Why proofs by contradiction are nice

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Abstract

What can be shown cannot be said.

Definition 1 $\diamondsuit$ *is the statement that there exists a sequence* $\langle A_\alpha | \alpha < \aleph_1 \rangle$ *such that for every* $A \subset \aleph_1$ *the set* $\left\{ \alpha < \aleph_1 \middle| A_\alpha = A \cap \alpha \right\}$ *is stationary.*

Definition 2 $\diamondsuit^+$ *is the assertion, that there is a sequence* $\langle A_\alpha | \alpha < \aleph_1 \rangle$ *such that*

- $\forall \alpha < \aleph_1 : A_\alpha \in [\mathfrak{P}(\alpha)]^{<\omega_1}$
- $\forall A \subset \aleph_1 \exists C$ *club in* $\aleph_1 \, \forall \alpha (\alpha \in C \rightarrow (A \cap \alpha \in A_\alpha \wedge C \cap \alpha \in A_\alpha)$

Theorem 3 $\diamondsuit^+ \Longrightarrow \diamondsuit$

Proof. Suppose towards a contradiction, that this is wrong, that is suppose $\diamondsuit^+ \wedge \neg\diamondsuit$. That means, that on the one hand

- $\exists \langle A_\alpha | \alpha < \aleph_1 \rangle (\forall \alpha < \aleph_1 : A_\alpha \in [\mathfrak{P}(\alpha)]^{<\omega_1} \wedge \forall A \subset \aleph_1 \exists C \text{ club in } \aleph_1 \, \forall \alpha \in C : A \cap \alpha \in A_\alpha)$,
  but on the other hand
- $\forall \langle B_\alpha | \alpha < \aleph_1 \rangle \, \exists B \subset \aleph_1, C \text{ club in } \aleph_1 \, \forall \alpha \in C : B \cap \alpha \neq B_\alpha$,

Let $\langle A_\alpha | \alpha < \aleph_1 \rangle$ be a witness for the truth of the first clause and let $e_\alpha : \omega \longrightarrow A_\alpha$ be enumerations of the sets A_α. Now define for $\alpha < \aleph_1$ and $n < \omega$ sets $B^n_\alpha := \left\{ \beta \middle| \beta < \aleph_1 \wedge \omega \cdot \beta + n \in e_\alpha(n) \right\}$ and thus sequences $\langle B^n_\alpha | \alpha < \aleph_1 \rangle$. Now the second clause yields the existence of $\aleph_0$ witnesses $B_n \subset \aleph_1$ for $n < \omega$ and clubs $C_n \subset \aleph_1$ such that $\forall n < \omega, \alpha \in C_n : B_n \cap \alpha \neq B^n_\alpha$. Now set $A := \left\{ \omega \cdot \beta + n \middle| \beta < \aleph_1 \wedge n < \omega \wedge \beta \in B_n \right\}$, let C_ω witness the truth of the first clause for $A \subset \aleph_1$ and finally let $C_{\omega+1} := \left\{ \omega^{\omega^\alpha} \middle| \alpha \in \aleph_1 \setminus 1 \right\}$. Set $C := \bigcap_{\gamma < \omega+2} C_\gamma$. C is clearly club. Take a $\delta \in C$. Then $A \cap \delta \in A_\delta$. This is witnessed by a natural number m, i.e. an $m < \omega$ such that $e_\delta(m) = A \cap \delta$. Take now a witness η for $B_m \cap \delta \neq B^m_\delta$ that is an $\eta \in B^m_\delta \triangle (B_m \cap \delta)$. Then immediately it follows, that $\omega \cdot \eta + m \in e_\delta(m) \triangle (A \cap \delta) = \emptyset$. $\lightning$ $\dashv$