $L_1 < s_h$. Take

$$b_i \in b^{-1}``(\{\langle v_i, e^{-1}(a_i)\rangle\}) \cap s_{h+2m_i+1} \setminus s_{h+2m_i}{}^3.$$

By claims 2.7 and 2.8 and condition 1. of the inductive hypothesis we have chosen a_i and b_i such that $a_i < U_\kappa$ and $b_i < U_1$. So by definition of the bounds L_κ, U_κ, L_1 and U_1 conditions 2. to 4. of our inductive hypothesis are fulfilled. We also made sure that 1. holds true for the next step.

But then, after n steps, by the definition of our coloured digraph C, we see that $\{\kappa a_i + b_i | i < n\}$ is homogeneous of colour 1.

$\diamondsuit$ Now suppose towards a contradiction that there is a a 3-coloured digraph $D = \langle l, A, c\rangle$ of size l such that both all its independent sets have size less than m and every subtournament S of D of size n contains a disagreeable triple.

Now we define a colouring χ as follows.

$$\chi : [\omega^2 l]^2 \longrightarrow 2$$

$$\{\omega^2 e + \omega g + i, \omega^2 f + \omega h + j\} \longmapsto \begin{cases} 1 \text{ if } g < h < j < i \text{ and } e \mapsto_b f, \\ 1 \text{ if } g < h < i < j \text{ and } e \mapsto_g f, \\ 1 \text{ if } g < i < h < j \text{ and } e \mapsto_d f, \\ 0 \text{ if } e \text{ and } f \text{ are unconnected}, \\ 0 \text{ if } \overline{\overline{\{g, h, i, j\}}} < 4, \\ 0 \text{ if } e = f \text{ or } i < g \text{ or } j < h. \end{cases}$$

In the naming of the pair of ordinals we can of course assume without loss of generality that $g \leqslant h$.

Now if there were a set $H \in [\kappa^2 l]^{\kappa^2 m}$ which is 0-homogeneous for χ there would be a set $S \in [l]^m$ such that for all $s \in S$ we have $\operatorname{otyp}(\{\kappa^2 s + \alpha \in H | \alpha < \kappa^2\}) = \kappa^2$. Since all independent sets in D have size less than m it follows that we have $e \mapsto f$ for $e, f \in S$. Now if $e \mapsto_d f$, choose first $g \in \{k | k < \kappa \wedge \overline{\overline{\{k' | k' < \kappa \wedge \kappa^2 e + \kappa k + k' \in H\}}} = \kappa\}$, then $i \in \{k | k \in \kappa \setminus (g+1) \wedge \kappa^2 e + \kappa g + k \in H\}$, then $h \in \{k | k \in \kappa \setminus (i+1) \wedge \overline{\overline{\{k' | k' < \kappa \wedge \kappa^2 f + \kappa k + k' \in H\}}} = \kappa\}$ and finally $j \in \{k | k \in \kappa \setminus (h+1) \wedge \kappa^2 f + \kappa h + k \in H\}$. But now we have found $\kappa^2 e + \kappa g + i, \kappa^2 f + \kappa h + j \in H$ such that $\chi(\{\kappa^2 e + \kappa g + i, \kappa^2 f + \kappa h + j\}) = 1$ which contradicts our assumption that H was 0-homogeneous. If instead of $e \mapsto_d f$ we have $e \mapsto_g f$ or $e \mapsto_b f$ we can argue similarly, we only have to change the sequence of choices.

So assume that there is a set $H \in [\omega^2 l]^n$ which is 1-homogeneous for χ. Considering the last clause in the definition of χ we conclude that no κ^2-block can contain more than one element of H, i.e. we have $\overline{\overline{H \cap \kappa^2(k+1) \setminus \kappa^2 k}} < 2$ for every $k < l$. So $S := \{s | s <$

[3]There are basically two ideas behind this abundance of case distinctions. First one can conceive of a completeness theorem for linear orders. Unless one cannot derive a contradiction there has to be a model. The other is the fact that if a tournament has a cycle it has one of length 3. Here it only gets a little bit more involved since one has to simultaneously consider the ω-coordinates and the 1-coordinates.

$l \wedge H \cap \kappa^2(s+1) \setminus \kappa^2 s \supsetneq \emptyset\}$ has size n. Moreover the fourth clause in the definition of χ implies that every two elements of S are connected by an arrow. So S spans a subtournament of D. Hence S has to contain a disagreeable triple $T = \{t_0, t_1, t_2\} \in [S]^3$. Take $\kappa^2 t_0 + \kappa u_0 + v_0, \kappa^2 t_1 + \kappa u_1 + v_1, \kappa^2 t_2 + \kappa u_2 + v_2 \in H$. Now we distinguish four cases

- T is cyclic. Suppose we have $t_0 \mapsto t_1 \mapsto t_2 \mapsto t_0$. Considering the first three clauses of the definition of χ we may conclude that for the elements chosen above we have $u_0 < u_1 < u_2 < u_0$. Contradiction!
- Suppose $t_0 \mapsto_g t_1$ or $t_0 \mapsto_d t_1$, that $t_2 \mapsto_b t_1$ or $t_1 \mapsto_g t_2$ or $t_1 \mapsto_d t_2$ and that $t_0 \mapsto_b t_2$. Then using again the definition of χ we arrive at $v_0 < v_1 < v_2 < v_0$. Contradiction!
- Suppose that $t_0 \mapsto_d t_1$, that $t_1 \mapsto t_2$. and that $t_0 \mapsto_b t_2$ or $t_0 \mapsto_g t_2$. Then $v_0 < u_1 < u_2 < v_0$ follows. Contradiction!
- Suppose that $t_0 \mapsto_d t_1$, that $t_2 \mapsto_b t_1$ or $t_2 \mapsto_g t_1$ and that either $t_0 \mapsto_b t_2$ or $t_2 \mapsto_g t_0$ or $t_2 \mapsto_d t_0$. Then $v_0 < u_1 < v_2 < v_0$. Contradiction!

Note that modulo a renaming of t_0, t_1 and t_2 the above clauses cover all disagreeable triples. So this finishes the second part of the proof.

■

The author wants to point out that if you believe in enough reflection you might believe something like (There are stationarily many weakly compact cardinals in the universe.) so theorem 2.6 really says something about a lot of ordinals.

So we now can also state—not in complete earnest—a joint generalization of the theorems 1.4 and 2.6. In this artificial sense being agreeable is a generalization of being transitive.

Theorem 2.9. *For $i \in 3 \setminus 1$ and $\kappa \in \Omega \setminus 3$ such that $\kappa \to (\kappa)^2_2$ the partition relation $\kappa^i l \to (\kappa^i m, n)$ holds true if and only if every coloured digraph $C = \langle l, A, c\rangle$ with $\mathrm{ran}(c) = 2i - 1$ contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are agreeable.*

The special case the author was actually most interested in however is this one:

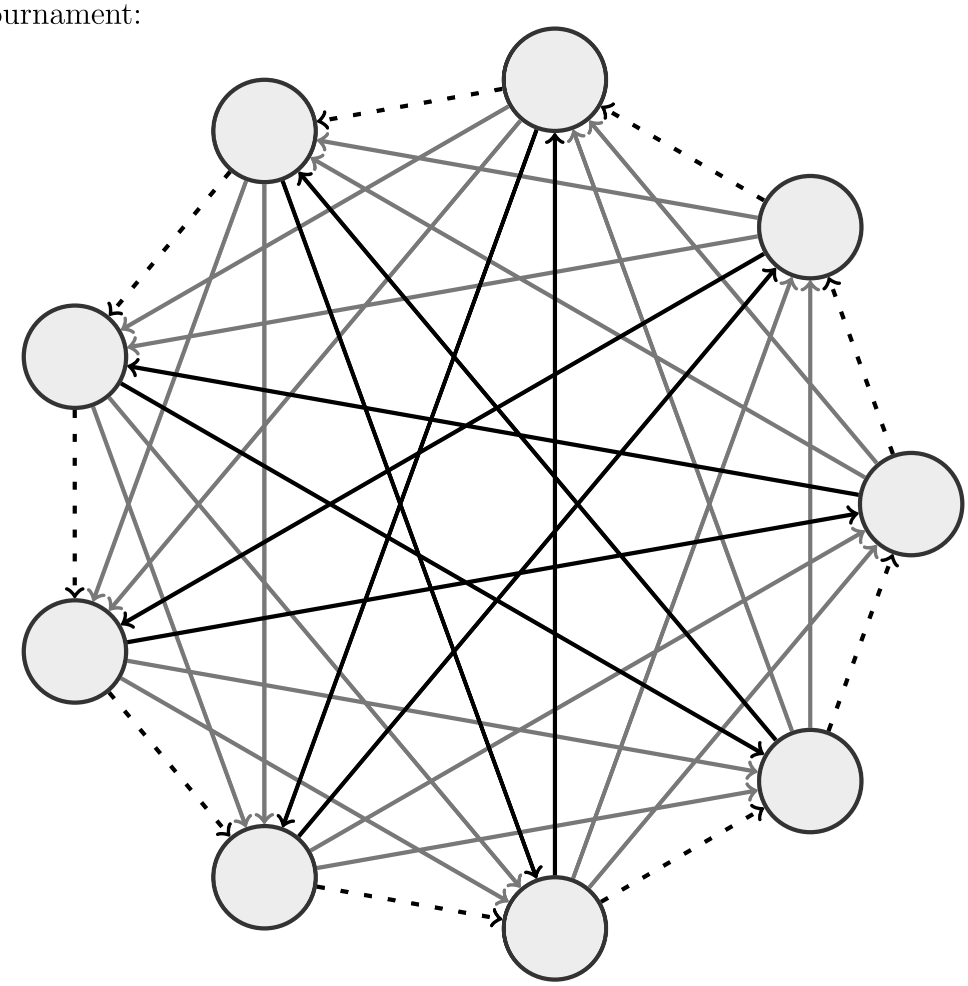
Corollary 2.10. *The partition relation $\omega^2 l \to (\omega^2 m, n)$ holds true if and only if every coloured digraph $C = \langle l, A, c\rangle$ with $\mathrm{ran}(c) = 3$ contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are agreeable.*

Henceforth A_n is a graph on n vertices such that all its induced subgraphs on three vertices are agreeable. This allows us to define the Ramsey number $r(I_m, A_n)$ to be the smallest natural number such that any digraph on $r(I_m, A_n)$ vertices either contains an independent set of size m or an induced subgraph on n vertices all triples of which are agreeable. Since A_n does not denote a single graph but a class of graphs this might be seen as a slight abuse of notation, or at least a shorthand. So the equivalence between the infinitary partition problems and finite digraph combinatorics above can be also condensed into the following formula:

$$r(\kappa^2 m, n) = \kappa^2 r(I_m, A_n) \text{ for } \kappa = \omega \text{ or } \kappa \text{ weakly compact.}$$

Theorem 2.11. $r(I_2, A_3) > 9$.

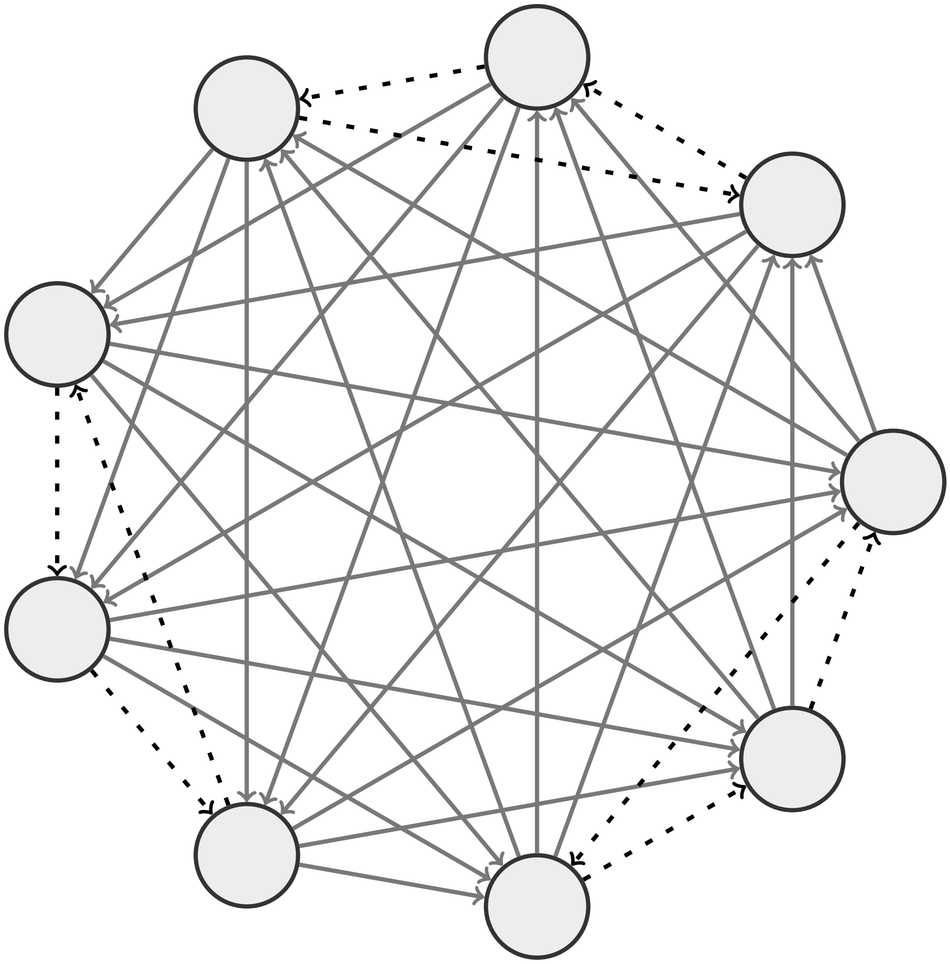
Proof. The statement above holds because there is the following tricoloured nine-person tournament:



Algebraically this can be seen as the tournament on $\mathbb{Z}_9$ where from every $i \in \mathbb{Z}_9$ there is a dashed black arrow pointing at $i+1$, solid grey arrows pointing both at $i+2$ and $i+3$ and a solid black arrow pointing at $i+4$.

A close observation reveals that every triple in this tournament is disagreeable thereby showing $\omega^2 9 \not\to (\omega^2 2, 3)$.

There is even an easier counterexample which is this one:



This is algebraically interpretable as the tournament on $\mathbb{Z}_3 \times \mathbb{Z}_3$ where from every $\langle i, j\rangle$ there is a dashed black arrow pointing at $\langle i+1, j\rangle$ and solid grey arrows pointing at $\langle i, j+1\rangle$ and $\langle i+1, j+1\rangle$ and $\langle i+2, j+1\rangle$.

Also this should be easily checkable. Either a triple is one of three completely coloured in dashed black—these are disagreeable—or it shares exactly one point with each of these triples—then it forms a disagreeable triple completely coloured in solid grey—or it shares two points with one and one with another—then it is either one of two dashed black/solid grey-coloured disagreeable triples. These counterexamples were not found in a very systematic way, so there might be others or there might not be others. ■

Now some easy observations quickly yield upper bounds for $r(I_2, A_3)$. For example by ordering the vertices and then colour a pair of them in one of six colours depending on both whether the direction of the arc agrees with the ordering or not and the original colour of the arc. Since all monochromatic transitive triples are agreeable this yields $r(I_2, A_3) \leqslant r(3, 3, 3, 3, 3, 3)$. Using $r(3, 3, 3) = 17$ from [55GG] together with the recursive formula we get $r(I_2, A_3) \leqslant 1898$.

A much better bound is found by observing that every four-person-tournament contains a transitive triple. So clearly $r(I_2, A_3) \leqslant r(4,4,4)$. The recursive formula together with the result that $r(3,3,4) \leqslant 31$ from [98PR] yields $r(4,4,4) \leqslant 236$ so $r(I_2, A_3) \leqslant 236$ follows. Finally by observing that every six-person-tournament coloured in either solid black or solid grey contains an agreeable triple we get $r(I_2, A_3) \leqslant r(4,6)$. In [97MR] it is shown that $r(4,6) \leqslant 41$ so we have $r(I_2, A_3) \leqslant 41$.

But far better results are possible.

Fact 2.12. *In a tournament on ten vertices containing no agreeable triple there can be at most ten dashed black arrows, thirty solid grey arrows and ten solid black arrows. So there are at least five dashed black, twenty-five solid grey and five solid black arrows.*

Proof. Take any tournament on ten vertices not containing any agreeable triple.

Suppose that there are eleven dashed black arrows. By the pigeonhole principle there has to be a vertex where two dashed black arrows leave. But then this vertex together with the targets of the two dashed black arrows forms an agreeable triple.

Suppose that there are eleven solid black arrows. Again by the pigeonhole principle there has to be a vertex where two solid black arrows leave. Analogously this vertex together with the targets of the two solid black arrows forms an agreeable triple.

Suppose that there are thirty-one solid grey arrows. By the pigeonhole principle there has to be a vertex where four solid grey arrows leave. Then all of the targets of these arrows have to be connected by dashed black arrows to one another. Fix one of these target-vertices v. Since it is adjacent to at least three dashed black arrows at least two dashed black arrows are leaving from there or two dashed black arrows are arriving there. Suppose two dashed black arrows are leaving. Then v together with the targets of these arrows forms an agreeable triple. If on the other hand two dashed black arrows are arriving then again v together with the sources of these arrows forms an agreeable triple.

The second statement follows immediately after one observes that a tournament on ten vertices has $\binom{10}{2}$, i.e. 45 arrows. ■

Lemma 2.13. *Any tournament T on ten vertices containing no agreeable triple contains three dashed black triples, i.e. we have $v_0, \dots, v_9 \in V_T$ with $v_1 \mapsto_d v_2 \mapsto_d v_3 \mapsto_d v_1$ and $v_4 \mapsto_d v_5 \mapsto_d v_6 \mapsto_d v_4$ and $v_7 \mapsto_d v_8 \mapsto_d v_9 \mapsto_d v_7$.*

Proof. First observe that any such tournament has to contain two such dashed black triples. This is because by fact 2.12 there have to be at least twenty-five solid grey arrows and hence five vertices where three solid grey arrows leave. Since the targets of the three solid grey arrows leaving such a vertex have to form a dashed black triple and since any vertex in such a dashed black triple may be the target of at most three solid grey arrows the pigeonhole principle implies that there have to be at least two dashed black triples.

Now suppose towards a contradiction that there are exactly two such dashed black triples $\{v_0, v_1, v_2\}$ and $\{v_3, v_4, v_5\}$ and that there are six vertices where three solid grey arrows leave.

Again the targets of these solid grey arrows have to form dashed black triples. Since two dashed black triples cannot overlap without producing an agreeable triple we can apply the pigeonhole principle and conclude that each vertex in each of both triples has to receive three solid grey arrows. Now since three solid grey arrows each are hitting v_0, v_1 and v_2 the sources of them again form a dashed black triple. Since there are only two of them and v_0, v_1 and v_2 are connected to each other by dashed black arrows we know that the sources of them can only be v_3, v_4 and v_5. So we have $v_i \mapsto_g v_j$ for every $i \in 6 \setminus 3$ and $j \in 3$. But since v_3 is also hit by three solid grey arrows and is already connected in the opposite direction with v_0, v_1 and v_2 there has to be a third dashed black triple.

An analogous argument works for the case that there are six vertices where three solid grey arrows *leave*.

So there are at most five vertices where three solid grey arrows leave and at most five vertices where three solid grey arrows arrive. So there are at most twenty-five solid grey arrows. By fact 2.12 we may restrict our attention to the case of ten dashed black, twenty-five solid grey and ten solid black arrows. Since no vertex is the target of more than one dashed black arrow and no vertex is the source of more than one dashed black arrow the remaining four dashed black arrows have to form a cycle. So without loss of generality we have $v_6 \mapsto_d v_7 \mapsto_d v_8 \mapsto_d v_9 \mapsto_d v_6$.

Now we argue as before. Consider now again the five vertices where three solid grey arrows are leaving. The targets of the three solid grey arrows leaving such a vertex must form a dashed black triple. By the pigeonhole principle every vertex in one of the two dashed black triples has to receive three solid grey arrows each. But then their sources again form a dashed black triple. This shows that without loss of generality we have $v_i \mapsto_g v_j$ for every $i \in 6 \setminus 3$ and $j \in 3$.

In total there are five vertices with three solid grey arrows leaving and there are five vertices with three solid grey arrows arriving. Both the targets of the former and the sources of the latter have to form dashed black triples. Since every vertex may emit or absorb at most three solid grey arrows it follows that there are two-element-sets $\{w_0, w_1\}$ and $\{w_2, w_3\}$ with $w_0 \mapsto_d w_1$ and $w_2 \mapsto_d w_3$ such that $v_i \mapsto_g w_j$ for every $i \in 3$ and $j \in 2$ and $w_i \mapsto_g v_j$ for every $i \in 4 \setminus 2$ and $j \in 6 \setminus 3$.

Now we are going to distinguish some—not altogether different—cases. Note that since there are ten solid black arrows and a vertex can emit at most one, every vertex has to emit exactly one.

$\diamondsuit$ $\overline{\overline{\{w_0, w_1, w_2, w_3\}}} = 2$. Since there are only two possible targets—the elements of the set $\{v_6, \ldots, v_9\} \setminus \{w_0, w_1\}$—for the solid black arrows emitted by the vertices $v_0, \ldots, v_5, w_0, w_1$ one of them has to be target of at least four solid black arrows which contradicts fact 2.12.

$\diamondsuit$ $\overline{\overline{\{w_0, w_1, w_2, w_3\}}} = 3$. Let $X := \{w_0, w_1\} \cap \{w_2, w_3\}$. Obviously $\overline{\overline{X}} = 1$. Let $w := \bigcup X$. The only possible target for the solid black arrow leaving w is $v := \bigcup(\{v_6, \ldots, v_9\} \setminus \{w_0, w_1, w_2, w_3\})$ since for each $i \in 4$ we either have $w = w_i$ or $w \mapsto_d w_i$ or $w_i \mapsto_d w$ and for all $i \in 6$ we either have $w \mapsto_g v_i$ or $v_i \mapsto_g w$. Now since the vertices $v_0, \ldots, v_5$ all also have to emit a solid black arrow and the sources of solid black arrows pointing at the same vertex have to be connected by dashed black arrows, lest there be an agreeable triple, we know that none of these solid black arrows can point at v. None can point at w either since w was already shown to be connected differently to every other vertex. So let $x := \bigcup(\{w_0, w_1\} \setminus \{v, w\})$ and $y := \bigcup(\{w_2, w_3\} \setminus \{v, w\})$. Since $v_i \mapsto_g x$ for any $i \in 3$ it follows that $v_i \mapsto_b y$ for any $i \in 3$ Similarly $y \mapsto_g v_i$ for any $i \in 6 \setminus 3$ implies $v_i \mapsto_b x$ for any $i \in 6 \setminus 3$. But now we see that not both x and y can emit a solid black arrow hence we have a contradiction.

◇ $\overline{\overline{\{w_0, w_1, w_2, w_3\}}} = 4$. Remember that $v_6, \dots, v_9$ form a dashed black four-cycle. Since $\{v_6, \dots, v_9\} = \{w_0, \dots, w_3\}$ and $w_0 \mapsto_b w_1$ and $w_2 \mapsto_d w_3$ we can conclude $w_0 \mapsto_d w_1 \mapsto_d w_2 \mapsto_d w_3 \mapsto_d w_0$. Now recall that we focused on the case where there are five vertices emitting three solid grey arrows. Since in total there are twenty-five solid grey arrows every arrow has to emit at least two solid grey arrows. Now consider the vertex w_0. Since it can only send one solid grey arrow to w_2 and none to v_0, v_1, v_2, w_1 or w_3 it has to send one to v_3, v_4 or v_5. But then w_0 and w_2 would have to be connected by a dashed black arrow making either $\{w_0, w_1, w_2\}$ or $\{w_2, w_3, w_0\}$ an agreeable triple. Again we have a contradiction.

■

Theorem 2.14. *$r(I_2, A_3) \leqslant 10$.*

Proof. Suppose towards a contradiction that there is a tricoloured tournament on ten vertices $v_0, \dots, v_9$ not containing any agreeable triple. By lemma 2.13 we know that there are three dashed black triples $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$. Since no vertex can emit more than one dashed black arrow and no vertex can absorb more than one dashed black arrow either, we know that v_0 has to be connected to $v_1, \dots, v_9$ purely by solid black and solid grey arrows. In the situation in which all triples are agreeable we know that the targets of three solid grey arrows starting at v_0 have to form a dashed black triple. The same holds for the sources of three solid grey arrows ending at v_0 and the sources of three solid black arrows ending at v_0. Moreover we know that v_0 may emit at most one solid black arrow. So we may distinguish three cases up to isomorphism:

◇ $v_1, v_2, v_3 \mapsto_b v_0$,
$v_4, v_5, v_6 \mapsto_g v_0$
$v_0 \mapsto_g v_7, v_8$ and $v_0 \mapsto_b v_9$.
In this case $\{v_0, v_7, v_9\}$ is agreeable.

◇ $v_1, v_2, v_3 \mapsto_b v_0$,
$v_0 \mapsto_g v_4, v_5, v_6$
$v_7, v_8 \mapsto_g v_0$ and $v_0 \mapsto_b v_9$.
Here $\{v_0, v_8, v_9\}$ is agreeable.

$\diamondsuit$ $v_1, v_2, v_3 \mapsto_g v_0$,
$v_0 \mapsto_g v_4, v_5, v_6$,
$v_7, v_8 \mapsto_b v_0$(and either $v_9 \mapsto_b v_0$ or $v_0 \mapsto_b v_9$).

Now we prove two claims:

Claim 2.15. *The dashed black triple $\{v_7, v_8, v_9\}$ is connected purely by solid grey arrows to one of the other two dashed black triples.*

Proof of Claim: First note that no solid grey arrow can point from a v_i with $i \in 4 \setminus 1$ to a v_j with $j \in 10 \setminus 7$, otherwise $\{v_0, v_i, v_j\}$ would be agreeable. Similarly no solid grey arrow may point from a v_i with $i \in 10 \setminus 7$ to a v_j with $j \in 7 \setminus 4$, otherwise $\{v_0, v_i, v_j\}$ would be agreeable. Since at least two solid black arrows are leaving the set $\{v_7, v_8, v_9\}$ heading towards v_0 and there can be at most one solid black arrow leaving a vertex we conclude that there can be at most one solid black arrow leaving $\{v_7, v_8, v_9\}$ in the direction of any vertex v_i with $i \in 10 \setminus 1$. So we have either that all arrows between $\{v_7, v_8, v_9\}$ and $\{v_4, v_5, v_6\}$ are either incoming solid grey or incoming solid black or that all arrows between $\{v_7, v_8, v_9\}$ and $\{v_1, v_2, v_3\}$ are either outgoing solid grey or incoming solid black. Suppose the latter happens. Assume towards a contradiction that $v_i \mapsto_b v_j$ for $i \in 4 \setminus 1$ and $j \in 10 \setminus 7$. Since at most one solid black arrow may leave a vertex we know that $v_k \mapsto_g v_i$ for both $k \in 10 \setminus (7 \cup \{j\})$. Now choose $k \in 10 \setminus (7 \cup \{j\})$ such that $v_k \mapsto_d v_j$. Then $\{v_i, v_j, v_k\}$ is agreeable, contradiction! So now assume that the former happens, i.e. that all arrows between $\{v_7, v_8, v_9\}$ and $\{v_4, v_5, v_6\}$ are either incoming solid grey or incoming solid black. Assume towards a contradiction that for some $i \in 7 \setminus 4$ and some $j \in 10 \setminus 7$ we have $v_i \mapsto_b v_j$. Since at most one solid black arrow may leave any vertex it follows that $v_i \mapsto_g v_k$ for $k \in 10 \setminus (7 \cup \{j\})$. Choose $k \in 10 \setminus (7 \cup \{j\})$ such that $v_j \mapsto_d v_k$. Then $\{v_i, v_j, v_k\}$ is agreeable, contradiction! $\blacksquare$(Claim)

Claim 2.16. *At both the dashed black triples $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ there are at least two vertices connected by a solid grey arrow to a vertex in the other dashed black triple.*

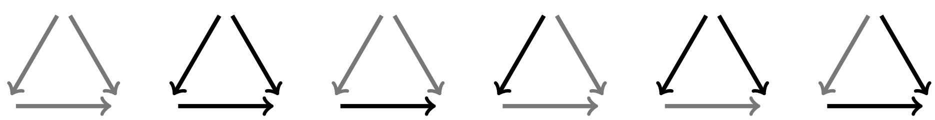
Proof of Claim: Suppose this would fail for $\{v_1, v_2, v_3\}$. Then two vertices, v_i and v_j with $\{i, j\} \in [4 \setminus 1]^2$ would be connected to $\{v_4, v_5, v_6\}$ purely by solid black arrows. But there are six such connections to be made and since at most one solid black arrow may leave a vertex only the usage of five solid black arrows is allowed here—contradiction. The proof for the other case works totally analogously. $\blacksquare$(Claim)

Now we are moving ever closer to a contradiction. By claim 2.15 we know that the dashed black triple $\{v_7, v_8, v_9\}$ is connected purely by solid grey arrows to one of the other two dashed black triples. Suppose $\{v_1, v_2, v_3\}$ is this triple. Since at most one solid black arrow may leave the dashed black triple $\{v_7, v_8, v_9\}$ in the direction of any vertex v_i with $i \in 10 \setminus 1$ and at most one solid black arrow can leave each vertex v_i with $i \in 7 \setminus 4$ we know that five arrows between $\{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$ have to be solid grey. So in particular at least two vertices in $\{v_4, v_5, v_6\}$ have to have solid grey connections to vertices in the dashed black triple $\{v_7, v_8, v_9\}$. Using claim 2.16 and a variant of the pigeonhole principle we may choose a vertex v_i with $i \in 7 \setminus 4$ which has a solid grey connection both to a vertex v_j with $j \in 4 \setminus 1$ and to a vertex v_k with $k \in 10 \setminus 7$. Since we supposed the truth of claim 2.15 to be witnessed by $\{v_1, v_2, v_3\}$ we know that also v_j and v_k are connected by solid grey arrows. If we now remember that all solid grey arrows between $\{v_1, v_2, v_3\}$ and $\{v_7, v_8, v_9\}$ have to point from the latter to the former we know that in order for $\{v_i, v_j, v_k\}$ to be disagreeable the only possibility left is $v_i \mapsto_g v_k \mapsto_g v_j \mapsto_g v_i$. But then we have $v_j \mapsto_g v_0 \mapsto_g v_i$ and $v_j \mapsto_g v_i$ hence $\{v_0, v_i, v_j\}$ is agreeable. Contradiction. If the truth of claim 2.15 is witnessed by $\{v_4, v_5, v_6\}$ instead the proof works totally analogously. So this proves the theorem.

■

3 Strong agreeability

Definition 3.1. *A triple is called* strongly agreeable *if and only if it is agreeable and does not contain any dashed black arrow. So it is strongly agreeable precisely if it is one of these:*



The proof of the following theorem is heavily inspired by [74Ba].

Theorem 3.2. *Let κ be weakly compact and $\lambda \in \kappa \setminus \omega$ be a cardinal. The partition relation $\kappa\lambda l \to (\kappa\lambda m, n)$ holds true if and only if every edge-coloured digraph $C = \langle l, A, c\rangle$ with* $\mathrm{ran}(c) = 2$ *contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are strongly agreeable.*

Proof.

◇ Towards a contradiction let us assume that the finite combinatorial characterization above holds true yet there is neither a 0-homogeneous set of size $\kappa\lambda m$ nor a 1-homogeneous n-tuple. Let furthermore $\chi : [\kappa\lambda l]^2 \longrightarrow 2$ be any colouring. W.l.o.g. we may assume that $\chi``(\kappa(\lambda k + \alpha + 1) \setminus \kappa(\lambda k + \alpha)) = 1$ for any $\alpha < \lambda$ and $k < l$. We define a new colouring ξ as follows:

$$\begin{aligned}\xi : [\kappa]^2 &\longrightarrow^{[\lambda l]^2} 4\\ \{\alpha,\beta\}_< &\longmapsto \langle\{\gamma,\delta\}_<, 2\chi(\{\kappa\gamma+\alpha,\kappa\delta+\beta\})+\chi(\{\kappa\gamma+\beta,\kappa\delta+\alpha\})\rangle\end{aligned}$$

Now we use fact 2.5 in the form $\kappa \to (\kappa)^2_{2^\lambda}$ thereby finding an $X \in [\kappa]^\kappa$ homogeneous for ξ. Set $\psi := \bigcup \xi``X$. We have $\psi : [\lambda l]^2 \longrightarrow 4$.

Let $\langle\langle b_j, c_j\rangle | j < 2l(l-1)\rangle$ be an enumeration of $l^2 \setminus \Delta$.

We are going to inductively construct for every $i < l$ a sequence of sets $\langle Y_i^j | j \leqslant 2l(l-1)\rangle$ by letting $Y_i^0 := \lambda(i+1) \setminus \lambda i$ and for any $j < l(l-1)$ letting $Y_{b_j}^{2j+1} := B$ and $Y_{c_j}^{2j+1} := C$ if there are sets $B \in [Y_{b_j}^{2j}]^\lambda$, $C \in [Y_{c_j}^{2j}]^\lambda$ with $\overline{\overline{\{y \in C | \psi(\{x,y\}) \equiv 1(2)\}}} < \lambda$ for all $x \in B$. If there are no such sets let $Y_{b_j}^{2j+1} := Y_{b_j}^{2j}$ and $Y_{c_j}^{2j+1} := Y_{c_j}^{2j}$. Similarly let $Y_{b_j}^{2j+2} := B$ and $Y_{c_j}^{2j+2} := C$ if there are sets $B \in [Y_{b_j}^{2j+1}]^\lambda$, $C \in [Y_{c_j}^{2j+1}]^\lambda$ with $\overline{\overline{\{y \in C | \psi(\{x,y\}) \in 4 \setminus 2\}}} < \lambda$ for all $x \in B$. If there are no such sets let $Y_{b_j}^{2j+2} := Y_{b_j}^{2j+1}$ and $Y_{c_j}^{2j+2} := Y_{c_j}^{2j+1}$. Generally let $Y_i^{2j+2} := Y_i^{2j}$ for $j < l(l-1)$ and $i \in l \setminus \{b_j, c_j\}$. Finally we let $Z_i := Y_i^{2l(l-1)}$ for all $i < l$.

Now we define a digraph $D = \langle l, A, c\rangle$ by setting for $i, j < l$:

$$\begin{aligned} i \mapsto_b j \Longrightarrow & \exists \alpha \in Z_i : \overline{\overline{\{\beta \in Z_j | \psi(\{\alpha,\beta\}) \equiv 1(2)\}}} = \lambda && (1)\\ \Longleftrightarrow & \forall C \in [Z_j]^\lambda : \overline{\overline{\{\eta \in Z_i | \overline{\overline{\{\nu \in C | \psi(\{\eta,\nu\}) \equiv 1(2)\}}} < \lambda\}}} < \lambda && (2)\\ i \mapsto_g j \Longrightarrow & \exists \alpha \in Z_i : \overline{\overline{\{\beta \in Z_j | \psi(\{\alpha,\beta\}) \in 4 \setminus 2\}}} = \lambda && (3)\\ \Longleftrightarrow & \forall C \in [Z_j]^\lambda \overline{\overline{\{\eta \in Z_i | \overline{\overline{\{\nu \in C | \psi(\{\eta,\nu\}) \in 4 \setminus 2\}}} < \lambda\}}} < \lambda && (4)\end{aligned}$$

We have $\Rightarrow$ instead of $\Leftrightarrow$ here for a reason. This definition neither necessarily tells us the direction of an arrow—since i and j could change roles—nor does it determine its colour uniquely—since we have $3 \in \{x \in 4 \setminus 2 | n \equiv 1(2)\}$. We only have to follow the principle that whenever these conditions allow us to set an arrow we do so. The equivalences hold because of the definition of the Z_i's. We will use the negations of (1) and (3) in the case where we define the 0-homogeneous set and (2) and (4) in the case in which we define the 1-homogeneous set.

After having so defined the digraph we may employ our finitary hypothesis and distinguish two cases:

- There is $N \in [l]^n$ such that all elements of $[N]^3$ are strongly agreeable. We are going to inductively construct a 1-homogeneous set H. Since no strongly agreeable triple is cyclic we may choose an enumeration $e : n \longleftrightarrow N$ in such a way that $i < j$ if and only if $e(i) \mapsto e(j)$. Now we proceed by induction. We may, using (2) and (4), choose an $\alpha_i \in Z_{e(i)}$ such that

◇ $\forall j < i(e(i) \mapsto_b e(j) \to \psi(\{\alpha_j, \alpha_i) \equiv 1(2))$.
◇ $\forall j < i(e(i) \mapsto_g e(j) \to \psi(\{\alpha_j, \alpha_i) \in 4 \setminus 2)$.
◇ $\forall j \in n \setminus (i+1)(e(i) \mapsto_b e(j) \to \overline{\overline{\{\beta \in Z_{e(j)} | \psi(\{\alpha_0, \beta\}) \equiv 1(2)\}}} = \lambda)$
◇ $\forall j \in n \setminus (i+1)(e(i) \mapsto_g e(j) \to \overline{\overline{\{\beta \in Z_{e(j)} | \psi(\{\alpha_0, \beta\}) \in 4 \setminus 2\}}} = \lambda)$

Now choose this end let $S \in [\kappa]^{2^n}$ be such that $\forall \eta \in S \forall \nu \in \eta \cap S \exists \zeta \in X \cap \eta \setminus \nu$. Let $\langle s_\eta | \eta < 2^n \rangle$ be the increasing enumeration of S and let $e : n \longleftrightarrow N$. Now let $k_{-2} := 0$, $k_{-1} := 2^n$ and $k_0 := 2^{n-1}$. By induction we choose $\gamma_i \in X \cap s_{k_i} \setminus s_{k_i - 1}$ and k_{i+1} such that the following conditions are met.

◇ If $e(j) \mapsto_b e(i+1)$ then $k_{i+1} < k_j$.
◇ If $e(j) \mapsto_g e(i+1)$ then $k_{i+1} > k_j$.
◇ $|k_{i+1} - k_j| \geqslant 2^{n-i-1}$ for every $j \leqslant i$.

This we do for every $i < n$. Note that we are always able to choose the parameters in this way because all elements of $[N]^3$ are strongly agreeable. Finally we may define $H := \{\kappa\alpha_i + \gamma_i | i < n\}$. H is 1-homogeneous.

- There is an independent $M \in [l]^m$. Again we have to distinguish two cases:

◇ λ is regular. This case is easy. Let $Y_0 := \emptyset$ and for limit ordinals ν let $Y_\nu := \bigcup_{\eta<\nu} X_\eta$. For any $\eta < \lambda$, $i \in M$ and an enumeration $f : m \longleftrightarrow M$ let

$$Y_{m\eta+i+1} := Y_{m\eta+i} \cup \{\min(\{\nu \in Z_{f(i)} | \forall \zeta \in Y_{m\eta+i} : \psi(\{\nu, \zeta\}) = 0\})\} \tag{5}$$

At the end let $Y := Y_\lambda$. M is independent so the formulas on the right of (1) and (3) are jointly negated.

The regularity of λ now ensures that the minimum referred to in (5) always exists.

◇ λ is singular. Here we have to be a bit more careful. We have to make sure that we do not spoil our chances of defining a homogeneous set before our inductive construction is finished. So we thin out the Z_i's further as follows:

For each $i \in M$ let $\eta_i \in \lambda\,\mathrm{cf}(\lambda) + 1 \setminus \lambda$ be the least ordinal such that there exists an enumeration $e_i : \eta_i \longleftrightarrow Z_i$ such that

$$\nu \longmapsto \max(\{\overline{\overline{\{\zeta \in Z_j | \psi(\{e_i(\nu), \zeta\}) > 0\}}} | j \in M \setminus \{i\}\})$$

is nondecreasing. Fix such η_i and e_i. Similar to the case before we set $Y_0 := \emptyset$, for limit ordinals ν we let $Y_\nu := \bigcup_{\zeta<\nu} X_\zeta$ and for $\nu < \lambda$, $i \in M$ and an enumeration $f : m \longleftrightarrow M$ we define

$$Y_{m\nu+i+1} := Y_{m\nu+i} \cup \{e_{f(i)}(\min(\{\zeta < \lambda | \forall \rho \in Y_{m\nu+i} : \psi(\{e_{f(i)}(\zeta), \rho\}) = 0\}))\}. \tag{6}$$

Once more set $Y := Y_\lambda$. Again the independence of M implies that the right sides of (1) and (3) are jointly negated. By rearranging the Z_i's using the e_i's we ensured that the minimum in (6) always exists.

Let $\langle X_\eta | \eta < \lambda l\rangle$ be a partition of X into λ sets of size κ. We set $H := \{\kappa\alpha + \gamma | \alpha \in Y \wedge \gamma \in X_\alpha\}$. A little contemplation shows that H is 0-homogeneous for χ.

$\diamondsuit$ Suppose that the finite combinatorial characterization above fails, i.e. there is an edge-2-coloured digraph $D = \langle l, A, c\rangle$ neither containing an independent set of size m nor a set of size n only inducing strongly agreeable triples. We define a colouring $\chi : [\kappa\lambda l]^2 \longrightarrow 2$ as follows.

$$\chi : [\kappa\lambda l]^2 \longrightarrow 2$$
$$\{\kappa(\lambda j + \alpha) + \gamma, \kappa(\lambda k + \beta) + \delta\}_< \longmapsto \begin{cases} 1 \text{ if } j \mapsto_b k \wedge \alpha < \beta \wedge \delta < \gamma \\ 1 \text{ if } k \mapsto_b j \wedge \beta < \alpha \wedge \gamma < \delta \\ 1 \text{ if } j \mapsto_g k \wedge \alpha < \beta \wedge \gamma < \delta \\ 1 \text{ if } k \mapsto_g j \wedge \beta < \alpha \wedge \delta < \gamma \\ 0 \text{ else.} \end{cases}$$

- Suppose that χ would admit a 0-homogeneous set $X \in [\kappa\lambda l]^{\kappa\lambda m}$. Then there would have to be a set $Y \in [l]^m$ such that $\operatorname{otyp}(X \cap \kappa\lambda(k+1) \setminus \kappa\lambda k) = \kappa\lambda$ for any $k \in Y$. Since D contains no independent set of size m we may choose $\{j, k\} \in [Y]^2$ with $j \mapsto k$. Let $\alpha := \min\{\eta < \lambda | \operatorname{otyp}(X \cap \kappa(\lambda j + \eta + 1) \setminus \kappa(\lambda j + \eta)) = \kappa\}$ and let $\beta := \min\{\eta \in \lambda \setminus (\alpha + 1) | \operatorname{otyp}(X \cap \kappa(\lambda k + \eta + 1) \setminus \kappa(\lambda k + \eta) = \kappa\}$.

  We distinguish two cases:

  $\diamond$ $c(\{j,k\}) = 0$. Let $\gamma := \min\{\eta < \kappa | \kappa(\lambda k + \beta) + \eta \in X\}$ and let $\delta := \min\{\eta \in \kappa \setminus (\gamma + 1) | \kappa(\lambda j + \alpha) + \eta \in X\}$. Obviously $\chi(\{\kappa(\lambda j + \alpha) + \delta, \kappa(\lambda k + \beta) + \gamma\}) = 1$ contradicting the 0-homogeneity of X.

  $\diamond$ $c(\{j,k\}) = 1$. Let $\gamma := \min\{\eta < \kappa | \kappa(\lambda j + \alpha) + \eta \in X\}$ and let $\delta := \min\{\eta \in \kappa \setminus (\gamma + 1) | \kappa(\lambda k + \beta) + \eta \in X\}$. Obviously $\chi(\{\kappa(\lambda j + \alpha) + \gamma, \kappa(\lambda k + \beta) + \delta\}) = 1$ contradicting the 0-homogeneity of X.

- Now suppose that χ would admit a 1-homogeneous n-tuple $X \in [\kappa\lambda l]^n$. Per definitionem of χ we have that $\{\kappa(\lambda j + \alpha) + \gamma, \kappa(\lambda k + \beta) + \delta\} \in [X]^2$ for $j, k < l$; $\alpha, \beta < \lambda$ and $\gamma, \delta < \kappa$ implies $j \neq k$; $\alpha \neq \beta$ and $\gamma \neq \delta$. Consider $Y := \{k < l | \exists \eta < \kappa\lambda : \kappa\lambda + \eta \in X\}$. We have $Y \in [l]^n$ and in fact Y induces a subdigraph S of D. W.l.o.g. we may assume that S is a tournament because if $\{j, k\} \in [Y]^2$ is independent we have that $X \cap ((\kappa\lambda(j+1) \setminus \kappa\lambda j) \cup (\kappa\lambda(k+1) \setminus \kappa\lambda k) \subset X$ is 0-homogeneous.

  So Y is a tournament and contains a triple $\{i, j, k\} \in [Y]^3$ which is not strongly agreeable. Per definitionem of Y we may conclude that there are $\alpha, \beta, \gamma < \lambda$ and $\delta, \eta, \zeta < \kappa$ such that $\kappa(\lambda i + \alpha) + \delta, \kappa(\lambda j + \beta) + \eta, \kappa(\lambda k + \gamma) + \zeta \in X$. We may distinguish three cases:

  $\diamond$ It is a 3-cycle. Assume $i \mapsto j \mapsto k \mapsto i$. Since X is 1-homogeneous we may, per definitionem of χ, conclude that $\alpha < \beta$, $\beta < \gamma$ and $\gamma < \alpha$. Contradiction!

- ⋄ We have $i \mapsto_b j$, $j \mapsto_b k$ and $i \mapsto_g k$. By 1-homogeneity of X and per definitionem of χ we may conclude that $\delta > \eta$, $\eta > \zeta$ and $\delta < \zeta$. Contradiction!
- ⋄ We have $i \mapsto_g j$, $j \mapsto_g k$ and $i \mapsto_b k$. Again we may use the 1-homogeneity of X and the definition of χ to conclude that $\delta < \eta$, $\eta < \zeta$ and $\delta > \zeta$. Contradiction!

Note that modulo a renaming of i, j and k these three cases exhaust all possibilities.

This finishes the second part of the proof.

■

Similar to before we denote by S_n the class of graphs G on n vertices such that all triples in G are strongly agreeable and by $r(I_m, S_n)$ the smallest natural number l such that all digraphs on l vertices either contain an independent set of size m or an induced subgraph on n vertices all triples of which are strongly agreeable. As before we may use this notation to condense the equivalences proved above into a single formula:

$$r(\kappa\lambda m, n) = \kappa\lambda r(I_m, S_n) \text{ for } \kappa \text{ weakly compact and any cardinal } \lambda \in \kappa \setminus \omega.$$

Analogous to [89Ba] we can prove another nice theorem.

Theorem 3.3. *If* $\mathrm{MA}_{\aleph_1}$ *holds true then* $r(\omega_1\omega m, n) = \omega_1\omega r(I_m, S_n)$.

Let $\chi : [\omega_1\omega l]^2 \longrightarrow 2$ be any colouring. We define $E := \{P \in [\omega_1\omega l]^2 | \chi(P) = 1\}$.

Let $Z := \bigcup_{\alpha<\Omega}[\omega_1(\alpha+1) \setminus \omega_1\alpha]^{\aleph_1}$

For $X \in Z$ let $\alpha_X < \Omega$ be such that $X \in [\omega_1(\alpha_X + 1) \setminus \omega_1\alpha_X]^{\aleph_1}$. Let $A_X := \{\xi < \omega_1 | \omega_1\alpha_X + \xi \in X\}$. We say that the pair $\langle X, Y\rangle$ is

♢ in constellation *black* iff $X, Y \in Z$ and

$$\forall B \in [X]^{\aleph_1}, C \in [Y]^{\aleph_1} \exists \gamma \in A_C, \beta \in A_B \setminus (\gamma+1) : \{\omega_1\alpha_X + \beta, \omega_1\alpha_Y + \gamma\} \in E.$$

♢ in constellation *grey* iff $X, Y \in Z$ and

$$\forall B \in [X]^{\aleph_1}, C \in [Y]^{\aleph_1} \exists \beta \in A_B, \gamma \in A_C \setminus (\beta+1) : \{\omega_1\alpha_X + \beta, \omega_1\alpha_Y + \gamma\} \in E.$$

♢ *free* iff $\forall B \in [X]^{\aleph_1}, C \in [Y]^{\aleph_1} \exists D \in [B]^{\aleph_1}, F \in [C]^{\aleph_1} \forall \delta \in D, \eta \in F : \{\delta, \eta\} \notin E.$

Let us call a set $S \subset \Omega$ weakly independent if for all $\alpha, \beta < \Omega$ with $X = S \cap \omega_1(\alpha+1) \setminus \omega_1\alpha$ and $Y = S \cap \omega_1(\beta+1) \setminus \omega_1\beta$ and $\overline{\overline{X}} = \overline{\overline{Y}} = \aleph_1$ we have that $\langle X, Y\rangle$ is free.

For the proof we are going to use theorem 3.1 from [89Ba]. By limiting its scope to the case $\alpha = \omega$ we get the following theorem.

Theorem 3.4. *Assume* $\mathrm{MA}_{\aleph_1}$*. Let* $E \subset [\omega_1\omega]^2$ *be a graph and let* $W \in [\omega_1\omega]^{\omega_1\omega}$ *be weakly independent. Then there exists* $H \in [W]^{\omega_1\omega}$ *which is independent such that* $\forall n < \omega$: $\overline{\overline{H \cap \omega_1(n+1) \setminus \omega_1 n}} \in \{0, \aleph_1\}$ *and* $H \cap \omega_1(n+1) \setminus \omega_1 n \supsetneq \emptyset$ *implies* $\overline{\overline{W \cap \omega_1(n+1) \setminus \omega_1 n}} \leqslant \aleph_0$ *for all natural* n*.*

Now we prove theorem 3.3.

Proof. Even without $\mathrm{MA}_{\aleph_1}$ it is easy to see that $r(\omega_1\omega m, n) \geqslant \omega_1\omega r(I_m, S_n)$. Let D be a 2-coloured digraph showing $l < r(I_m, S_n)$. Now consider the following colouring:

$$\chi : [\omega_1\omega l]^2 \longrightarrow 2$$

$$\{\omega_1\omega h + \omega_1 j + \alpha, \omega_1\omega i + \omega_1 k + \beta\}_< \longmapsto \begin{cases} 1 \text{ if } h \mapsto_b i \wedge j < k \wedge \beta < \alpha \\ 1 \text{ if } i \mapsto_b h \wedge k < j \wedge \alpha < \beta \\ 1 \text{ if } h \mapsto_g i \wedge j < k \wedge \alpha < \beta \\ 1 \text{ if } i \mapsto_g h \wedge k < j \wedge \beta < \alpha \\ 0 \text{ else.} \end{cases}$$

Analogously to the proof of theorem 3.2 this now shows $\omega_1\omega l \not\to (\omega_1\omega m, n)$.

Now assume $\mathrm{MA}_{\aleph_1}$. Let $l := r(I_m, S_n)$. We are going to show $\omega_1\omega l \to (\omega_1\omega m, n)$ by adapting the method of [89Ba] for our purposes.

Note that if a pair $\langle X, Y\rangle$ is not free there are $X' \in [X]^{\aleph_1}$ and $Y' \in [Y]^{\aleph_1}$ such that $\langle X', Y'\rangle$ is in one of the two other constellations. If at least one of these three properties applies to a pair then we call it *decided*.

We are choosing l-tuples $\langle X_0^F, \dots, X_{l-1}^F\rangle$ with $F \in [\omega]^{<\omega}$ by recursion on $\max(F)$.

- ♢ If $\overline{\overline{F}} = 1$ and $i < l$ let $X_i^F \in [\omega_1(\omega i + \bigcup F + 1) \setminus \omega_1(\omega i + \bigcup F)]^{\aleph_1}$ such that for all $G \in [\bigcup F]^{<\omega}$ and $j \in l \setminus \{i\}$ either $\langle X_j^G, X_i^F\rangle$ is free or there is an $X \in [X_j^G]^{\aleph_1}$ such that $\langle X, X_i^F\rangle$ is in one of the other two constellations above. This can be achieved by thinning out repeatedly.
- ♢ If $\overline{\overline{F}} \in \omega \setminus 2$ let $\mu := \max(F)$ and $G := F \setminus \{\mu\}$. Now by again thinning out repeatedly for all $i < l$ choose $X_i^F \subset X_i^G$ such for all $j \in l \setminus \{i\}$ we have that $\langle X_i^F, X_j^{\{\mu\}}\rangle$ is decided.

Claim 3.5. *For every natural* f *there is an* $F \in [\omega]^f$ *and an* $H \in [\omega]^\omega$ *such that for all* $g, h < l$ *the facts both whether or not for* $x, y \in [H]^i$ *with* $i \in F$ *and* $\max(x) < \min(y)$ *we have that* $\langle X_g^x, X_h^y\rangle$ *is free or not free and what the constellation of a pair* $\langle X_g^x, X_h^y\rangle$ *with* $\min(y) \in x$ *and* $x, y \in [H]^i$ *is only depends on* g *and* h*.*

Proof of Claim: We repeatedly use Ramsey's theorem to arrive at F and H.

Using Ramsey's theorem we thin ω to some $H_0 \in [\omega]^\omega$ on which the freeness of some such $\langle X_g^x, Y_h^y\rangle$ for $x, y \in [H_0]^1$ only depends on $\{g, h\}$ and whether $\max(x) < \min(y)$ or $\max(y) < \min(x)$.

Because of the nature of our recursive construction the constellation of a pair $\langle X_g^x, X_h^y\rangle$ with $\min(y) \in x$ is determined by the constellation of the pair $\langle X_g^{x \cap (\min(y)+1)}, X_h^{\{\min(y)\}}\rangle$. The constellation of this pair however can be made canonical as well using Ramsey's theorem. So inductively do this for $x \cap \min(y) = i$ to arrive at some H_i' for $i < \omega$ and after that find an $H_{i+1} \in [H_i']^\omega$ on which the freeness/non-freeness for $x, y \in [H_{i+1}]^{i+1}$ with $\max(x) < \min(y)$ is canonical, i.e. only depends on g and h. This yields a sequence $H_0 \supset H_0' \supset H_1 \supset H_1' \supset \ldots$.

Since there are only finitely many possible patterns of both freeness/non-freeness and constellations we may find an $F \in [\omega]^f$ and an H in the sequence which is sufficiently thin such that these patterns are the same for all $i \in F$. ■(Claim)

Use the claim to find F of a sufficiently(for what folllows) large finite size $2L$, say $2^{n+1}-2$. Now we define a digraph by letting there be arrows as follows:

$$g \mapsto_b h \text{ implies } \langle X_g^x, X_h^{\{\max(x)\}}\rangle \text{ generally is in constellation } \textit{black},$$
$$g \mapsto_g h \text{ implies } \langle X_g^x, X_h^{\{\max(x)\}}\rangle \text{ generally is in constellation } \textit{grey}.$$

Moreover whenever one of these conditions applies an arrow is set.

Now first suppose there were an independent set $I \in [l]^m$. Let $\langle h_i | i < \omega\rangle$ be the ascending enumeration of H and pick some $f \in F$. Let $Z := \{\{h_{fi}, \ldots, h_{f(i+1)-1}\} | i < \omega\}$. Then the union of all the X_g^x's for $g \in I$ and $x \in Z$ is a weakly independent set of size $\omega_1\omega m$. This is because by our recursive construction and by definition of our digraph $\langle X_g^{x\setminus\{\max(x)\}}, X_h^y\rangle$ is free whenever $\max(x) \in y$ and there is no arrow between g and h . We now may use theorem 3.4 to thin this out to some truly independent set of the same order-type, i.e. some set of order-type $\omega_1\omega m$ which is homogeneous for χ in colour 0.

Now suppose that our digraph contains no independent set of size m. Since we chose l to be $r(I_m, S_n)$ we know that there is a strongly agreeable set $S \in [l]^n$.

Let $\langle f_i | i < L\rangle$ be an ascending enumeration of F. Define $a_i := \{h_{f_i}, \ldots, h_{f_L}\rangle$ and let $A_g^i := X_g^{a_i}$.

Claim 3.6. *Let $g, h \in S$, $\{i,j\} \in [L]^2$ and $i < j$. Now if $g \mapsto_b h$ then $\langle A_g^i, A_h^j\rangle$ is in constellation black and if $g \mapsto_g h$ then it is in constellation grey.*

Proof of Claim: Consider $x := \{h_{f_i}, \ldots, h_{f_{j-1}}\}$ and $y := \{h_{f_j}, \ldots, h_{f_{2j-i-1}}\}$.

We know that $\langle X_g^x, X_h^y\rangle$ is not free. Because of $X_g^y \subset X_h^{\{h_{f_j}\}}$ we have that $\langle X_g^x, X^{\{h_{f_j}\}}\rangle$ is not free. By our recursive construction we get that $\langle X_g^{x\cup\{h_{f_j}\}}, X_h^{\{h_{f_j}\}}\rangle$ is decided. The colour of the arrow between g and h in S tells us how. Since $A_h^i \subset X_h^{x\cup\{h_{f_j}\}}$ and $A_g^j \subset X_g^{\{h_{f_j}\}}$ the claim follows. ■(Claim)

Now we are going to construct the complete graph $G \in [\omega_1\omega l]^n$ with $[G]^2 \subset E$. To this end let $\langle s_i | i < n\rangle$ be an enumeration of S such that $s_i \mapsto_g s_j$ or $s_j \mapsto_b s_i$ for all $i < j < n$. This is possible since all triples in S are strongly agreeable. To start the construction now, i.e. in step 0 choose the middle element j_0 of L. This divides L into two intervals of size, say, $2^{n-1}-1$. Let us call $\{-1, L\}$ *the border*.

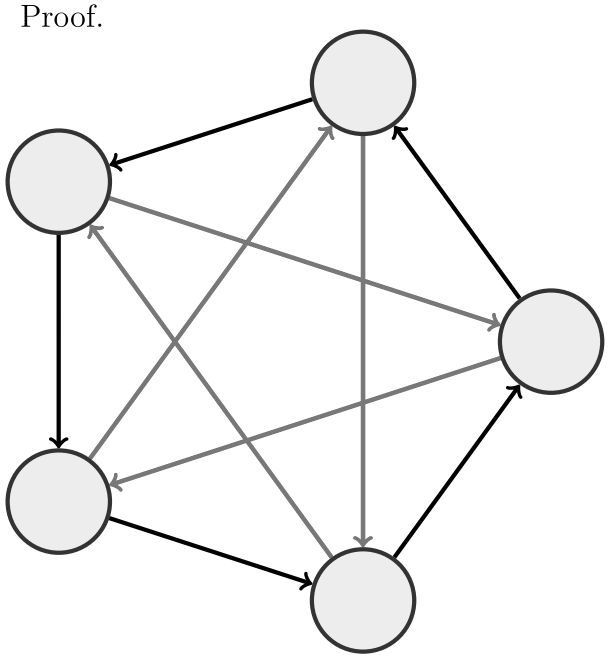
Now inductively in step $i < n$ we haven chosen j_k's with $k < i$ such that the minimal distance between either of them and between one of them and the border is $2^{n-i}-1$. Now we choose an j_{i+1} such that the minimal distance between two of the j_{k+1}'s for $k \leqslant i$ and between of them and the border is $2^{n-i-1}-1$ and such that $j_{i+1} > j_k$ iff $s_k \mapsto s_i$ for all $k \leqslant i$. Again strong agreeability allows us to do this.

Finally we choose in step 0 an ordinal $\alpha_0 \in A^{j_0}_{s_0}$ such that for all $i \in n \setminus 1$ we have $\overline{\overline{\{\alpha \in A^{j_i}_{s_i} | \{\alpha_0, \alpha\} \in E\}}} = \aleph_1$. Such an ordinal exists for otherwise there would be an $i < n$ and uncountably many $\alpha \in A^{j_0}_{s_0}$ which are only connected to countably many things in $A^{j_i}_{s_i}$ by E. Because of the way in which we sorted the elements of S this would contradict the constellation of $\langle A^{j_0}_{s_0}, A^{j_i}_{s_i}\rangle$. Now thin out the $A^{j_i}_{s_i}$ for $i \in n \setminus 1$ to those uncountable subsets of ordinals the unordered pairs with α_0 of which are in E. Now generally in step $i < n$ choose an ordinal $\alpha_i \in A^{j_i}_{s_i}$ such that $\{\alpha_k, \alpha_i\} \in E$ for all $k < i$ and thin out the $A^{j_k}_{s_k}$'s for $k \in n \setminus (i+1)$. In this way we may construct our complete graph of size n, i.e. our set homogeneous for χ in colour 1. ■

Now we may take a look at the easiest examples.

Theorem 3.7. $r(I_2, S_3) = 6$.

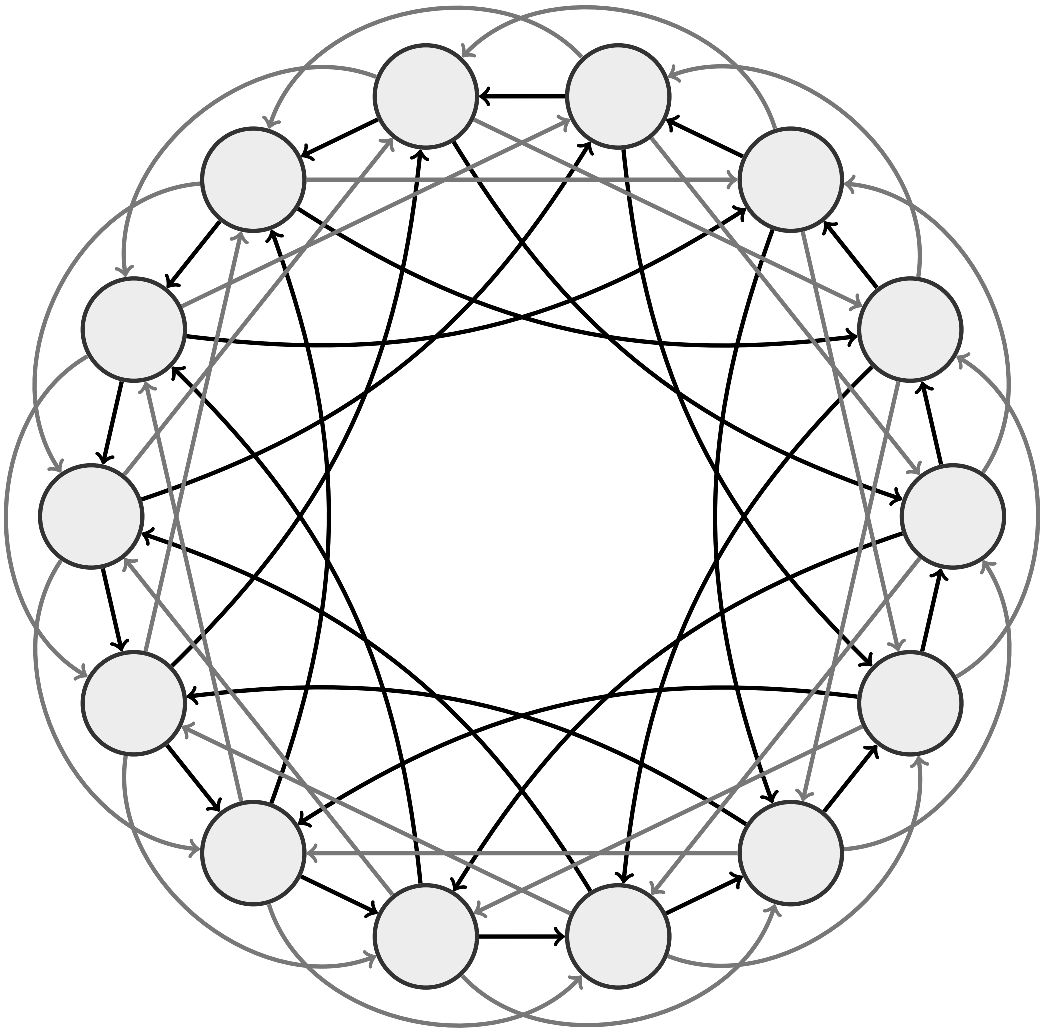
Proof.



The lower bound is provided by the counterexample on the right, for the upper bound one can argue as follows:
Consider any vertex v in a six-element tournament—if the digraph is not a tournament we are finished immediately. Now $N^-_B(v) + N^+_B(v) + N^-_G(v) + N^+_G(v) \geqslant 5$ so we may conclude using the pigeonhole principle that one of these four neighborhoods contains at least two elements. But then v together with this neighborhood is strongly agreeable.

■

Theorem 3.8. $r(I_3, S_3) = 15$.



Proof.

The lower bound is given by the counterexample above. It can be viewed as a digraph on $\mathbb{Z}_{14}$ where $v \mapsto_b v+1$, $v \mapsto_g v+2$, $v \mapsto_g v-3$ and $v \mapsto_b v-4$ for all $v \in \mathbb{Z}_{14}$.

■

The upper bound is established by theorem 4.4 below.

Corollary 3.9.
Let κ be the least weakly compact cardinal. Then $r(\kappa\aleph_{\varepsilon_{\omega 7+9}}3, 3) = \kappa\aleph_{\varepsilon_{\omega 7+9}}15$.

4 Upper bounds

4.1 Upper bounds for independent versus strongly agreeable sets.

We now are going to provide several general upper bounds for various Ramsey numbers. We start with something easy—an upper bound for the Ramsey numbers $r(I_m, S_3)$.

Lemma 4.1. $r(I_{m+1}, S_3) \leqslant r(I_m, S_3) + 4m + 1$ *for all* $m \in \omega \setminus 2$.

Proof. Let D be any edge-bicoloured digraph on $r(I_m, S_3) + 4m + 1$ vertices. Fix any vertex v. Since $\overline{\overline{N_B^+(v)}} + \overline{\overline{N_B^-(v)}} + \overline{\overline{N_G^+(v)}} + \overline{\overline{N_G^-(v)}} + \overline{\overline{I(v)}} = r(I_m, S_3) + 4m$ it follows that one of the following cases has to obtain:

- ♢ $\overline{\overline{N_B^+(v)}} \geqslant m + 1$.
- ♢ $\overline{\overline{N_B^-(v)}} \geqslant m + 1$.
- ♢ $\overline{\overline{N_G^+(v)}} \geqslant m + 1$.
- ♢ $\overline{\overline{N_G^-(v)}} \geqslant m + 1$.
- ♢ $\overline{\overline{I(v)}} \geqslant r(I_m, S_3)$.

In the first four cases either the respective neighbourhood is independent or two vertices in it are connected by an arrow in which case they form an S_3 together with v. In the last case the neighbourhood either contains an S_3 or an independent set of size m which together with v would form an independent set of size $m + 1$. ■

Theorem 4.2. *For all* $m \in \omega \setminus 2$ *we have* $r(I_m, S_3) \leqslant m(2m - 1)$.

Proof. The base case is established by theorem 3.7. For the inductive step by lemma 4.1 we only have to check that

$$(m + 1)(2(m + 1) - 1) = m(2m - 1) + 4m + 1$$

which is true. ■

We continue now by providing an upper bound for $r(I_2, S_n)$.

Lemma 4.3. *For all* $n \in \omega \setminus 3$ *we have* $r(I_2, S_{n+1}) \leqslant 4r(I_2, S_n) - 2$.

Proof. Let $T = \langle V, A, c\rangle$ be a bicoloured tournament on $4r(I_2, S_n) - 2$ vertices. Fix any vertex $v \in V$. Since $\overline{\overline{N_B^-(v) + N_B^+(v) + N_G^-(v) + N_G^+(v)}} = 4r(I_2, S_n) - 3$ by the pigeonhole principle one of these neighbourhoods N has cardinality at least $r(I_2, S_n)$. Hence there is an $S \in [N]^n$ in which all triples are strongly agreeable. So if we set $X := S \cup \{v\}$ we have $X \in [V]^{n+1}$ and all triples in X are strongly agreeable. ■

Theorem 4.4. *For any $n \in \omega \setminus 3$ we have*

$$r(I_2, S_n) \leqslant \frac{4^{n-1} + 2}{3}.$$

Proof. By induction. The base case is given by theorem 3.7. The induction step also works:

$$4 \cdot \frac{4^{n-1} + 2}{3} - 2 = \frac{4^{n-1+1} + 4 \cdot 2 - 2 \cdot 3}{3} = \frac{4^{n+1-1} + 2(4-3)}{3}$$

■

Again, note that for any integer n also $\frac{4^{n-1}+2}{3}$ is an integer. It is easy to see by induction that $4^{n-1} \equiv 1(3)$ for any $n \in \omega \setminus 1$. Hence $4^{n-1} + 2 \equiv 0(3)$ for any $n \in \omega \setminus 1$.

Now we want to give a general upper bound for $r(I_m, S_n)$.

Lemma 4.5. *$r(I_{m+1}, S_{n+1}) \leqslant r(I_m, S_{n+1}) + 4r(I_{m+1}, S_n) - 3$ for all $m \in \omega \setminus 2$ and all $n \in \omega \setminus 3$.*

Proof. Immediate. ■

Theorem 4.6. *For all $m \in \omega \setminus 2$ and all $n \in \omega \setminus 3$ we have $r(I_m, S_n) \leqslant u(m, n)$ where*

$$u(m, n) := \frac{1}{4}\Big(3 + \sum_{i=0}^{n-1} \binom{i + m - 2}{i} 4^i\Big). \tag{7}$$

Proof. By induction. First we verify that $u(2, 3) = 6$. Recall that we have $r(I_2, S_3) = 6$ by theorem 3.7. Next we check that for $u(2, n)$ agrees with the formula from theorem 4.4. We have

$$u(2, n) = \frac{1}{4}\big(3 + \sum_{i=0}^{n-1} 4^i\big).$$

Consider $v(n) := u(2, n+1) - u(2, n)$. It is easy to see that $v(n) = 4^{n-1}$. But it is also easy to see that

$$\frac{4^n + 2}{3} - \frac{4^{n-1} + 2}{3} = 4^{n-1}.$$

Induction on n provides what was demanded.

It is even easier to see, using well-known properties of the binomial coefficients that $u(m, 3) = m(2m - 1)$.

Now we can check that in fact

$$u(m+1, n+1) = u(m, n+1) + 4u(m+1, n) - 3. \tag{8}$$

Then with lemma 4.5 the theorem will follow. The following calculation proves (8).

$$\begin{aligned}
& u(m, n+1) + 4u(m+1, n) - 3 \\
= \ & \frac{1}{4}\Big(3 + \sum_{i=0}^{n} \binom{i+m-2}{i} 4^i + 4\sum_{i=0}^{n-1} \binom{i+m-1}{i} 4^i\Big) \\
= \ & \frac{1}{4}\Big(3 + \sum_{i=0}^{n} \binom{i+m-2}{i} 4^i + \sum_{i=1}^{n} \binom{i+m-2}{i-1} 4^i\Big) \\
= \ & \frac{1}{4}\Big(4 + \sum_{i=1}^{n} \Big(\binom{i+m-2}{i} + \binom{i+m-2}{i-1}\Big) 4^i\Big) \\
= \ & \frac{1}{4}\Big(4 + \sum_{i=1}^{n} \binom{i+m-1}{i} 4^i\Big) \\
= \ & \frac{1}{4}\Big(3 + \sum_{i=0}^{n} \binom{i+m-1}{i} 4^i\Big) \\
= \ & u(m+1, n+1).
\end{aligned}$$

■

4.2 Upper bounds for independent versus agreeable sets.

We are now going to provide an upper bound for the Ramsey numbers $r(I_n, A_3)$.

Lemma 4.7. *$r(I_{n+1}, A_3) \leqslant r(I_n, A_3) + 2r(I_{n+1}, L_3) + 4n - 1$ for all $n \in \omega \setminus 2$.*

Proof. Let there be an edge-tricoloured digraph $\langle V, A\rangle$ with $r(I_n, A_3) + 2r(I_{n+1}, L_3) + 4n - 1$ vertices. Pick any vertex v such that $N_B^+(v)$ is maximal. Note that $\overline{\overline{V \setminus \{v\}}} = r(I_n, A_3) + 2r(I_{n+1}, L_3) + 4n - 2$! Now we may distinguish several cases. Note that necessarily at least one of them must hold!

◇ $\overline{\overline{N_B^-(v)}} \geqslant n + 1$. Since we chose v to have maximum solid black out-degree we might immediately hand over to the next case.

◇ $\overline{\overline{N_B^+(v)}} \geqslant n+1$. In this case either there are two vertices $w, x \in N_B^+(v)$ which are connected by an arrow—in which case $\{v, w, x\}$ is an agreeable triple—or $N_B^+(v)$ forms an independent set of size $n+1$. In both cases we found a set homogeneous in the sought-after-sense.

◇ $\overline{\overline{N_G^-(v)}} \geqslant r(I_{n+1}, L_3)$. If there are $w, x \in N_G^-(v)$ which are connected by either a solid black or a solid grey arrow then $\{v, w, x\}$ is an agreeable triple. So let us assume that all pairs of vertices in $N_G^-(v)$ are either connected by dashed black arrows or not connected at all. But now our case condition implies that we either have an independent set of size $n+1$ or a transitive dashed black triple—which is agreeable.

◇ $\overline{\overline{N_G^+(v)}} \geqslant r(I_{n+1}, L_3)$. Works exactly like the case before.

◇ $\overline{\overline{N_D^-(v)}} \geqslant n+1$. This and the next case work like the second. Either there are two vertices $w, x \in N_D^-(v)$ which are connected by an arrow—then $\{v, w, x\}$ is agreeable—or $N_D^-(v)$ is an independent set of size $n+1$.

◇ $\overline{\overline{N_D^+(v)}} \geqslant n + 1$. Again either there are two vertices which are connected which then form an agreeable triple together with v or $N_D^+(v)$ is independent.

◇ $\overline{\overline{I(v)}} \geqslant r(I_n, A_3)$. Now in the last case $I(v)$ either contains an agreeable triple or is itself independent of size n. In the first case we are finished immediately and in the second $I(v) \cup \{v\}$ is an independent set of size $n+1$.

■

Corollary 4.8. *$r(\omega^2(n+1), 3) \leqslant r(\omega^2 n, 3) + \omega 2r(\omega(n+1), 3) + \omega^2(4n-1)$ for all $n \in \omega \setminus 2$.*

Theorem 4.9. *For all $n \in \omega \setminus 2$ we have*

$$r(I_n, A_3) \leqslant \frac{(2n+1)(n^2+4n-6)}{3}. \tag{9}$$

Proof. By induction. The basis case is $n = 2$, formula 9 is verified by theorem 2.14 in it. So let us assume that the formula holds up to some $n \in \omega \setminus 2$. Using lemma 4.7 we get

$$r(I_{n+1}, A_3) \leqslant r(I_n, A_3) + 2r(I_{n+1}, L_3) + 4n - 1.$$

Larson and Mitchell—see Lemma 4.1 of [97LM]—showed that $r(I_n, L_3) \leqslant n^2$, so we may bound this from above as follows:

$$
\begin{aligned}
&\leqslant \frac{(2n+1)(n^2+4n-6)}{3} + 2(n+1)^2 + 4n - 1 \\
&= \frac{(2(n+1)+1)((n+1)^2+4(n+1)-6)}{3}
\end{aligned}
$$

■

Note that the fraction in theorem 4.9 is an integer if n is one. If $n \equiv 1(3)$ then $2n+1 \equiv 0(3)$ and we are fine. If, however, $n \equiv 0(3)$ then $n^2+4n-6 \equiv 0(3)$ and if $n \equiv 2(3)$ then $n^2+4n-6 \equiv 0(3)$ too.

5 A semiconstructive lower bound

Theorem 5.1. $r(I_{mn+1}, A_3) > (r(I_{m+1}, L_3) - 1)(r(I_{n+1}, L_3) - 1)$.

Proof. Let $D_0 = \langle V_0, A_0 \rangle$ be a digraph on $r(I_m, L_3) - 1$ vertices neither containing a transitive triple nor an independent set of size $m+1$ and let $D_1 = \langle V_1, A_1 \rangle$ be a digraph on $r(I_n, L_3) - 1$ vertices neither containing a transitive triple nor an independent set of size $n+1$. Then consider the coloured digraph $D_0 * D_1 := \langle V_0 \times V_1, A_2, c \rangle$ where $A_3 = \{\langle \langle v_0, v_1 \rangle, \langle v_2, v_1 \rangle \rangle | v_0, v_2 \in V_0 \wedge v_1 \in V_1 \wedge \langle v_0, v_2 \rangle \in A_0\}$, $A_4 = \{\langle \langle v_0, v_1 \rangle, \langle v_2, v_3 \rangle \rangle | v_0, v_2 \in V_0 \wedge v_1, v_3 \in V_1 \wedge \langle v_1, v_3 \rangle \in A_1\}$, $A_2 = A_3 \cup A_4$, $c``A_3 = \{1\}$ and $c``A_4 = \{2\}$. Clearly the independent sets in $D_0 * D_1$ have size at most mn and the only transitive triples in $D_0 * D_1$ contain two solid grey arrows which either originate from the same vertex or point at the same. Both types of triples are disagreeable. ■

6 Some tables

At the end we take the liberty of supplying a table of some Ramsey numbers $r(\alpha, n)$ known today for countable ordinals α and natural numbers n. Following the example of Radziszowski in [94Ra] we also supply a complementary table containing references. If a reference is given above another, then the upper one gives the upper bound and the lower one gives the lower bound.

	3	4	5	6	7	8	9	m
3	6	9	14	18	23	28	36	
4	9	18	25					
ω	ω	ω	ω	ω	ω	ω	ω	ω
$\omega+1$	$\omega2+1$	$\omega3+1$	$\omega4+1$	$\omega5+1$	$\omega6+1$	$\omega7+1$	$\omega8+1$	$\omega(m-1)+1$
$\omega+2$	$\omega2+4$	$\omega3+7$	$\omega4+11$	$\omega5+16$	$\omega6+22$	$\omega7+29$	$\omega8+37$	$\omega(m-1)+\frac{m(m-1)}{2}+1$
$\omega+3$	$\omega2+8$	$\omega3+16$						
$\omega+n$	$\omega2+r(n,3)+n-1$							
$\lambda2$	$\lambda4$	$\lambda8$	$\lambda14$	$\lambda28$				
$\lambda3$	$\lambda9$							
ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
ω^2+1	ω^22+1	ω^23+1	ω^24+1	ω^25+1	ω^26+1	ω^27+1	ω^28+1	$\omega^2(m-1)+1$
ω^2+2	ω^22+4	ω^23+7	ω^24+11	ω^25+16	ω^26+22	ω^27+29	ω^28+37	$\omega^2(m-1)+\frac{m(m-1)}{2}+1$
ω^2+3	ω^22+8	ω^23+16						
ω^2+n	$\omega^22+r(n,3)+n-1$							
$\omega^2+\omega$	$\omega^24+\omega$							
ω^22	ω^210							
ω^3	ω^4	ω^4	ω^5	ω^5	ω^5	ω^5	ω^6	$\omega^{2+\lceil \mathrm{ld}(m)\rceil}$
ω^3+n	$\omega^4+r(n,3)$	$\omega^4+\omega^3+r(n,4)+n-1$						
ω^4	ω^7	ω^7	ω^{10}	ω^{10}	ω^{10}	ω^{10}		
ω^{5+n}	ω^{9+2n}	ω^{9+2n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{17+4n}	$\omega^{1+(4+n)\lceil \mathrm{ld}(m)\rceil}$
ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω
ω^{ω^2}	ω^{ω^2}	ω^{ω^2}						
$\kappa\lambda2$	$\kappa\lambda6$							
$\kappa\lambda3$	$\kappa\lambda15$							

	3	4	5	6	7	8	9	10+m
3		[55GG]	[55GG]	[64Ké]	[68GY] [66Ka]	[92MM] [82GR]	[82GR] [66Ka]	
4	[55GG]	[55GG]	[95MR] [65Ka]					
ω	[30Ra]	[30Ra]	[30Ra]	[30Ra]	[30Ra]	[30Ra]	[30Ra]	[30Ra]
$\omega+1$	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]
$\omega+2$	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]
$\omega+3$	[69HSc]	[69HSc]						
$\omega+n$	[69HSc]							
$\lambda 2$	[56ER]/[74Ba]	[64EM]/[74Ba]	[70RP]/[74Ba]	[70RP]/[74Ba]				
$\lambda 3$	[74Be]/[74Ba]							
ω^2	[57Sp]	[57Sp]	[57Sp]	[57Sp]	[57Sp]	[57Sp]	[57Sp]	[57Sp]
ω^2+1	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]
ω^2+2	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]	[69HSc]
ω^2+3	[69HSc]	[69HSc]						
ω^2+n	[69HSc]							
$\omega^2+\omega$	[69HSb]							
$\omega^2 2$	[69HSa]							
ω^3	[57Sp]	[69HSb]	[74No]	[74No]	[74No]	[74No]	[74No]	[74No]
ω^3+n	[69Mi]	[69HSb]						
ω^4	[75No]	[75No]	[10HL]	[10HL]	[10HL]	[10HL]		
ω^{5+n}	[79No]	[79No]	[79No]	[79No]	[79No]	[79No]	[79No]	[79No]
ω^ω	[74La]	[74La]	[74La]	[74La]	[74La]	[74La]	[74La]	[74La]
ω^{ω^2}	[upDL]	[upDL]						
$\kappa\lambda 2$	This paper!							
$\kappa\lambda 3$	This paper!							

Combinations of the methods used to find the numbers above would probably generalize some results.

7 Problems, Questions & Conjectures

To the author the most interesting open question now is the following:

Question 7.1. *What is the order of growth of the function $n \mapsto r(I_n, A_3)$?*

The author conjectures that he not being able to present a better upper bound is at most partly due to his personal incompetence. So he states the following conjecture—which to his mind seems very modest indeed.

Conjecture 7.2. *There exists an $\varepsilon > 0$ such that $\varepsilon n^{2+\varepsilon} < r(I_n, A_3)$ for all natural n.*

The author was considerably surprised by the second counterexample showing $r(I_2, A_3) > 9$ since there was no solid black arrow in it. Hence the following seems to be a natural question to the author:

Question 7.3. *Are sharp lower bounds for $r(I_m, A_n)$ for m and n natural numbers always attainable by counterexamples using only solid grey and dashed black arrows?*

The author conjectures the answer to this question to be "No.".

The author was not able to come up with a formula for the upper bounds given by the recurrence relations for $r(I_m, A_n)$. Perhaps an expert on recurrence relations can find one.

Problem 7.4. *Find a nice formula for a general upper bound for $r(I_m, A_n)$!*

It is fairly obvious that the methods of this paper could be extended to determine Ramsey numbers for even more decomposable ordinals. The related finite combinatorics might turn out to be extremely difficult though. We mention just one example as an open problem.

Problem 7.5. *Determine $r(\kappa\lambda\omega 2, 3)$ for $\kappa > \lambda$ both weakly compact!*

The motivation behind this research was mainly an interest in countable ordinals. So although we have not discussed Ramsey numbers involving ordinals in $\omega_1 \setminus \omega^3$ here at all there are many open problems there. As a teaser we mention the following:

Problem 7.6. *Determine $r(\omega^{\omega 2}, 3)$!*

The author in vain tried to strengthen theorem 3.3 by replacing $\mathrm{MA}_{\aleph_1}$ by "$\omega_1\omega$ is a partition ordinal.". So he has to leave this as an open question.

Question 7.7. *Does $r(\omega_1\omega m, n) = \omega_1\omega r(I_m, S_n)$ whenever $\omega_1\omega$ is a partition ordinal?*

A variant of this problem is the following:

Question 7.8. *Does $r(\omega_1\omega m, n) = \omega_1\omega r(I_m, S_n)$ whenever $\omega_1\omega^2$ is a partition ordinal?*

At the end we want to mention that there are problems in Ramsey theory which to the author's mind could have been attacked algorithmically but were not. The author conjectures that $r(I_4, L_3)$ and $r(I_3, L_4)$ could easily be algorithmically determined. The same should hold true for $r(I_2, S_4)$. Meanwhile the author is rather sceptical about the possibilities to determine $r(I_3, A_3)$ or $r(I_2, A_4)$ without considerable further mathematical insight.

The author wishes to thank at this point Rene Schipperus for getting him interested in this subject, his PhD advisor Stefan Geschke for allowing him to venture into the realms of it, Benjamin Seyfferth for his help in reading parts of the french articles [69HSc], [69HSa] and [69HSb], Andrés Caicedo, Peter Komjath, Lionel Ngyuen Van The, Philipp Schlicht, Stevo Todorčević, Wolfgang Wohofsky and in particular Jean Larson for discussing this work with them. Moreover [98CG] was a great source of inspiration for the author. At last I would also like to thank the referee her/his input.[4]

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[4]The research presented in this article was supported by the DFG project "Continuous Ramsey theory in higher dimension".

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