NEW BOUNDS ON THE RAMSEY NUMBER $r(I_m, L_n)$

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Abstract. We investigate the Ramsey numbers $r(I_m, L_n)$ which is the minimal natural number k such that every oriented graph on k vertices contains either an independent set of size m or a transitive tournament on n vertices. Apart from the finitary combinatorial interest, these Ramsey numbers are of interest to set theorists since it is known that $r(\omega m, n) = \omega r(I_m, L_n)$, where ω is the lowest transfinite ordinal number, and $r(\kappa m, n) = \kappa r(I_m, L_n)$ for all initial ordinals κ. Continuing the research by Bermond from 1974 who did show $r(I_3, L_3) = 9$, we prove $r(I_4, L_3) = 15$ and $r(I_5, L_3) = 23$. The upper bounds for both the estimates above are obtained by improving the upper bound of m^2 on $r(I_m, L_3)$ due to Larson and Mitchell (1997) to $m^2 - m + 3$. Additionally, we provide asymptotic upper bounds on $r(I_m, L_n)$ for all $n \geq 3$. In particular, we show that $r(I_m, L_3) \in \Theta(m^2/\log m)$.

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1. Introduction

Ramsey Theory, sometimes described as the collection of mathematics conforming to the slogan that total disorder is impossible, can be traced back to [930Rams]. In this paper Ramsey proved the theorem now named after him. Shortly thereafter, Paul Erdős and George Szekeres, in [935ES], answered a question of Esther Klein affirmatively. She had previously shown that among any five points in the plane one can find four forming a convex quadrilateral and asked whether this result could be generalised. After a few more years, Erdős and Rado proved a cornucopia of theorems in their seminal paper [956ER]. This paper also saw the introduction of the *partition symbol* whose definition we restate here. We also use $[X]^n$ to denote the family of all n-element subsets of X while ω denotes the order-type of the natural numbers.

Definition 1.1. $\alpha \longrightarrow (\beta_0, \ldots, \beta_k)^n$ *means that for any set X of size α and any function (which may be called colouring) $\chi : [X]^n \longrightarrow k+1$ there is an $i \leqslant k$ and a $Y \subset X$ of size β_i which is* homogeneous *for χ, i.e. the colouring χ is constant on $[Y]^n$. We call α the* source *and $\beta_0, \ldots, \beta_k$ the* targets *of the relation.*

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The notion of size in this definition is usually interpreted to mean cardinality but one can in principle employ any notion of size for the formulation of a corresponding partition relation. The main alternative is to consider the notion of linear order-type. This is particularly attractive since there is exactly one linear order-type for every finite cardinal (also known as natural number) n. Also, assuming the Axiom of Choice, every set can be well-ordered, so replacing every cardinal in the relation above by its corresponding initial ordinal (the smallest ordinal of that cardinality), yields an equivalent statement. Therefore replacing cardinality by linear order-type amounts to a proper extension of the scope of Definition 1.1. Now Ramsey's Theorem for two colours may be stated as follows:

Theorem 1.2. *$\omega \longrightarrow (\omega, \omega)^n$ for all natural numbers n.*

There exists an alternative notation which is more common in papers on finitary combinatorics:

Definition 1.3. *$r_n(\beta_0, \ldots, \beta_k)$ is the smallest α such that $\alpha \longrightarrow (\beta_0, \ldots, \beta_k)^n$ but $\gamma \not\longrightarrow (\beta_0, \ldots, \beta_k)^n$ for all γ smaller than α. The number $r(\beta_0, \ldots, \beta_k)$ is understood to be $r_2(\beta_0, \ldots, \beta_k)$.*

As the finite linear orders are themselves well-ordered by proper inclusion, for finite α the relation $\alpha \longrightarrow (\beta_0, \ldots, \beta_k)^n$ is equivalent to $r_n(\beta_0, \ldots, \beta_k) \geqslant \alpha$.

In this paper we denote a set of m pairwise independent vertices by I_m, the undirected complete graph on n vertices by K_n and the transitive tournament on n vertices by L_n. By an oriented graph we mean a graph without loops or double edges all edges of which have an orientation.

If an oriented graph D does not contain an oriented graph S as an induced subgraph, then we say that D is S-free. If S_0, S_1 are oriented graphs and D contains neither as an induced subgraph, we say that D is (S_0, S_1)-free. For natural numbers m and n let $r(I_m, L_n)$ denote the minimal natural number k such that no oriented graph on k vertices is (I_m, L_n)-free. In this spirit, the classical undirected Ramsey numbers $r(m, n)$, which is the minimal natural number k such that no undirected graph on k vertices is (I_m, K_n)-free will be denoted here as $r(I_m, K_n)$.[1]

Then [956ER, Theorem 25] amounts to the following:

Theorem 1.4 (Erdős and Rado [956ER])**.** *$r(\omega m, n) = \omega r(I_m, L_n)$.*

In [967ER] Erdős and Rado showed that for any infinite initial ordinal κ and any natural numbers m and n there is a natural number ℓ such that $r(\kappa m, n) \leqslant \kappa\ell$. They

[1] Notice that some authors use $r(H_1, H_2)$ to denote the minimal natural number k such that for every graph G on k vertices, either G contains H_1 or the complement of G contains H_2. But we do not follow that notation in this paper.

conjectured that ℓ never depends on κ. In [974Baum] Baumgartner proved this. In fact, we have the following:

Theorem 1.5 (Baumgartner [974Baum]). *$r(\kappa m, n) = \kappa r(I_m, L_n)$ for all infinite initial ordinals κ.*

The case $m = 2$ received a decent amount of attention, c.f. [994SF], also the third author published results along the lines of [974Baum] involving products of more than one infinite initial ordinal in [014Wein]. Yet the last paper published on the numbers $r(I_m, L_n)$ which is known to the authors was published two decades ago by Larson and Mitchell, c.f. [997LM]. In this paper they proved $r(I_m, L_3) \leqslant m^2$ using a degree argument and provided a counterexample showing $r(I_4, L_3) > 13$. This left open three possibilities for the number $r(I_4, L_3)$, the arguably easiest case among the hitherto open ones. It is known that $r(I_2, L_3) = 4$, c.f. [956ER], that $r(I_2, L_4) = 8$, c.f. [964EM] and that $r(I_2, L_5) = 14$ and $r(I_2, L_6) = 28$, c.f. [970RP]. It is also known that $r(I_3, L_3) = 9$, c.f. [974Berm].

This research is motivated by the wide gap between the knowledge of the undirected Ramsey numbers $r(I_m, K_n)$ and that of the oriented Ramsey numbers $r(I_m, L_n)$. The numbers $r(I_m, K_3)$ are known for $n < 10$, they are $3, 6, 9, 14, 18, 23, 28, 36$ the last of these values was established in 1982 by Grinstead and Roberts in [982GR0]. It is known that $r(I_4, K_4) = 18$[955GG] and $r(I_5, K_4) = 25$[995MR]. Moreover good asymptotic estimates are available for undirected Ramsey numbers. Ajtai, Komlós, and Szemeredi proved in [980AKS] that $r(I_m, K_3) \in O(m^2/\log m)$ and that for each $n \geq 3$, $r(I_m, K_n) \leq c_n m^{n-1}/(\log m)^{n-2}$. Kim showed in [995Kim] that $r(m, 3) \in \Theta(m^2/\log m)$.

In this paper, we determine the values $r(I_4, L_3)$ and $r(I_5, L_3)$. For each $n \geq 3$, we give asymptotic upper bounds on $r(I_m, L_n)$ which are of the same order as the best known upper bounds for $r(I_m, K_n)$.

In Section 3, we improve Larson's and Mitchell's upper bound of m^2 for the numbers $r(I_m, L_3)$ to $m^2 - m + 3$. In Section 4, we construct oriented graphs witnessing $r(I_4, L_3) > 14$ and $r(I_5, L_3) > 22$. Thereby we prove

Theorem 1.6. *$r(I_4, L_3) = 15$ and $r(I_5, L_3) = 23$.*

In Section 5 we use results of Alon and Kim to show

Theorem 1.7. *$r(I_m, L_3) \in \Theta(m^2/\log m)$.*

Then we follow an argument of Ajtai, Komlós, and Szemeredi to extend the above result to

Theorem 1.8. *For each $n \geq 3$, $r(I_m, L_n) \leq C_n m^{n-1}/(\log m)^{n-2}$, where C_n is constant for each n.*

Notice that, though the upper bound in Theorem 1.7 is asymptotically better to $m^2 - m + 3$, the latter is much smaller for small values of m. In particular, the former fails to allow for an exact determination of any of $r(I_4, L_3)$ and $r(I_5, L_3)$. Finally—in an appendix—we provide a formula which gives the best known upper bounds for small values of m and n.

2. Preliminaries

Let v be a vertex of $D = (V, A)$. We denote the *in-neighbourhood* of v in by $N^-(v)$ and the *out-neighbourhood* of v by $N^+(v)$. Formally, we have $N^-(v) = \{w \in V : (v, w) \in A\}$ and $N^+(v) = \{w \in V : (w, v) \in A\}$. We denote the vertices non-adjacent to v by $I(v)$, formally we have $I(v) = V \setminus (\{v\} \cup N^-(v) \cup N^+(v))$. We denote $|N^-(v)|$ by $d^-(v)$ and $|N^+(v)|$ by $d^+(v)$.

For the remaining section we assume that $D = (V, A)$ is a (I_m, L_n)-free graph.

Lemma 2.1. *Let $v \in V$. Then the following holds:*

(1) *The induced subgraphs on $N^-(v)$ and $N^+(v)$ are (I_m, L_{n-1})-free.*
(2) *The induced subgraph on $I(v)$ is (I_{m-1}, L_n)-free.*

Proof. To show the first assertion suppose towards a contradiction that $N^-(v)$ contains a set of vertices T such that the induced subgraph on T is a tournament of size $n-1$. Then $\{v\} \cup T$ is a tournament of size n. This contradicts that D is L_n-free.

To show the second assertion suppose towards a contradiction that $I(v)$ contains an independent set I of size $m-1$. Then $\{v\} \cup I$ is an indepedent set of size m. This contradicts that D is I_m-free. ■

This has the following consequences for the case $n = 3$.

Corollary 2.2. *Let $v \in V$. Suppose that D is L_3-free. Then $N^-(v)$ and, respectively, $N^+(v)$ are independent sets. Particularly, $d^-(v), d^+(v) \leqslant m-1$.*

We now provide a recursive upper bound for $r(I_m, L_n)$.

Lemma 2.3. *We have $r(I_{m+1}, L_{n+1}) \leqslant 2r(I_{m+1}, L_n) + r(I_m, L_{n+1}) - 1$ for all natural numbers m and n. Furthermore, if an (I_{m+1}, I_{n+1})-free graph $D = (V, A)$ has size $2r(I_{m+1}, L_n) + r(I_m, L_{n+1}) - 2$, then all $v \in V$ satisfy*

(1) *$d^-(v) = d^+(v) = r(I_{m+1}, L_n) - 1$ and*
(2) *$|I(v)| = r(I_m, L_{n+1}) - 1$.*

Proof. Let D be a (I_{m+1}, L_{n+1})-free graph. Let $v \in D$. By Lemma 2.1, $N^-(v)$ and $N^+(v)$ have at most size $r(I_{m+1}, L_n) - 1$ each, and $I(v)$ has at most size $r(I_m, L_{n+1}) - 1$.

Hence,

$$\begin{aligned}|V| &\leqslant |\{v\}| + |N^-(v)| + |N^+(v)| + |I(v)| \\ &\leqslant 1 + 2(r(I_{m+1}, L_n) - 1) + r(I_m, L_{n+1}) - 1 \\ &= 2r(I_{m+1}, L_n) + r(I_m, L_{n+1}) - 2.\end{aligned}$$

This implies the assertion. ■

The following Lemma is proved in [997LM] and follows from Corollary 2.2 and Lemma 2.3. Proposition 3.4 is an improvement of it.

Lemma 2.4. $r(I_m, L_3) \leqslant m^2$.

A proof of the following Lemma can be found in [959Stea] but we reprove it here.

Lemma 2.5. $r(I_2, L_n) \leqslant 2^{n-1}$.

Proof. Suppose that the statement of the Lemma would fail. Then there is a smallest natural number n for which it does so. Let T be a tournament on 2^{n-1} vertices witnessing this. Pick any vertex $v \in T$. Clearly, one of $N^-(v)$ and $N^+(v)$ must have at least 2^{n-2} elements. Without loss of generality suppose that $N^-(v)$ does. By minimality of n, there is an $X \in [N^-(v)]^{n-1}$ inducing a transitive subtournament, a contradiction. ■

Remark 2.6. *Lemma 2.5 implies*

$$r(m, n) \leqslant r(I_m, L_n) \leqslant r(m, 2^{n-1}).$$

In Section 5 we will provide some evidence that $r(I_m, L_n)$ behaves similarly to the lower bound, at least asymptotically.

3. Improving the Larson-Mitchell Upper Bound

In this section we improve Lemma 2.4 and show that $r(I_m, L_3) \leqslant m^2 - (m-3)$ for all $m \geqslant 3$. This upper bound turns out to be tight for $m \in \{3, 4, 5\}$.

If $D = (V, A)$ is a oriented graph and $B, C \subset V$, let $E(B, C)$ denote the set of edges between vertices in B and vertices in C, irrespective of their direction. Formally we have $E(B, C) = A \cap ((B \times C) \cup (C \times B))$.

Lemma 3.1. *Every (I_3, L_3)-free oriented graph D on eight vertices has the following properties:*

(1) *D is 4-regular,*
(2) *every triple of vertices of D contains at least one edge,*
(3) *the non-neighbourhood of any vertex of D induces a triangle, and*
(4) *any set of 5 vertices in D either contains a triangle or the induced underlying unoriented graph is isomorphic to C_5.*

(5) *any set of* 6 *vertices in* D *contains a triangle,*
(6) D *is unique up to isomorphism.*

Proof. As $r(I_2, L_3) = 4$ and $r(I_3, L_2) = 3$, the bound in Lemma 2.3 is tight. Hence, the graph is 4-regular. The second and third assertion follow from D being I_3-free.

Let M be a set of five vertices of D. We assume that M does not contain a triangle. We can ignore the orientation of the edges. Let $x \in M$. By part 3, $|I(x) \cap M| \leqslant 2$. If $|I(x) \cap M| \leqslant 1$, then $|M \cap (N^+(x) \cup N^-(x)| \geqslant 3$, so part 2 implies the assertion. Hence, $|I(x) \cap M| = 2$ for all $x \in M$. Hence, the induced subgraph on M is isomorphic to a cycle of length 5. This implies the fourth assertion. The fifth assertion follows similarly.

Now we show the uniqueness of D. W.l.o.g. the vertex set of D is $\{0,1,2,3,4,5,6,7\}$, where $N^+(0) = \{1, 2\}$, $N^-(0) = \{3, 4\}$, and $I(0) = \{5, 6, 7\}$. As $I(0)$ is I_2-free, w.l.o.g. we have the edges

$$(5, 6), (6, 7), \text{ and } (7, 5)$$

in D. By definition, we have

$$(0, 1), (0, 2), (3, 0), \text{ and } (4, 0)$$

in D. As every vertex in D has degree 4,

$$|E(N^+(0) \cup N^-(0), I(0))| = 4|I(0)| - 2|E(I(0), I(0))| = 12 - 6 = 6.$$

Hence,

$$2|E(N^+(0), N^-(0))| = 4 \cdot 3 - |E(N^+(0) \cup N^-(0), I(0))| = 6.$$

As D is L_3-free, the three edges in $E(N^+(0), N^-(0))$ go from $N^+(0)$ to $N^-(0)$, so w.l.o.g. we can assume that the edges

$$(1, 3), (2, 3), \text{ and } (2, 4)$$

are in D. As 2 has in-degree 2, there is one edge from $I(0)$ to 2, w.l.o.g. that is $(5, 2)$. As $I(2)$ is (I_2, L_3)-free, the edges $(1, 6)$ and $(7, 1)$ are in D. Similarly, $I(1)$ is (I_2, L_3)-free, so $(4, 5)$ is an edge of D. As the out- and in-degrees of all vertices are 2, the edges $(3, 7)$ and $(6, 4)$ are in D. Now we have given all 16 oriented edges of D without loss of generality. ■

The unique oriented (I_3, L_3)-free graph may be defined on $\mathbb{Z}_8$ by setting both $x \mapsto x + 1$ and $x \mapsto x - 2$, see Figure 1.

Lemma 3.2. *An* (I_4, L_3)*-free oriented graph on fourteen vertices contains at least* 38 *edges.*

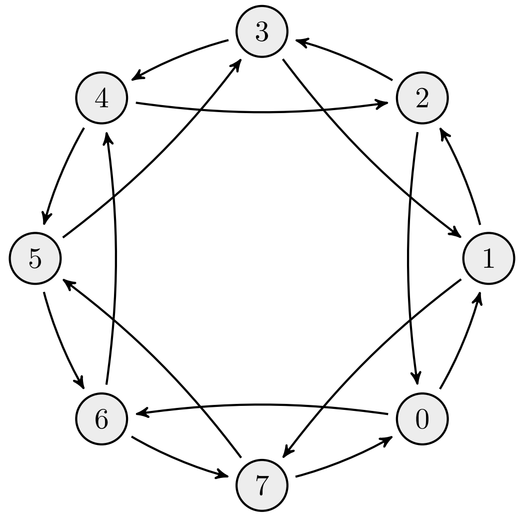

FIGURE 1. The unique (I_3, L_3)-free graph on 8 vertices.

Proof. Suppose towards a contradiction that $D = (V, A)$ was a 14-vertex (I_4, L_3)-free oriented graph with at most 37 edges. Recall $r(I_3, L_3) = 9$. By Lemma 2.1, $d^-(v) + d^+(v) \geqslant 14 - r(I_3, L_3) = 5$. By Corollary 2.2, $d^-(v) + d^+(v) \leqslant 6$. Let x be the number of vertices v such that $d^-(v) + d^+(v) = 5$. As we have at most 37 edges and 14 vertices, we obtain

$$5x + 6(14 - x) \leqslant 37 \cdot 2.$$

This implies that there are at least ten vertices $v \in V$ such that $d^-(v) + d^+(v) = 5$. Pick one of them with $d^-(v) = 2$. Clearly, $d^+(v) = 3$. Since $N^+(v)$ is already an independent set of size 3, and D is I_4-free, all the 8 vertices in $I(v)$ are adjacent to at least one vertex of $N^+(v)$. Hence $|E(N^+(v), I(v))| \geqslant 8$. Similarly, since any four vertices of $I(v)$ contain two independent vertices, we get $|E(N^-(v), I(v))| \geqslant 5$ lest there be an independent quadruple. By Lemma 3.1, $|E(I(v), I(v))| = 16$. Let y be the number of vertices u in $I(v)$ such that $d^+(u) + d^-(u) = 6$. Then

$$45 \leqslant |E(N^+(v), I(v))| + |E(N^-(v), I(v))| + 2 \cdot |E(I(v), I(v))| = 6y + 5(8 - y).$$

Hence, $y \geqslant 5$. This contradicts that we have at least ten vertices u with $d^-(u)+d^+(u) = 5$. ■

Lemma 3.3. *Let $D = (V, A)$ be an (I_m, L_3)-free oriented graph. Let $v \in V$ with $d^-(v) = d^+(v) = m - 1$. Then*

$$|E(N^-(v) \cup N^+(v), I(v))| \geqslant 4(|I(v)| - m + 1).$$

Proof. By Corollary 2.2, $N^-(v)$ and $N^+(v)$ are independent sets of size $m - 1$. Notice that $x \in I(v)$ is adjacent to at least one vertex of $N^-(v)$ as otherwise $N^-(v) \cup \{x\}$ is an independent set of size m. We call $x \in I(v)$ a private neighbour (with respect to $N^-(v)$) if x has exactly one neighbour in $N^-(v)$. We claim that a vertex $u \in N^-(v)$ is adjacent to at most two private neighbours.

Suppose that u is adjacent to three private neighbours, call them x, y and z. If x, y and z are all connected, then the induced subgraph on $\{u, x, y, z\}$ is a (I_2, L_3)-free graph. This contradicts $r(I_2, L_3) = 4$. If without loss of generality x and y are not connected, then $\{x, y\} \cup N^-(v) \setminus \{u\}$ is an independent set of size m. This contradicts thats D is I_m-free. This shows our claim.

Hence, each $u \in N^-(v)$ is adjacent to at most two private neighbours in $I(v)$. Hence, at most $2d^-(v)$ vertices of $I(v)$ have one neighbour in $N^-(v)$, while at least $|I(v)| - 2d^-(v)$ vertices in $I(v)$ have at least two neighbours in $N^-(v)$. Hence,

$$|E(N^-(v), I(v))| \geqslant 2d^-(v) + 2(|I(v)| - 2d^-(v)) = 2(|I(v)| - m + 1).$$

An analogous argument shows

$$|E(N^+(v), I(v))| \geqslant 2(|I(v)| - m + 1).$$

The assertion follows. ■

Proposition 3.4. *If m is a natural number, where $m \geqslant 3$, then $r(I_m, L_3) \leqslant m^2 - m + 3$ and every (I_m, L_3)-free oriented graph on $m^2 - m + 2$ vertices has at least $(m^2 - m + 2)(2m - 3)/2$ edges.*

Proof. We prove this by induction on m. The truth of the proposition for $m = 3$ follows from Lemmas 2.4 and 3.1. Assume that $r(I_m, L_3) \leqslant m^2 - m + 3$ and every (I_m, L_3)-free oriented graph on $m^2 - m + 2$ vertices has at least $(m^2 - m + 2)(2m - 3)/2$ edges.

Assume towards a contradiction that D is an (I_{m+1}, L_3)-free oriented graph $D = (V, A)$ on $(m + 1)^2 - (m + 1) + 3 = m^2 + m + 3$ vertices. As $r(I_{m+1}, L_2) = m + 1$ and, by induction hypothesis, $r(I_m, L_3) \leqslant m^2 - m + 3$, we have equality in Lemma 2.3. By Lemma 2.3, $d^-(v) = d^+(v) = m$ and $I(v) = m^2 - m + 2$ for all $v \in V$.

Let us fix v. As $d^-(v) + d^+(v) = 2m$, we can apply Lemma 3.3 and obtain

$$|E(N^-(v) \cup N^+(v), I(v))| \geqslant 4(|I(v)| - m) = 4(m^2 - 2m + 2). \tag{1}$$

As $d^-(w) + d^+(w) = 2m$ for $w \in I(v)$, we have that

$$|E(N^-(v) \cup N^+(v), I(v))| + 2|E(I(v), I(v)| = 2m \cdot |I(v)| = 2m(m^2 - m + 2). \tag{2}$$

Now we will employ our knowledge about the degrees and the induction hypothesis for the number of edges in a (I_m, L_3)-free oriented graph on $m^2 - m + 2$ vertices. We distinguish three cases:

- If $m = 3$, then, by Lemma 3.1, $|E(I(v), I(v))| \geqslant 16$. By Equation (2),
$$|E(N^-(v) \cup N^+(v), I(v))| \leqslant 16.$$
By Equation (1),
$$|E(N^-(v) \cup N^+(v), I(v))| \geqslant 20.$$
This is clearly a contradiction.
- If $m = 4$, then by Lemma 3.2, $|E(I(v), I(v))| \geqslant 38$. By Equation (2),
$$|E(N^-(v) \cup N^+(v), I(v))| \leqslant 36.$$
By Equation (1),
$$|E(N^-(v) \cup N^+(v), I(v))| \geqslant 40.$$
Again, this is a contradiction.
- If $m > 4$, then we have $|E(I(v), I(v))| \geqslant (2m-3)(m^2-m+2)/2$ by the induction hypothesis. By Equation (2),
$$|E(N^-(v) \cup N^+(v), I(v))| \leqslant 3m^2 - 3m + 6.$$
As $m > 4$, this contradicts Equation (1).

It remains to show the claim on the minimum number of edges. By Lemma 3.2, we only have to consider the case $m > 3$. Let $D = (V, A)$ a (I_m, L_3)-free graph with $m^2 - m + 2$ vertices. Following the arguments in Lemma 2.3, we have equality in our bound if and only if each vertex $v \in V$ satisfies $d^-(v) + d^+(v) \geqslant 2m - 3$. Hence, we have at least
$$|V| \cdot (2m-1)/2 = (m^2 - m + 2)(2m - 3)/2$$
edges in D. ■

4. Constructive Lower Bounds

Observation 4.1. $r(I_4, L_3) = 15$.

Proof. By Proposition 3.4, $r(I_4, L_3) \leqslant 15$. The oriented (I_4, L_3)-free graph on page 10 may be defined on $\mathbb{Z}_{14}$ by setting both $x \mapsto x+1$ and $x \mapsto x-2$ for all $x \in \mathbb{Z}_{14}$ and moreover $x \mapsto x+4$ if x is even and $x \mapsto x-6$ if x is odd. ■

We want to remark that there is no oriented (I_4, L_3)-free Cayley graph on 14 vertices.

Observation 4.2. $r(I_5, L_3) = 23$.

Proof. By Proposition 3.4, $r(I_5, L_3) \leqslant 23$. The oriented (I_5, L_3)-free graph on page 11 may be defined on $\mathbb{Z}_{22}$ by setting both $x \mapsto x+1$, $x \mapsto x+4$, $x \mapsto x-5$ and $x \mapsto x+10$ for all $x \in \mathbb{Z}_{22}$. ■

Both obersvations together imply Theorem 1.6.

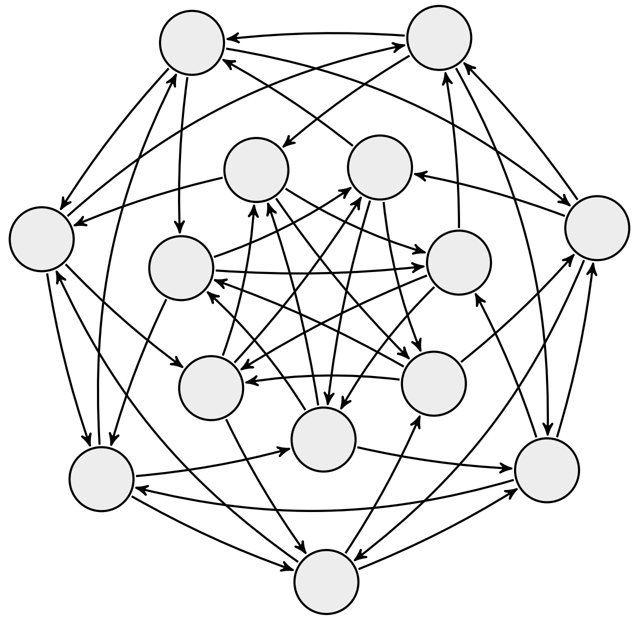

FIGURE 2. An oriented graph showing $r(I_4, L_3) > 14$.

5. Probabilistic Upper Bounds

In this section, we use a result of Alon to show that $r(I_m, L_3)$ is in $O(m^2/\log m)$. This bound is better than the one in Proposition 3.4 for large enough m. Moreover, this is tight upto multiplicative constants since $r(I_m, L_3) \geqslant r(I_m, K_3)$. Then we follow an upper bound argument of Ajtai, Komlós and Szemeredi for $r(I_m, K_n)$ to obtain upper bounds of commensurate order for $r(I_m, L_n)$.

Proposition 5.1 ([996Alon, Prop. 2.1])**.** *Let $G = (V, E)$ be a graph on v vertices with maximum degree $d \geqslant 1$, in which the neighbourhood of any vertex is r-colourable. Then*

$$\alpha(G) \geqslant \frac{v \operatorname{ld} d}{160 d \operatorname{ld}(r+1)}.$$

Corollary 5.2.

$$r(I_m, L_3) \leqslant \frac{2^9 m^2}{\operatorname{ld} m}.$$

Proof. Assume towards a contradiction that there is a natural number m and a oriented graph D on $v := 2^9 m^2/\operatorname{ld} m$ vertices with no transitive triangle and no independent set of size m. Let G be the undirected graph attained from D by forgetting the directions of the edges. Let d denote the maximal degree of a vertex in G. Note that the neighbourhood of any vertex x is 2-colourable since it consists of the in- and

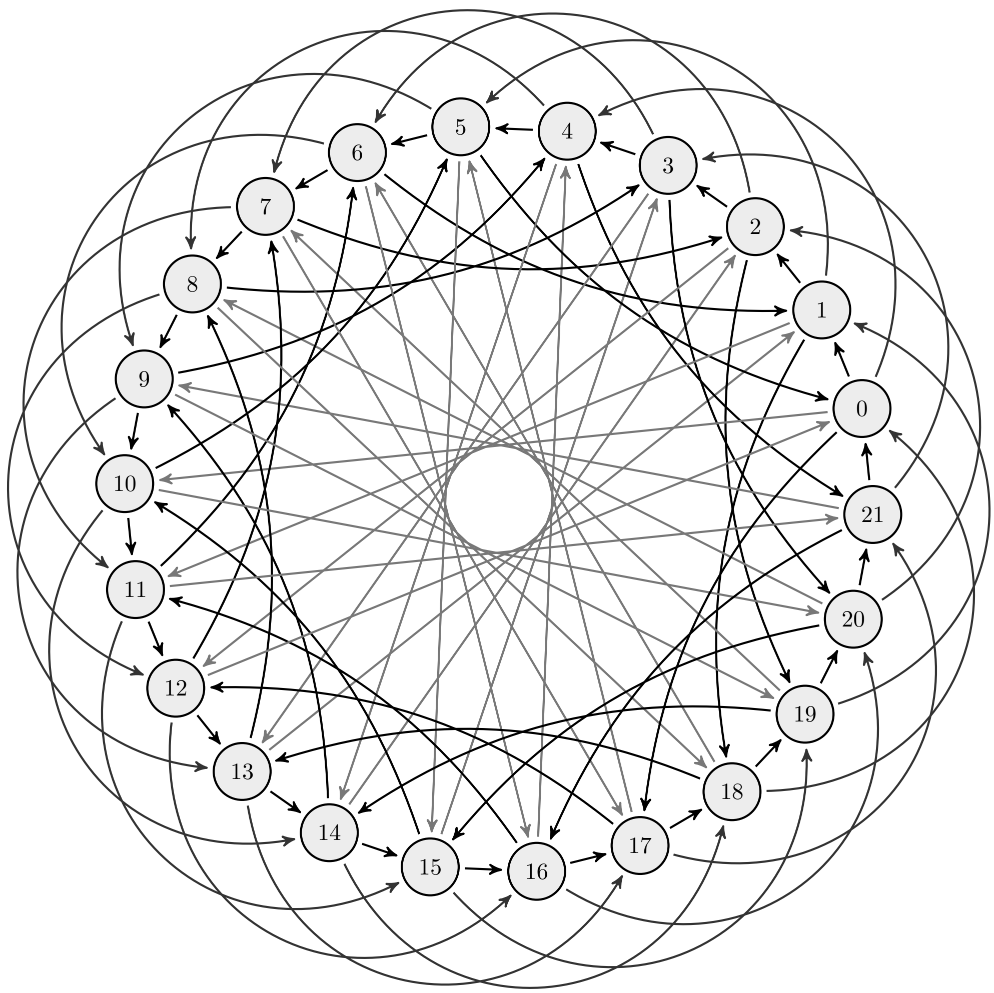

FIGURE 3. An oriented graph showing $r(I_5, L_3) > 22$.

out-neighbourhoods of x, both of which are independent sets. Since

$$\frac{v \operatorname{ld} d}{160 d \operatorname{ld} 3}$$

is decreasing in d and, by Corollary 2.2, $d < 2m$, by Proposition 5.1, we may conclude that

$$m > \alpha(G) \geqslant \frac{2^9 m^2}{\operatorname{ld} m} \cdot \frac{\operatorname{ld}(2m)}{160 \cdot 2m \operatorname{ld} 3} = \frac{2^9 m(\operatorname{ld} 2 + \operatorname{ld} m)}{320 \operatorname{ld} 3 \operatorname{ld} m} = \frac{2^9 m}{320 \operatorname{ld} 3}(1 + \frac{1}{\operatorname{ld} m}).$$

It follows that $2^9 < 320 \operatorname{ld} 3$ which is a contradiction. ∎

Due to Kim [995Kim], $r(I_m, K_3) \geqslant \Theta(m^2/\log m)$. Since $r(I_m, L_3) \geqslant r(I_m, K_3)$, the bound in Corollary 5.2 is asymptotically tight up to a multiplicative constant.

We follow an argument by Ajtai, Komlós, and Szemeredi for (I_m, K_n)-free graphs [980AKS] to obtain another upper bound for (I_m, L_n)-free graphs.

Following the proof of in [980AKS, Lemma 4], we obtain the following lemma.

Lemma 5.3. *Let D be an oriented graph with v vertices, e edges, h transitive triangles, and average degree d. Let $0 < p < 1$. Then there exists an induced subgraph D' of D with v' vertices, e' edges, h' transitive triangles, and average degree d' satisfying*

$$v' \geqslant vp/2, \qquad e' \leqslant 3ep^2 \qquad h' \leqslant 3hp^3, \qquad d' \leqslant 6dp.$$

We also need the average version of Alon's bound. The constant in the bound can be easily verified from the proof there.

Theorem 5.4 ([996Alon, Theorem 1.1])**.** *Let $G = (V, E)$ be a graph on v vertices with average degree $d \geqslant 1$, in which the neighbourhood of any vertex is r-colourable. Then*

$$\alpha(G) \geqslant \frac{v \operatorname{ld} 2d}{640 d \operatorname{ld}(r+1)}.$$

Lemma 5.5. *Let $\varepsilon > 0$. Let D be a graph with v vertices, average degree d, and $h \leqslant vd^{2-\varepsilon}$ transitive triangles. If d is sufficienty large, then*

$$\alpha(D) \geqslant 2^{-16} \varepsilon \frac{v}{d} \operatorname{ld} d.$$

Proof. By Lemma 5.3, we obtain a graph D' with $v' \geqslant vp/2$ vertices, average degree $d' \leqslant 6dp$, and $h' \leqslant 3hp^3$ transitive triangles. If we choose $p = \sqrt{d^{\varepsilon-2}/12}$, then we get $h' \leqslant \frac{1}{4} vp \leqslant \frac{1}{2} v'$. Deleting one vertex from each of the transitive triangles gives us an oriented L_3-free graph D'' on $v'' \geqslant \frac{1}{4} vp$ vertices. By Corollary 5.1,

$$\begin{aligned} \alpha(D) &\geqslant \alpha(D'') \\ &\geqslant \frac{v'' \operatorname{ld} d'}{160 d' \operatorname{ld} 3} \\ &\geqslant \frac{vp \operatorname{ld} 6dp}{4 \cdot 640 \cdot 6dp \operatorname{ld} 3} \\ &\geqslant 2^{-16} \varepsilon \cdot \frac{v \operatorname{ld} d}{d}. \end{aligned}$$

■

Theorem 5.6. *For every $m, n \geqslant 2$,*

$$r(I_m, L_n) \leqslant 2^{19n} \cdot \frac{m^{n-1}}{(\operatorname{ld} m)^{n-2}}.$$

Proof. We proof the bound by induction on n. We already know that $r(I_m, L_2) \leqslant m$ and, by Corollary 5.2, $r(I_m, L_3) \leqslant 2^9 \cdot \frac{m^2}{\operatorname{ld} m}$. Fix $n \geqslant 4$ and assume that the claim is true for $n-1$ and $n-2$.

Suppose that D is an oriented L_n-free graph on

$$v \geqslant 2^{19n} \cdot \frac{m^{n-1}}{(\operatorname{ld} m)^{n-2}}$$

vertices. We will argue that $\alpha(D) \geqslant m$. Let $\varepsilon = 1/n$.
Case 1. The number of transitive triangles in D is less than $v \cdot d^{2-\varepsilon}$. By Lemma 2.3, the maximum degree d in D is less than

$$2r(I_m, L_{n-1}) \leqslant 2^{19n-18} \cdot \frac{m^{n-2}}{(\operatorname{ld} m)^{n-3}}.$$

By Lemma 5.5,

$$\alpha(D) \geqslant 2^{-16}\varepsilon \cdot v \cdot \frac{\operatorname{ld} d}{d} \geqslant m.$$

Case 2. The number of transitive triangles in D is at least $v \cdot d^{2-\varepsilon}$. The graph D contains at most $vd/2$ edges. By double counting there exists an oriented edge $e = (x, y)$ in D such that (a, b) lies in at least

$$\frac{vd^{2-\varepsilon}}{vd/2}$$

transitive triangles of the form $\{(a, b), (b, v), (a, v)\}$. Let V_e denote the set of vertices v such that $\{(a, b), (b, v), (a, v)\}$ is a transitive triangle of D. Then $|V_e| \geqslant 2d^{1-\varepsilon}$.

If there is an oriented subgraph H isomorphic to L_{n-2} in the subgraph D' induced on V_e, then the induced subgraph on $\{a, b\} \cup H$ is isomorphic to L_n. Hence, D' is (I_m, L_{n-2})-free. Hence,

$$2d^{1-\varepsilon} \leqslant |V_e| < r(I_m, L_{n-2}).$$

Hence,

$$d < r(I_m, L_{n-2})^{1/(1-\varepsilon)} < 2^{19(n-1)} \cdot \frac{m^{n-2}}{(\operatorname{ld} m)^{n-3}}.$$

By Turan's bound,

$$\alpha(D) \geqslant \frac{v}{d+1} \geqslant m.$$

■

This implies Theorem 1.7.

6. Coda

There are more open problems in finite combinatorics stemming from set theory. Determining $r(I_3, L_4)$ would continue our work and seems feasible given the size of the candidates for examples of (I_3, L_4)-free graphs.

In [014Wein] the Ramsey numbers of the form $r(\omega^2 m, n)$ were characterised by finite Ramsey numbers concerned with edge-coloured oriented graphs. We have $r(\omega^2 m, n) = \omega^2 r(I_m, A_n)$ where A_n is a class of n-vertex edge-coloured transitive tournaments satisfying some additional restrictions. It would be nice to improve the following asymptotic bounds on $r(I_m, A_3)$ established there.

$$r(I_m, A_3) \in O\big(m^3\big) \cap \Omega\Big(\frac{m^2}{\operatorname{ld} m}\Big).$$

Finally, for the Ramsey numbers $r(\omega^m, n)$ formulae have been found for all natural numbers $m \neq 4$ and all natural numbers n by Nosal in [975Nosa, 979Nosa]. The determination of the numbers $r(\omega^4, n)$ by a formula, however, has still to be accomplished.

Appendix A. A Formula for Small m and n

We provide the following—admittedly slightly baroque—formula. It gives asymptotically suboptimal upper bounds for $r(I_m, L_n)$ but provides the state of the art for small m and n.

Theorem A.1. *$r(I_m, L_n) \leqslant v(m, n)$ for all $m \in \omega \setminus 2$ and $n \in \omega \setminus 3$ where*

$$v(m,n) := \sum_{i=0}^{n-2} \binom{i+m-1}{i+1} 2^i - \binom{m+n-6}{m-4} 2^{n-3} + 1. \tag{3}$$

Proof. We are going to prove the Theorem using a two-dimensional induction. To this end, we have to check three things:

(1) First we show that (3) agrees with Lemma 2.5.

$$\begin{aligned} v(2,n) &= \sum_{i=0}^{n-2} \binom{i+1}{i+1} 2^i - \binom{n-4}{-2} 2^{n-3} + 1 \\ &= \sum_{i=0}^{n-2} 2^i + 1 \\ &= (2^{n-1} - 1) + 1 = 2^{n-1}. \end{aligned}$$

(2) Now we prove that (3) agrees with Proposition 3.4 for $m \in \omega \setminus 3$. As $\binom{m-3}{m-4} = m-3$ for $m \geqslant 3$,

$$
\begin{aligned}
v(m,3) =& \sum_{i=0}^{1} \binom{i+m-1}{i+1} 2^i - \binom{m-3}{m-4} 2^0 + 1 \\
=& (m-1) + m(m-1) - (m-3) + 1 \\
=& m^2 - m + 3.
\end{aligned}
$$

(3) Finally we prove that $v(m+1,n+1) = 2v(m+1,n) + v(m,n+1) - 1$:

$$
\begin{aligned}
& 2v(m+1,n) + v(m,n+1) - 1 \\
=& \sum_{i=0}^{n-2} \binom{i+m}{i+1} 2^{i+1} - \binom{m+n-5}{m-3} 2^{n-2} + 2 \\
& + \sum_{i=0}^{n-1} \binom{i+m-1}{i+1} 2^i - \binom{m+n-5}{m-4} 2^{n-2} + 1 - 1 \\
=& \sum_{i=0}^{n-1} \binom{i+m-1}{i} 2^i - \left(\binom{m+n-5}{m-4} + \binom{m+n-5}{m-3} \right) 2^{n-2} + 1 \\
& + \sum_{i=0}^{n-1} \binom{i+m-1}{i+1} 2^i \\
=& \sum_{i=0}^{n-1} \left(\binom{i+m-1}{i} + \binom{i+m-1}{i+1} \right) 2^i - \binom{m+n-4}{m-3} 2^{n-2} + 1 \\
=& \sum_{i=0}^{n-1} \binom{i+m}{i+1} 2^i - \binom{m+n-4}{m-3} 2^{n-2} + 1 = v(m+1,n+1).
\end{aligned}
$$

■

References

[014Wein] T. V. Weinert. Idiosynchromatic poetry. *Combinatorica*, 34(6):707–742, 2014, doi:10.1007/s00493-011-2980-1, `https://www.math.bgu.ac.il/~weinert/Poetry_colour0.pdf`.

[013Jone] A. L. Jones. Even more on partitioning triples of countable ordinals. Preprint, 2013.

[997LM] J. A. Larson and W. J. Mitchell. On a problem of Erdős and Rado. *Ann. Comb.*, 1(3):245–252, 1997, doi:10.1007/BF02558478, `http://dx.doi.org/10.1007/BF02558478`.

[996Alon] N. Alon. Independence numbers of locally sparse graphs and a Ramsey type problem. *Random Structures Algorithms*, 9(3):271–278, 1996, doi:10.1002/(SICI)1098-2418(199610)9:3<271::AID-RSA1>3.0.CO;2-U, `http://dx.doi.org/10.1002/(SICI)1098-2418(199610)9:3<271::AID-RSA1>3.0.CO;2-U`.

[995Kim] J. H. Kim. The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$. *Random Structures Algorithms*, 7(3):173–207, 1995, doi:10.1002/rsa.3240070302, `http://dx.doi.org/10.1002/rsa.3240070302`.

[995MR] B. D. McKay and S. P. Radziszowski. $R(4,5) = 25$. *J. Graph Theory*, 19(3):309–322, 1995, doi:10.1002/jgt.3190190304, http://dx.doi.org/10.1002/jgt.3190190304.

[994Radz] S. P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30 pp. (electronic), 1994, http://www.combinatorics.org/Surveys/index.html.

[994SF] A. Sánchez-Flores. On tournaments and their largest transitive subtournaments. *Graphs Combin.*, 10(4):367–376, 1994, doi:10.1007/BF02986687, http://dx.doi.org/10.1007/BF02986687.

[991MRO] B.D. McKay and S.P. Radziszowski. The First Classical Ramsey Number for Hypergraphs is Computed. *Proceedings of the Second Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA91, San Francisco, (1991) 304–308.

[982GRO] C. Grinstead and S. Roberts. On the Ramsey Numbers R (3,8) and R (3,9). *Journal of Combinatorial Theory, Ser. B*, 33 (1982) 27–51, doi:10.1016/0095-8956(82)90055-7, http://dx.doi.org/10.1016/0095-8956(82)90055-7.

[980AKS] M. Ajtai, J. Komlós and E. Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, 29(3):354–360, 1980, doi:10.1016/0097-3165(80)90030-8, http://dx.doi.org/10.1016/0097-3165(80)90030-8.

[979Nosa] E. Nosal. Partition relations for denumerable ordinals. *J. Combin. Theory Ser. B*, 27(2):190–197, 1979, doi:10.1016/0095-8956(79)90080-7, http://dx.doi.org/10.1016/0095-8956(79)90080-7.

[975Nosa] E. Nosal. *On arrow relations w(k,w) 2(2) (m,n,k less than w): A study in the partition calculus.* ProQuest LLC, Ann Arbor, MI, 1975, http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:NK25043. Thesis (Ph.D.)–University of Calgary (Canada).

[974Baum] J. E. Baumgartner. Improvement of a partition theorem of Erdös and Rado. *J. Combinatorial Theory Ser. A*, 17:134–137, 1974.

[974Berm] J.-C. Bermond. Some Ramsey numbers for directed graphs. *Discrete Math.*, 9:313–321, 1974.

[970RP] K. B. Reid, Jr. and E. T. Parker. Disproof of a conjecture of Erdős and Moser on tournaments. *J. Combinatorial Theory*, 9:225–238, 1970.

[967ER] P. Erdős and R. Rado. Partition relations and transitivity domains of binary relations. *J. London Math. Soc.*, 42:624–633, 1967.

[964EM] P. Erdős and L. Moser. On the representation of directed graphs as unions of orderings. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 9:125–132, 1964, http://www.renyi.hu/~p_erdos/1964-22.pdf.

[961Shoe] J. R. Shoenfield. The problem of predicativity. In *Essays on the foundations of mathematics*, pages 132–139. Magnes Press, Hebrew Univ., Jerusalem, 1961.

[959Stea] R. Stearns. The voting problem. *Amer. Math. Monthly*, 66:761–763, 1959.

[956ER] P. Erdős and R. Rado. A partition calculus in set theory. *Bull. Amer. Math. Soc.*, 62:427–489, 1956, http://www.ams.org/journals/bull/1956-62-05/S0002-9904-1956-10036-0/S0002-9904-1956-10036-0.pdf.

[955GG] R. E. Greenwood, Jr. and A. M. Gleason. Combinatorial relations and chromatic graphs. *Canad. J. Math.*, 7:1–7, 1955, http://cms.math.ca/cjm/v7/cjm1955v07.0001-0007.pdf.

[935ES] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935, http://www.numdam.org/item?id=CM_1935__2__463_0.

[930Rams] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30:264–286, 1930, http://plms.oxfordjournals.org/content/s2-30/1/264.full.pdf.

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