

DEFINABLE MAD FAMILIES AND FORCING AXIOMS

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ABSTRACT. We show that the consistency strength of $\text{ZFC} + \text{BPFA}$ (i.e., the Bounded Proper Forcing Axiom) + “there are no Σ_2^1 infinite MAD families” is at least $\text{ZFC} +$ a remarkable cardinal. Equivalently, under BPFA and an anti-large cardinal assumption there is a Σ_2^1 infinite MAD family.

1. INTRODUCTION

A. By a MAD family, we mean a collection \mathcal{A} with the following two properties: Firstly, \mathcal{A} is an almost disjoint (short: a.d.) family, that is, \mathcal{A} consists of infinite subsets of ω and any two distinct $a, a' \in \mathcal{A}$ are almost disjoint, i.e., $a \cap a'$ is finite. Secondly, for any infinite set $b \subseteq \omega$ there is $a \in \mathcal{A}$ such that $|a \cap b| = \aleph_0$; that is, \mathcal{A} is *maximal* among a.d. families under inclusion.

While finite MAD families exist trivially, infinite MAD families can be constructed using the Axiom of Choice. This makes them an example of an irregular set somewhat analogous to a set without the Baire property or a non-measurable set.

As is well-known, Mathias [14, 13] proved that no infinite MAD family can be analytic. On the other hand, Arnie W. Miller in [15] constructed a co-analytic infinite MAD family under the assumption that $\mathbf{V} = \mathbf{L}$, showing that Mathias’ result is optimal.

Mathias [14] also produced a model of $\text{ZF} + \text{DC}$ in which there are no infinite MAD families, starting from the assumption of a Mahlo cardinal. Much later, Törnquist showed that there are no infinite MAD families in Solovay’s model [23], and Horowitz and Shelah produced a model of $\text{ZF} +$ “there are no infinite MAD families” without making any large cardinal assumption [9].

The definability of MAD families has been investigated under many natural extensions of the axiomatic system ZFC . It was shown recently by Neeman and Norwood and independently by Bakke-Haga, Törnquist and the second author, that under the Axiom of Determinacy (AD) no infinite MAD family can be an element of $\mathbf{L}(\mathbb{R})$; and under the Axiom of Projective Determinacy and the Axiom of Dependent Choice (DC) there is no projective infinite MAD family [17, 2, 20]. In fact, Neeman and Norwood show the slightly stronger result that under AD^+ (a technical strengthening of AD introduced by Woodin) there are no infinite MAD families.

Another natural family of extensions of ZFC are *forcing axioms*. Definability properties of irregular sets of reals under such axioms have long been investigated. An early example is work of Martin and Solovay [12] showing that Martin’s Axiom for sets of size \aleph_1 (MA_{\aleph_1}) implies that all Σ_2^1 sets are measurable and have the Baire property. It was shown by Törnquist [23] that similarly MA_{\aleph_1} rules out the existence of infinite MAD families which are Σ_2^1 in terms of definitional complexity.

Key words and phrases. Maximal almost disjoint families, MAD families, Bounded proper forcing axiom, remarkable cardinals.

As is well-known, MA_{\aleph_1} is equiconsistent with ZFC; adding assumptions about regularity of definable sets of reals can increase this consistency strength. For instance it was shown by Harrington and Shelah [8] that MA_{\aleph_1} together with “all projective sets are measurable and have the Baire property” is equiconsistent with ZFC+ there is a weakly compact cardinal (in fact, it suffices to add “all Δ_3^1 sets have the Baire property” or “all Δ_3^1 sets are measurable” to MA_{\aleph_1} to drive up its consistency strength).¹

On the other hand, the theory MA_{\aleph_1} + “there is no infinite projective MAD family” (and even MA_{\aleph_1} + “there is no infinite MAD family which is definable from a parameter in ${}^\omega\text{On}$ ”) is equiconsistent with ZFC; this is a by-product of Horowitz and Shelah’s construction in [9] of a model of ZF + “there are no infinite MAD families” (since as part of this construction they force MA_{\aleph_1} to hold).

Stronger forcing axioms such as the Proper Forcing Axiom (PFA) imply that AD holds in $\mathbf{L}(\mathbb{R})$, as was shown by Steel [21]. Thus under PFA, just as under AD, set of reals in $\mathbf{L}(\mathbb{R})$ are regular: They are measurable, have the Baire property, and no infinite MAD families can be found among them. As we shall show, this is not the case for the so-called *bounded* version of this axiom, the Bounded Proper Forcing Axiom (BPFA). This axiom was introduced by Goldstern and Shelah in [7] and later recognized to be equivalent to a principle of generic absoluteness by Bagaria [1]; it is quite a bit stronger than MA_{\aleph_1} but much weaker than PFA.

In the present paper, we show that the assumption that there are no infinite MAD families with a simple definition drives up the consistency strength of BPFA.

Theorem 1.1. *The theory ZFC + BPFA+ “there is no infinite Π_2^1 MAD family” has consistency strength of at least a remarkable cardinal.*

This is indeed remarkable, since ZFC + BPFA alone is known to have consistency strength of a Σ_1 -reflecting cardinal, which is weaker than a remarkable cardinal. As we have already mentioned, it is known that BPFA (in fact, just MA_{\aleph_1}) rules out the existence of Σ_2^1 (in fact, of ω_1 -Suslin) infinite MAD families [23], so the complexity of Π_2^1 in the above theorem cannot be improved.

We can also view this result from a different perspective. We have seen that forcing axioms which are strong enough to imply $\text{AD}^{\mathbf{L}(\mathbb{R})}$ rule out the existence of definable infinite MAD families. Our result shows that under an *anti-large cardinal assumption*, forcing axioms can lead to the opposite result: They imply the existence of infinite MAD families at a rather low level of the projective hierarchy.

Theorem 1.2. *Suppose BPFA holds and that ω_1 is not remarkable in \mathbf{L} . Then there is an infinite Π_2^1 MAD family.*

We take this as evidence that under certain forcing axioms and anti-large cardinal assumptions, the universe behaves somewhat like \mathbf{L} (as in \mathbf{L} there are infinite Π_1^1 MAD families). This idea is also corroborated by the proof of the above theorem.

An obvious question is to which degree the above theorem can be generalized in the sense of replacing “almost-disjointness” by other relations. To state this question precisely, let us introduce some terminology: Let E be a binary, symmetric, and anti-reflexive relation on a Polish space X . We view $G = \langle X, E \rangle$ as a simple graph with vertex set X

¹The reader can find a wealth of further results in [3, Chapter 9].

and edge relation E . To say such G is Borel, Σ_1^1, \dots means that E is Borel, Σ_1^1, \dots as a subset of X^2 (a Polish space, with the product topology). A set $D \subseteq X$ is called G -discrete if no two of its elements are E -related, and *maximal discrete* if it is G -discrete and maximal with respect to \subseteq among G -discrete subsets of X .

We can now ask: Under the same hypothesis as in Theorem 1.2, which Borel graphs on Polish spaces have an infinite Π_2^1 maximal discrete set? What about Σ_3^1 graphs?

That the answer is not “all of them” is obvious from the fact that maximal discrete sets for the relation $xEy \iff (x \neq y \wedge |x\Delta y| < \aleph_0)$ on $\mathcal{P}(\omega)$ cannot be measurable and hence not Π_1^1 . Likewise, under MA_{\aleph_1} no Π_2^1 maximal discrete set for this relation can exist, since under MA_{\aleph_1} all Σ_2^1 sets are measurable.

The obstacle to generalizing our construction to arbitrary Borel (or Σ_3^1) graphs is the coding mechanism in Fact 4.5 which relies heavily on the combinatorics of our specific graph. Vidnyánszky [24] has found a large class of graphs which admit a co-analytic maximal discrete set if $\mathcal{P}(\omega) \subseteq \mathbf{L}$: For instance, this holds for Borel graphs $G = \langle X, E \rangle$ with the property that for each countable $D \subseteq X$ and $d \in X$ such that $D \cup \{d\}$ is G -discrete, the set

$$\{d' \in X \mid D \cup \{d'\} \text{ is } G\text{-discrete and } D \cup \{d, d'\} \text{ is not}\}$$

is cofinal in the hyperarithmetic degrees. Vidnyánszky’s construction uses properties specific to Π_1^1 sets, and it is not clear how to carry out this argument at the level of Π_2^1 sets (under BPFA).

B. Our work has some precursors in the literature: In [4] it is shown that under BPFA, if ω_1 is not remarkable in \mathbf{L} every predicate on $\mathcal{P}(\omega)$ which has a Σ_1 definition in $\mathbf{H}(\omega_2)$ also has a Σ_3^1 definition.

It was shown by Asger Törnquist in [22] that if there is an infinite Σ_2^1 MAD family, there is an infinite Π_1^1 MAD family. Unfortunately, the latter proof does not lift to show that there exists a Π_2^1 infinite MAD under BPFA + ω_1 is not remarkable in \mathbf{L} . The reason for this is that Törnquist’s proof relies on properties of Σ_2^1 and Π_1^1 sets which do not hold for Σ_3^1 and Π_2^1 sets.

C. The paper is organized as follows. In section §2 we discuss a result of Caicedo and Velickovic which can be summed up as follows: BPFA implies that there is a well-ordering of $\mathcal{P}(\omega)$ of length ω_2 with definable initial segments. In §3 we discuss the role of the anti-large cardinal assumption, referring to work of Schindler, and discuss a technique of localization which we have used before (e.g., [6]) and which takes a particularly simple form under BFA + ω_1 is not remarkable in \mathbf{L} . Finally, in §4 we prove Theorem 1.2. We close with open questions in §5.

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2. A WELL-ORDERING WITH DEFINABLE INITIAL SEGMENTS

It was shown by Moore [16] that under BPFA there is a well-ordering of $\mathcal{P}(\omega)$ of order-type ω_2 . Improving Moore’s result, Caicedo and Velickovic [5] obtained, under BPFA, such a well-ordering which is definable by a Σ_1 formula with a parameter from $\mathcal{P}(\omega_1)$.

Their well-ordering has the following property which will be crucial to our argument.

Theorem 2.1. *Under BPFA there is a well-ordering \prec of $\mathcal{P}(\omega)$ such that for some Σ_1 formula $\Phi_{\prec}(u, v, w)$ and some parameter $c_{\prec} \subseteq \omega_1$,*

$$(\forall x \in \mathcal{P}(\omega))(\forall I) \left(\Phi_{\prec}(x, I, c_{\prec}) \iff I = \{y \in \mathcal{P}(\omega) : y \prec x\} \right)$$

Such a well-ordering is obviously very useful when one is interested in devising a recursive definition of optimal complexity. For convenience, we give a name to this type of well-order:

Definition 2.2. *We say a well-order \prec of $\mathcal{P}(\omega)$ with the property from Theorem 2.1 has $(\Sigma_1, \mathcal{P}(\omega_1))$ -definable good initial segments.*

In fact, what Caicedo and Velickovic show in their article [5] is that $\mathcal{P}(\omega)$ carries a well-order \prec with the properties (i) and (ii) in the fact below. Of course, this is equivalent to having $(\Sigma_1, \mathcal{P}(\omega_1))$ -definable initial segments.

Fact 2.3. *That a well-ordering of $\mathcal{P}(\omega)$ has $(\Sigma_1, \mathcal{P}(\omega_1))$ -definable initial segments (i.e., has the property from Theorem 2.1) is equivalent to the conjunction of the following:*

- (i) \prec is Σ_1 with a parameter $c_{\prec} \subseteq \omega_1$,
- (ii) There is a formula $\Phi_{\text{is}}(u)$ such that for any transitive model M with $c_{\prec} \in M$, $M \models \Phi_{\text{is}}(c_{\prec})$ if and only if \prec is absolute for M and $M \cap \mathcal{P}(\omega)$ is an initial segment of \prec .

Remark 2.4. The requirement in (ii) above that $M \models \Phi_{\text{is}}(c_{\prec})$ implies that \prec is absolute for M is redundant; it follows from Requirement (i) if we replace $\Phi_{\text{is}}(c_{\prec})$ by its conjunction with $(\forall x, y \in \mathcal{P}(\omega)) x \prec y \vee y \prec x$.

Proof. To see that (i) \wedge (ii) implies that \prec is a well-ordering with $(\Sigma_1, \mathcal{P}(\omega_1))$ -definable initial segments let $\Phi_{\prec}(x, I, c_{\prec})$ be the formula

$(\exists M)$ M is a transitive \in -model with $\{x, c_{\prec}, I\} \subseteq M$ and

$$M \models \text{“}\Phi_{\text{is}}(c_{\prec}) \wedge I = \{y \in \mathcal{P}(\omega) \mid y \prec x\}\text{”}$$

and observe $I = \{y \in \mathcal{P}(\omega) : y \prec x\} \iff \Phi_{\prec}(x, I, c_{\prec})$.

For the other direction, firstly observe that if \prec has $(\Sigma_1, \mathcal{P}(\omega_1))$ -definable initial segments then obviously \prec is Σ_1 in the parameter c_{\prec} . Secondly, let $\Phi_{\text{is}}(c_{\prec})$ be the formula

$$(\forall x, y \in \mathcal{P}(\omega)) x \prec y \vee y \prec x \wedge (\forall x \in \mathcal{P}(\omega))(\exists I) \Phi_{\prec}(x, I, c_{\prec}). \quad \square$$

For a proof that under BPFA there is such a well-ordering of $\mathcal{P}(\omega)$ with $(\Sigma_1, \mathcal{P}(\omega_1))$ -definable good initial segments, we refer the reader to the excellent exposition in [5].

3. CODING, RESHAPING, AND LOCALIZATION

We start by recalling the following well-known fact.

Fact 3.1. Let $\mathcal{B} = \langle b_{\xi} : \xi < \omega_1 \rangle$ be an arbitrary sequence of pairwise almost disjoint infinite subsets of ω . Under MA_{\aleph_1} , for every subset of $S \subseteq \omega_1$ there is a $c \subseteq \omega$ such that

$$(1) \quad S = \{\xi < \omega_1 : c \cap b_{\xi} \text{ is infinite}\}.$$

The proof of this fact is equally well-known; it uses Solovay’s almost disjoint coding (see [10] or, e.g., [11]).

We take the opportunity to introduce the following rather natural terminology:

Definition 3.2. We shall say that $c \subseteq \omega$ almost disjointly via \mathcal{B} codes the set S to mean precisely that (1) holds.

Our only use of the assumption that ω_1 is not remarkable in \mathbf{L} is in the following fact (this was shown by Ralf Schindler in [18]).

Fact 3.3. Suppose ω_1 is not remarkable in \mathbf{L} and BPFA holds. Then there exists $r \in \mathcal{P}(\omega)$ such that $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$.

Notation 3.4.

- (1) For the rest of this article, let us suppose that $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$ for some $r \in \mathcal{P}(\omega)$ which from now on shall remain fixed.
- (2) Fix an almost disjoint family $\mathcal{F} = \langle f_\xi : \xi < \omega_1 \rangle$ which has a Σ_1 definition in $\mathbf{L}[r]$ and such that for any $\alpha < \omega_1$, $\langle f_\xi : \xi < (\omega_1)^{\mathbf{L}_\alpha[r]} \rangle$ is the set satisfying this definition in $\mathbf{L}_\alpha[r]$.

It is a consequence of $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$ and MA_{\aleph_1} that any predicate which is Σ_1 in $\mathbf{H}(\omega_2)$ (with a parameter) can be localized in a strong sense. A version of this result can, e.g., be found in [4].

To state the following localization lemma, let us make a definition which will be used throughout the paper.

Definition 3.5 (Suitable models). A *suitable model* is a countable transitive \in -model N such that $r \in N$, $N \models \text{ZF}^-$ and “ ω_1 exists”.

Lemma 3.6 (A form of localization). *Suppose MA_{\aleph_1} holds (and recall that we are working under the assumption that $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$ made in 3.4). Let $\phi(y, \omega_1)$ be an arbitrary formula, where $y \in \mathcal{P}(\omega)$ and ω_1 are parameters, and suppose that for some transitive \in -model M with $\{\omega_1, y\} \in M$ it holds that $M \models \phi(y, \omega_1)$. Then there is $c \subseteq \omega$ such that the following holds:*

- (2) Given any suitable model N with $\{c, y\} \subseteq N$ the following must hold in N : “There is a transitive \in -model M^* such that $\{y, (\omega_1)^N\} \subseteq M^*$ and $M^* \models \phi(y, (\omega_1)^N)$ ”.

Proof. Fix a transitive model M as in the lemma. We can assume M to have size ω_1 . Find $S \subseteq \omega_1$ such that via Gödel pairing, S gives rise to a well-founded binary relation S^* on ω_1 whose transitive collapse is $\langle M, \in \upharpoonright M \rangle$. We can ask that y and ω_1 are mapped to specific points in $\langle \omega_1, S^* \rangle$ by the inverse of the collapsing map, say to 0 and 1.

Let

$$(3) \quad D = \{\beta \in \omega_1 : (\exists \mathcal{N}^*) \mathcal{N}^* \prec L_{\omega_2}[S^*, y], \{S^*, y\} \in \mathcal{N}^*, \beta = \omega_1 \cap \mathcal{N}^*\}.$$

For $Y \subseteq \text{On}$, let $\text{Even}(Y) = \{\xi : 2\xi \in Y\}$ and $\text{Odd}(Y) = \{\xi : 2\xi + 1 \in Y\}$. Choose Y to be any subset of ω_1 such that $\text{Even}(Y) = S^*$ and for each $\beta \in D$, the preimage under Gödel pairing of $\text{Odd}(Y) \cap [\beta, \beta + \omega)$ is a well-founded binary relation of rank at least $\min(D \setminus (\beta + 1))$.

Claim 3.7. $Y \subseteq \omega_1$ satisfies the following:

- (4) Given any suitable model N with $\{Y \cap (\omega_1)^N, y\} \subseteq N$ the following must hold in N : “There is a transitive \in -model M^* such that $\{y, (\omega_1)^N\} \subseteq M^*$ and $M^* \models \phi(y, (\omega_1)^N)$ ”.

Proof. To see that Y indeed satisfies (4) let N as in (4) be given. Letting $\beta = (\omega_1)^N$ it must hold that $\beta \in D$: For if $\beta' < \beta$, since $\beta \cap Y \in N$, this model contains a well-founded binary relation of length $\min(D \setminus (\beta' + 1))$ as an element, and so $\min(D \setminus (\beta' + 1)) < \beta = (\omega_1)^N$ because N is a model of \mathbf{ZF}^- . As D is closed, $\beta \in D$. It also follows that $S^* \cap \beta \in N$.

By definition of D we may pick \mathcal{N}^* as in (3). Letting $\overline{\mathcal{N}}^*$ be the transitive collapse of \mathcal{N}^* we obtain an elementary embedding $j : \overline{\mathcal{N}}^* \rightarrow L_{\omega_2}[S^*, y]$ with critical point $\beta = (\omega_1)^N$ (namely, the inverse of the collapsing map) such that $\{S^*, y, \omega_1\} \subseteq \text{ran}(j)$. By elementarity $\overline{\mathcal{N}}^* \models$ “The transitive collapse of $\langle \beta, S^* \upharpoonright \beta \rangle$ is a transitive \in -model M^* such that $M^* \models \Phi(y, \beta)$ ”. But this transitive collapse is also an element of N , so by absoluteness of Δ_1 formulas N must satisfy the same sentence. Claim 3.7. \square

Finally, we find $c \in \mathcal{P}(\omega)$ which almost disjointly via \mathcal{F} codes the set $Y \subseteq \omega_1$ constructed above. To see that c satisfies (2) let N as in (2) be given. By choice of \mathcal{F} (see Notation 3.4) $\langle f_\xi : \xi < (\omega_1)^N \rangle \in N$ and so since $N \models \mathbf{ZF}^-$ it holds that $Y \cap (\omega_1)^N \in N$. By (4) the sentence “there is a transitive \in -model M^* such that $\{y, (\omega_1)^N\} \subseteq M^*$ and $M^* \models \phi(y, (\omega_1)^N)$ ” holds in N , verifying (2). Lemma 3.6. \square

4. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2 in the following, slightly more general form:

Theorem 4.1. *Suppose there is a well-ordering of $\mathcal{P}(\omega)$ of length ω_2 with $(\Sigma_1, \mathcal{P}(\omega_1))$ -definable initial segments, \mathbf{MA}_{\aleph_1} holds, and $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$ where $r \in \mathcal{P}(\omega)$. Then there is an infinite $\mathbf{\Pi}_2^1$ MAD family.*

It is clear by Theorem 2.1 and Fact 3.3 that $\mathbf{BPFA} + \omega_1$ is not remarkable in \mathbf{L} implies the hypothesis, so proving the above theorem will indeed prove Theorem 1.2.

Notation 4.2. *From now on, we suppress the parameter r and assume $\omega_1 = (\omega_1)^{\mathbf{L}}$; our argument will relativize to r trivially.*

By Theorem 2.1 we can fix a well-ordering \prec of $\mathcal{P}(\omega)$ with $(\Sigma_1, \mathcal{P}(\omega_1))$ -definable initial segments, together with a parameter $c_\prec \subseteq \omega$ and a formula $\Phi_{\text{is}}(c_\prec)$ as in Fact 2.3.

We shall inductively construct a sequence $\langle a_\nu : \nu < \omega_2 \rangle$ such that $\mathcal{A} = \{a_\nu : \nu < \omega_2\}$ will be a $\mathbf{\Pi}_2^1$ MAD family.

The most straightforward formula defining a MAD family \mathcal{A} would express that $a \in \mathcal{A}$ iff there is an initial segment $\langle a_\nu : \nu \leq \xi \rangle$ of the construction with $a = a_\xi$; that is, assuming we can find a formula expressing that $\langle a_\nu : \nu \leq \xi \rangle$ is an initial segment of this construction. But of course it is not clear how any projective formula should express such a fact about $\langle a_\nu : \nu < \xi \rangle$, this being an object of size ω_1 . A first step towards a solution is that a_ξ should *code* certain sets of size ω_1 , including $\langle a_\nu : \nu < \xi \rangle$. Almost disjoint coding via \mathcal{F} (see Fact 3.1) allows us to find a real coding these large sets. We then want to find a real ‘localizing’ this coding, i.e., ensuring that the property of coding an initial segment of the construction is expressible by a $\mathbf{\Pi}_2^1$ formula. Using a variant of the coding from [15] we can then code these reals into a_ξ .

4.1. Coding into an almost disjoint family. We call the following fact from Miller’s article [15] to the reader’s attention.

Fact 4.3 (see [15, Lemma 8.24, p. 195]). Fix $z \in \mathcal{P}(\omega)$ and suppose $\vec{a} = \langle a_\nu : \nu < \xi \rangle$ is a countable sequence of pairwise almost disjoint infinite sets. For any $d \in [\omega]^\omega$ which is almost disjoint from every element of \vec{a} there is $a \in [\omega]^\omega$ such that

- $a \cap d$ is infinite,
- a is almost disjoint from each a_ν for $\nu < \xi$,
- and z is computable from a and $\vec{a} \upharpoonright \omega = \langle a_n : n < \omega \rangle$.

Using this fact, Miller succeeds in constructing a co-analytic MAD family in \mathbf{L} : he recursively constructs $\langle a_\nu : \nu < \omega_1 \rangle$ such that in the end, $\mathcal{A} = \{a_\nu : \nu < \omega_1\}$ turns out to be a $\mathbf{\Pi}_1^1$ MAD family. At some initial stage $\xi < \omega_1$ having constructed $\vec{a} = \langle a_\nu : \nu < \xi \rangle$ he considers a counterexample d to the maximality of the family $\{a_\nu : \nu < \xi\}$ constructed so far. Instead of adding this set d to \vec{a} , he adds a as in the fact above, which in addition codes some information z so as to bring down the definitional complexity of \mathcal{A} .

Since we shall need a variant of this type of coding, let us repeat Miller's proof of the above fact.

Proof of Fact 4.3. Let $\vec{b} = \langle b_n : n \in \mathbb{N} \rangle$ enumerate $\{a_\nu : \omega \leq \nu < \xi\}$. For each $n \in \omega$, choose a finite set $G_n \subseteq a_n \setminus \bigcup (\{b_k : k < n\} \cup \{a_k : k < n\})$ so that $|G_n \cup (a_n \cap d)|$ is even if $n \in z$, and odd otherwise. Finally, let $a = d \cup \bigcup \{G_n : n \in \omega\}$. \square

For our purposes the previous fact is useless, since as $2^\omega = \omega_2$ under BPFA we shall have to deal with uncountable sequences $\vec{a} = \langle a_\nu : \nu < \xi \rangle$. Interestingly, there is a variant of the above construction that allows us to deal with uncountable sequences.

Before we describe this variant let us commit, once and for all, to some sequence (to be used for coding purposes) as an initial segment of the MAD family we are about to construct.

Notation 4.4. *Let us fix, for the rest of this article, some sequence $\vec{a}_\omega = \langle a_n : n \in \omega \rangle$ of infinite sets any two of which are almost disjoint.*

We now state our variant of Miller's coding lemma. For this variant, we must make an additional assumption (the existence of c below) which in our case will easily be seen to follow from MA_{\aleph_1} (see Remark 4.6 below)

Fact 4.5. Suppose $\vec{a} = \langle a_\nu : \nu < \xi \rangle$ is a (possibly uncountable) sequence of pairwise almost disjoint infinite subsets of ω such that $\vec{a} \upharpoonright \omega = \vec{a}_\omega$. Further suppose we have $c \in [\omega]^\omega$ satisfying the following:

- c is almost disjoint from each a_ν , for $\omega \leq \nu < \xi$, and
- $c \cap a_n$ is infinite for each $n \in \omega$.

Then for any $z \in \mathcal{P}(\omega)$ and any $d \in [\omega]^\omega$ which is almost disjoint from every element of $\text{ran}(\vec{a})$ there is $a \in [\omega]^\omega$ such that

- $a \cap d$ is infinite,
- a is almost disjoint from each a_ν for $\nu < \xi$,
- and z is computable from a and $\vec{a} \upharpoonright \omega = \langle a_n : n < \omega \rangle$.

In fact there are functions $\text{dc}: \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ and $\text{cd}: \mathcal{P}(\omega)^3 \rightarrow \mathcal{P}(\omega)$, both of which are computable in \vec{a}_ω , such that a as above is given by $a = \text{cd}(d, c, z)$ and z can be recovered from a as $z = \text{dc}(a)$.

The name **dc** was chosen to remind us that this function will be used to ‘decode’ z from a , and likewise, the name **cd** should remind us that the function produces a ‘code’ (for z).

Remark 4.6. We will use Fact 4.5 in the situation where $\vec{a} = \langle a_\nu : \nu < \xi \rangle$ is of length $\xi < \omega_2$ and MA_{\aleph_1} holds. Then it is easy to see c as in Fact 4.5 exists: Just use Fact 3.1 to obtain c so that $\{\nu < \xi \mid c \cap a_\nu \text{ is infinite}\} = \omega$.

Proof of Fact 4.5. We define **cd**: $\mathcal{P}(\omega)^3 \rightarrow \mathcal{P}(\omega)$ as follows. Let F_n be the shortest finite initial segment of

$$c \cap a_n \setminus \bigcup \{a_k : k < n\}$$

such that $|F_n \cup (d \cap a_n)|$ is even if $n \in z$ and odd otherwise. Clearly, F_n can be found by a procedure which is computable in \vec{a}_ω , c , d , and z . Now define the function **cd** by

$$\text{cd}(d, c, z) = d \cup \bigcup \{F_n : n \in \omega\}.$$

Moreover, we define **dc**: $\mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ as follows: Given $a \in [\omega]^\omega$ let

$$\text{dc}(a) = \{n \in \omega : |a \cap a_n| \text{ is even}\}.$$

Clearly, these functions satisfy the conditions in the lemma. \square

4.2. Minimal local witnesses. The functions **cd** and **dc** together with the almost disjoint coding into reals of subsets of ω_1 via \mathcal{F} will help us arrange that a_ξ codes $\langle a_\nu : \nu < \xi \rangle$. But crucially, we need the fact that a_ξ codes an initial segment of the construction (up to stage ξ , some ordinal below ω_2) to be witnessed by a Π_2^1 formula (the same formula for all $\xi < \omega_2$). This involves uniquely selecting a real $c_\xi \in \mathcal{P}(\omega)$ which we call a *minimal local witness* and whose task is to *localize* the coding to suitable countable models. Uniquely selecting such a real is a non-trivial task, and to tackle it we introduce some terminology.

Notation 4.7. Let $F: \omega^2 \rightarrow \omega$ denote some fixed recursive bijection for the remainder of this article.

Definition 4.8.

- (1) Given $c \subseteq \omega$ and $n \in \omega$ we write $(c)_n$ for $\{m \in \omega : F(n, m) \in c\}$.
- (2) Given $c \subseteq \omega$ we write $\text{Seq}(c)$ for the sequence $\langle (c)_n : n \in \omega \rangle$.
- (3) Let $G: \text{On}^2 \rightarrow \text{On}$ denote the Gödel pairing function. We say $c \subseteq \omega$ *almost disjointly via \mathcal{F} codes the sequence \vec{b}* to mean that $c \subseteq \omega$ almost disjointly via \mathcal{F} codes a set $S \subseteq \omega_1$ and $\vec{b} = \langle b_\nu : \nu < \xi \rangle$ where:
 - For $\theta < \omega_1$, letting

$$S_\theta = \{\eta < \omega_1 \mid G(\theta, \eta) \in S\},$$

$$S_\theta^* = \{(\zeta_0, \zeta_1) \in (\omega_1)^2 \mid \omega + G(\zeta_0, \zeta_1) \in S_\theta\},$$

it holds that $\langle \omega_1, S_\theta^* \rangle$ is a well-ordering and $\xi = \{\text{otp } S_\theta^* \mid \theta < \omega_1, S_\theta \neq \emptyset\}$;

- For each $\nu < \xi$ there is exactly one θ such that $S_\theta \neq \emptyset$ and $\text{otp } S_\theta^* = \nu$, and for this θ it holds that

$$b_\nu = \omega \cap S_\theta.$$

The crucial definition for our proof of Theorem 4.1 (and thus, of Theorem 1.2) is that of *minimal local witness*.

Remark 4.9. In the end, our MAD family will be

$$\mathcal{A} = \{a_n : n \in \omega\} \cup \{a \in [\omega]^\omega : c = \text{dc}(a) \text{ is a minimal local witness and } a = \text{cd}((c)_0, (c)_1, c)\}.$$

We will show below that being a minimal local witness is expressible by a Π_2^1 formula. Thus, \mathcal{A} will be Π_2^1 . The low definitional complexity will be achieved through a careful recursive construction of \mathcal{A} . We will have $\mathcal{A} = \{a_\xi : \xi < \omega_2\}$ where letting $c_\xi = \text{dc}(a_\xi)$, $(c_\xi)_2$ almost disjointly via \mathcal{F} codes $\langle a_\nu : \nu < \xi \rangle$.

Before we introduce the notion of minimal local witnesses, we make another convenient definition, for which some motivation should be provided by the previous remark.

Definition 4.10. We shall say that a sequence $\vec{b} = \langle b_\nu : \nu < \xi \rangle$ is a *coherent candidate* if $\vec{a}_\omega \subseteq \vec{b}$ and moreover, for each $\nu < \xi$ it holds that $(\text{dc}(b_\nu))_2$ almost disjointly via \mathcal{F} codes the sequence $\vec{b} \upharpoonright \nu$.

We proceed towards the definition of minimal local witness, by defining the notions of k -witness, minimal k -witness and k -localizer, by induction on $k \in \omega$, $k \geq 3$.

Definition 4.11. We say $\bar{c} \in \mathcal{P}(\omega)^3$ is a *3-witness* if and only if

- (a) $\bar{c}(2)$ almost disjointly via \mathcal{F} codes a sequence $\vec{b} = \langle b_\nu : \nu < \xi \rangle$.
 - (b) \vec{b} is a coherent candidate.
 - (c) $\bar{c}(1)$ is subset of ω such that $c \cap b_\nu$ is infinite if $\nu < \omega$ and finite for all other $\nu < \xi$.
 - (d) $\bar{c}(0)$ is an element of $[\omega]^\omega$ which is almost disjoint from each b_ν for $\nu < \xi$;
- (*)₃

Remark 4.12. Clearly, the sequence \vec{b} from (a) is intended to be an initial segment of the MAD family under construction. We ask (b) as a first step towards ensuring that this is indeed the case. The reader will notice that in (c) we require that $\bar{c}(1)$ has the same properties as c in Fact 4.5, and in (d) we require that $\bar{c}(0)$ has the same properties as d in said fact. The reader may think of $\bar{c}(0)$ as a counterexample to maximality of \vec{b} which we wish to eliminate at stage ξ of our construction of \mathcal{A} by adding a ‘self-coding’ element to our MAD family which has infinite intersection with $\bar{c}(0)$.

We continue with the definition of minimal 3-witness to a sequence \vec{b} of subsets of ω .

Definition 4.13. For any 3-witness $\bar{c} \in \mathcal{P}(\omega)^3$, we say \bar{c} is a *witness to \vec{b}* if \vec{b} is the sequence coded by $\bar{c}(2)$ as in (a) above. We also write $\vec{b}(\bar{c}(2))$ for this sequence. Write \prec^3 for the lexicographic ordering on $\mathcal{P}(\omega)^3$ induced by \prec . We call a 3-witness $\bar{c} \in \mathcal{P}(\omega)^3$ *minimal* if it is \prec^3 -minimal among all 3-witnesses to the same sequence \vec{b} . This is the same as saying that $\bar{c}(2)$ is \prec -minimal satisfying (d) in (*)₃, $\bar{c}(1)$ is \prec -minimal satisfying (c), and $\bar{c}(0)$ is \prec -minimal satisfying (a).

As is not hard to see, the notion of minimal 3-witness is sufficiently absolute for our purposes:

Lemma 4.14. *The notion of 3-witness is absolute for transitive models M of ZF^- such that $\omega_1 \in M$ and the notion of minimal 3-witness is absolute for such models if in addition $M \models \Phi_{\text{is}}(c_\prec)$.*

Proof. The statement $(*)_3(a)$ that $\bar{c}(2)$ almost disjointly via \mathcal{F} codes a sequence \vec{b} is easily seen to be equivalent in \mathbf{ZF}^- to a Σ_1 property of $\bar{c}(2)$, allowing ω_1 as a parameter. Moreover, this Σ_1 statement is absolute for transitive models M of \mathbf{ZF}^- since a witness can be constructed inside M using the Replacement Axiom, and as $\mathcal{F} \in M$ by choice of \mathcal{F} . That \vec{b} is a coherent candidate is absolute for the same reasons. Statements (c) and (d) are obviously Δ_1 in the parameters \vec{b} and \bar{c} . This shows that the notion of 3-witness is absolute for transitive models M of \mathbf{ZF}^- .

Minimality of 3-witnesses is now easily seen to be absolute provided that in addition $M \models \Phi_{\text{is}}(c_{\prec})$ since \prec is absolute for such M and $\mathcal{P}(\omega) \cap M$ is an initial segment of \prec : If $\bar{c} \in M$ is a minimal 3-witness, $M \models \text{“}\bar{c} \text{ is a minimal 3-witness”}$ by absoluteness of \prec and of the notion of 3-witness. Vice versa, suppose $\bar{c} \in M$ and $M \models \text{“}\bar{c} \text{ is a minimal 3-witness”}$. Then $\bar{c}(0)$ must be minimal satisfying $(*)_3(a)$ since if there were $c' \prec \bar{c}(0)$ satisfying (a), it would have to be the case that $c' \in M$ since $\mathcal{P}(\omega) \cap M$ is a \prec -initial segment, contradicting $M \models \text{“}\bar{c} \text{ is a minimal 3-witness”}$. Likewise for $\bar{c}(1)$ and $\bar{c}(2)$. \square

We now give the crucial definition of a *localizer*—a real which ensures that minimal 3-witnesses can be recognized from a local (i.e., a $\mathbf{\Pi}_2^1$) property.

Definition 4.15. Given $\bar{c} \in \mathcal{P}(\omega)^3$ (a putative 3-witness) we say $c \in \mathcal{P}(\omega)$ is a *3-localizer* for \bar{c} if and only if:

- For any suitable model N with $\{\bar{c}, c, \vec{a}_\omega\} \subseteq N$, the following holds in N : There is a transitive model M of \mathbf{ZF}^- such that $M \models \Phi_{\text{is}}(c_{\prec})$, $\{\omega_1, \bar{c}, \vec{a}_\omega\} \subseteq M$, and
- (*)₄
- (a) $M \models \text{“}\bar{c} \text{ is a minimal 3-witness”}$.
 - (b) Writing $\vec{b}(\bar{c}(2))^M$ as $\langle b_\nu : \nu < \xi \rangle$ it holds that for each $\nu < \xi$, $M \models \text{“}\bar{c}_\nu^* \upharpoonright 3 \text{ is a minimal 3-witness”}$, where $\bar{c}_\nu^* = \text{Seq}(\text{dc}(b_\nu))$.

Remark 4.16. Note that “ $\bar{c}_\nu \upharpoonright 3$ is a minimal 3-witness” is a statement which uses \vec{a}_ω as a parameter.

We need the following crucial lemmas:

Lemma 4.17. *Suppose $\bar{c} \in \mathcal{P}(\omega)^3$ is a minimal 3-witness, $\vec{b}(\bar{c}(2)) = \langle b_\nu \mid \nu < \xi \rangle$ and for each $\nu < \xi$ it holds that $\text{Seq}(\text{dc}(b_\nu)) \upharpoonright 3$ is a minimal 3-witness. Then there exists a 3-localizer for \bar{c} .*

Proof. Suppose $\bar{c} \in \mathcal{P}(\omega)^3$ is as in the lemma. Fix a transitive model M of \mathbf{ZF}^- such that $\{\omega_1, \bar{c}\} \subseteq M$ and so that $M \models \Phi_{\text{is}}(c_{\prec})$. By Lemma 4.14 the property of being a minimal 3-witness is absolute for M , so $M \models \text{“}\bar{c} \text{ is a minimal 3-witness”}$ and $M \models \text{“}\bar{c}_\nu^* \upharpoonright 3 \text{ is a minimal 3-witness”}$ where $\bar{c}_\nu^* = \text{Seq}(\text{dc}(b_\nu))$ for each $\nu < \xi$.

Now as in the proof of Lemma 3.6, find c coding almost disjointly via \mathcal{F} a subset of ω_1 which is isomorphic to $\in \upharpoonright M$ and such that for any suitable model N , if $c, \bar{c} \in N$ then it holds in N that c codes a model M^* which witnesses the Σ_1 statement expressing \bar{c} and each $\bar{c}_\nu^* \upharpoonright 3$ are minimal 3-witnesses. Clearly, c is a 3-localizer for \bar{c} . \square

Lemma 4.18. *Suppose $\bar{c} \in \mathcal{P}(\omega)^3$. If there exists a 3-localizer for \bar{c} , then \bar{c} is a minimal 3-witness, and letting $\vec{b}(\bar{c}(2)) = \langle b_\nu \mid \nu < \xi \rangle$ it holds for each $\nu < \xi$ that $\text{Seq}(\text{dc}(b_\nu)) \upharpoonright 3$ is a minimal 3-witness.*

Proof. Suppose c is a 3-localizer for \bar{c} . Let \bar{N} be a countable elementary submodel of $\mathbf{L}_{\omega_2}[c, \bar{c}, \vec{a}_\omega]$ with $\{\omega_1, c, \bar{c}, \vec{a}_\omega\} \subseteq \bar{N}$ and let N be the transitive collapse of \bar{N} . Then N is

suitable, and so by $(*)_4$ the following holds in N : There is a transitive model M of \mathbf{ZF}^- such that $M \models \Phi_{\text{is}}(c_{\prec})$, $\{(\omega_1)^N, \bar{c}, \vec{a}_\omega\} \subseteq M$, and

- (a) $M \models \text{“}\bar{c} \text{ is a minimal 3-witness”}$.
- (b) Writing $\vec{b}(\bar{c}(2))^M$ as $\langle b_\nu : \nu < \xi \rangle$, for each $\nu < \xi$, $M \models \text{“}\bar{c}_\nu^* \upharpoonright 3 \text{ is a minimal 3-witness”}$, where $\bar{c}_\nu^* = \text{Seq}(\text{dc}(b_\nu))$.

By elementarity, there exists such a model M in $\mathbf{L}_{\omega_2}[c, \bar{c}, \vec{a}_\omega]$ with all of the above properties, where $(\omega_1)^N$ is replaced by ω_1 . Since $\omega_1 \in M$ and $M \models \mathbf{ZF}^- \wedge \Phi_{\text{is}}(c_{\prec})$, by Lemma 4.14 the property of being a minimal 3-witness is absolute for M . Hence \bar{c} and each $\bar{c}_\nu^* \upharpoonright 3$ for $\nu < \xi$ are a minimal 3-witnesses, finishing the proof. \square

Thus we have shown, roughly, that \bar{c} is a minimal 3-witness coding a coherent candidate consisting of minimal 3-witnesses if and only if there exists a 3-localizer for \bar{c} . Of course, there may be more than one 3-localizer for a given minimal 3-witness.

Definition 4.19. We say $\bar{c} \in \mathcal{P}(\omega)^4$ is a *minimal 4-witness* if and only if $\bar{c}(3)$ is the \prec -least localizer for $\bar{c} \upharpoonright 3$.

We now continue the definition of k -localizer and minimal $k+1$ -witness for elements of $\mathcal{P}(\omega)^k$ by induction on k , following the template given by the definition for $k=3$.

Definition 4.20. Let $k \in \omega \setminus 4$ and suppose we have already defined what it means to be a minimal k -witness for elements of $\mathcal{P}(\omega)^k$. Given $\bar{c} \in \mathcal{P}(\omega)^k$ (a putative k -witness) we say $c \in \mathcal{P}(\omega)$ is *a k -localizer for \bar{c}* if and only if the following holds:

- For any suitable model N with $\{\bar{c}, c, \vec{a}_\omega\} \subseteq N$, the following holds in N : There is a transitive model M of \mathbf{ZF}^- such that $M \models \Phi_{\text{is}}(c_{\prec})$, $\{\omega_1, \bar{c}, \vec{a}_\omega\} \subseteq M$, and
- $(*)_k$
 - (a) $M \models \text{“}\bar{c} \text{ is a minimal } k\text{-witness”}$,
 - (b) Writing $\vec{b}(\bar{c}(2))^M$ as $\langle b_\nu : \nu < \xi \rangle$, for each $\nu < \xi$ it holds that $M \models \text{“}\bar{c}_\nu \upharpoonright k \text{ is a minimal } k\text{-witness”}$, where $\bar{c}_\nu = \text{Seq}(\text{dc}(b_\nu))$.

Moreover, we say $\bar{c} \in \mathcal{P}(\omega)^{k+1}$ is a *minimal $(k+1)$ -witness* if and only if $\bar{c}(k)$ is the \prec -least k -localizer for $\bar{c} \upharpoonright k$.

Finally, we say $\bar{c} \in \mathcal{P}(\omega)^\omega$ is a *minimal local witness* if and only if

- $(**)$ for each $k \in \omega \setminus 3$, $\bar{c}(k)$ is a k -localizer for $\bar{c} \upharpoonright k$

and we say $c \in \mathcal{P}(\omega)$ is a *minimal local witness* if and only if $\text{Seq}(c)$ is a minimal local witness.

Given arbitrary $\bar{c} \in \mathcal{P}(\omega)^{\leq \omega}$ let us say \bar{c} *codes* \vec{b} if $\bar{c}(2)$ almost disjointly via \mathcal{F} codes the sequence \vec{b} . In this case let us also write $\vec{b}(\bar{c})$ for \vec{b} . We shall also say \bar{c} is a witness to \vec{b} to mean that \bar{c} is a $\text{lh}(\bar{c})$ -witness or, if $\text{lh}(\bar{c}) = \omega$, a minimal local witness, and $\vec{b}(\bar{c}) = \vec{b}$.

Just as before for $k=3$ we have the following crucial lemma:

Lemma 4.21. *Suppose $k \in \omega \setminus 4$ and $\bar{c} \in \mathcal{P}(\omega)^k$. There exists a k -localizer for \bar{c} if and only if \bar{c} is a minimal k -witness, and letting $\vec{b}(\bar{c}(2)) = \langle b_\nu \mid \nu < \xi \rangle$ it holds for each $\nu < \xi$ that $\text{Seq}(\text{dc}(b_\nu)) \upharpoonright k$ is a minimal k -witness.*

Proof. This is shown precisely as Lemmas 4.18 and 4.17 above. \square

In the next lemma, we verify for the reader's convenience that the local minimal witness to a sequence is uniquely determined by this sequence.

Lemma 4.22. *For each sequence $\vec{b} = \langle b_\xi : \xi < \nu \rangle$, there is at most one minimal local witness $\bar{c} \in \mathcal{P}(\omega)^\omega$ coding \vec{b} . Likewise, if two sequences \bar{c} and \bar{c}' are minimal local witnesses and $\bar{c}(2) = \bar{c}'(2)$, then $\bar{c} = \bar{c}'$.*

Proof. Suppose \bar{c} and \bar{c}' are minimal local witnesses coding \vec{b} . Since $\bar{c}(3)$ is a 3-localizer to $\bar{c} \upharpoonright 3$, by Lemma 4.18 the latter is a minimal 3-witness to \vec{b} . The same holds for \bar{c}' . But obviously, there is only one minimal 3-witness to \vec{b} , so $\bar{c} \upharpoonright 3 = \bar{c}' \upharpoonright 3$. But since $\bar{c}(4)$ is a 4-localizer for $\bar{c} \upharpoonright 4$, $\bar{c}(3)$ is the \prec -least 3-localizer by Lemma 4.21. Since the same holds for $\bar{c}'(4)$ we have $\bar{c}(3) = \bar{c}'(3)$. Continue by induction to obtain $\bar{c} = \bar{c}'$. The second statement follows, since if $\bar{c}(2) = \bar{c}'(2)$, also $\vec{b}(\bar{c}) = \vec{b}(\bar{c}')$. \square

We are now ready to begin the proof.

Proof of Theorems 1.2 and 4.1. As we have stated earlier, we shall inductively construct a sequence $\langle a_\nu : \nu < \omega_2 \rangle$ such that $\mathcal{A} = \{a_\nu : \nu < \omega_2\}$ will be a $\mathbf{\Pi}_2^1$ MAD family. For the first ω elements of $\langle a_\nu : \nu < \omega_2 \rangle$ take the sequence $\vec{a}_\omega = \langle a_k : k \in \omega \rangle$ fixed in 4.4 (since our coding functions cd and dc use \vec{a}_ω). Fix $c_{\mathcal{A}} \in \mathcal{P}(\omega)$ from which both \vec{a}_ω and c_{\prec} are computable; in the end \mathcal{A} will be $\mathbf{\Pi}_2^1(c_{\mathcal{A}})$.

Suppose we have already constructed $\langle a_\nu : \nu < \xi \rangle$ (where $\omega \leq \xi < \omega_2$) and assume as induction hypothesis that for each $\nu < \xi$, letting $c_\nu = \text{dc}(a_\nu)$ and $\bar{c}_\nu = \text{Seq}(c_\nu)$ we have that $a_\nu = \text{cd}(\bar{c}_\nu(0), \bar{c}_\nu(1), c_\nu)$ and \bar{c}_ν (or equivalently, c_ν) is a minimal local witness. Also, let us write $d_\nu = \bar{c}_\nu(0)$.

Write $\mathcal{A}_\xi = \{a_\nu : \nu < \xi\}$. We will now define a_ξ . First find d_ξ such that

- (5) d_ξ is the \prec -least element of $[\omega]^\omega$ which is almost disjoint from every element of \mathcal{A}_ξ .

Such d_ξ exists since BPFA implies that there is no MAD family of size less than ω_2 .

We now find a minimal local witness $\bar{c}_\xi \in \mathcal{P}(\omega)^\omega$ to $\langle a_\nu : \nu < \xi \rangle$ (see Definition 4.13).

- Of course, we let $\bar{c}_\xi(0) \equiv d_\xi$.
- By Fact 3.1 (see also Remark 4.6) there exists $c \in [\omega]^\omega$ satisfying the requirement from Fact 4.5 that $\{\nu < \xi : |c \cap a_\nu| < \omega\} = \xi \setminus \omega$. We let $\bar{c}_\xi(1)$ be the \prec -least such c .
- Also by Fact 3.1, there exists a subset of ω which almost disjointly via \mathcal{F} codes $\langle a_\nu : \nu < \xi \rangle$; let $\bar{c}_\xi(2)$ be the \prec -least such subset.

By construction $\bar{c}_\xi \upharpoonright 3$ is a minimal 3-witness. Let $\bar{c}_\xi(3)$ be the \prec -least 3-localizer for $\bar{c}_\xi \upharpoonright 3$, which exists by Lemma 4.18. Continue defining $\bar{c}_\xi \upharpoonright k + 1$ by recursion on k for $k > 3$, letting $\bar{c}_\xi(k)$ be the \prec -least k -localizer for $\bar{c}_\xi \upharpoonright k$, using Lemma 4.21, arriving at a minimal local witness \bar{c}_ξ to $\langle a_\nu : \nu < \xi \rangle$ with $\bar{c}_\xi(0) = d_\xi$.

Finally, we write c_ξ for the element of $\mathcal{P}(\omega)$ such that $\text{Seq}(c_\xi) = \bar{c}_\xi$ and define

$$a_\xi = \text{cd}(\bar{c}_\xi(0), \bar{c}_\xi(1), c_\xi),$$

finishing the recursive definition of $\langle a_\xi : \xi < \omega_2 \rangle$. Write $\mathcal{A} = \{a_\xi : \xi < \omega_2\}$. Clearly, by choice of $c_\xi(0) = d_\xi$ and $\bar{c}_\xi(1)$ and by the properties of the function cd from Fact 4.5, this is an almost disjoint family.

It is not hard to see that \mathcal{A} is maximal. We first point out the following simple observation:

Claim 4.23. *Whenever $\nu < \xi < \omega_2$, $d_\nu \prec d_\xi$.*

Proof. This is clear by the definition: Suppose otherwise that $d_\xi \preceq d_\nu$. Since $\mathcal{A}_\nu \subseteq \mathcal{A}_\xi$, d_ξ is almost disjoint from every set in \mathcal{A}_ν . So by minimality of d_ν , we infer $d_\nu = d_\xi$. But then since $d_\nu \cap a_\nu$ is infinite by the properties of the function cd from Fact 4.5, d_ξ is not almost disjoint from every element of \mathcal{A}_ξ , contradicting how d_ξ was chosen.

Claim 4.23. \square

Claim 4.24. *The set \mathcal{A} is a maximal almost disjoint family.*

Proof. Suppose towards a contradiction that $d \in [\omega]^\omega \setminus \mathcal{A}$ and $\mathcal{A} \cup \{d\}$ is an almost disjoint family. Let $\xi < \omega_2$ be the least ordinal such that $d \preceq d_\xi$; such an ordinal exists since \prec well-orders the reals in ordertype ω_2 and so the sequence $\langle d_\xi : \xi < \omega_2 \rangle$ is \prec -cofinal in $\mathcal{P}(\omega)$. But since at stage ξ in the construction of \mathcal{A} , d_ξ was chosen to be the least element almost disjoint from every element of $\{a_\nu : \nu < \xi\}$, we have $d = d_\xi$. Then since $a_\xi = \text{cd}(\bar{c}_\xi(0), \bar{c}_\xi(1), \bar{c}_\xi)$ and $\bar{c}_\xi(0) = d_\xi = d$, $a_\xi \cap d$ is infinite by the properties of the function cd from Fact 4.5, contradiction.

Claim 4.24. \square

We now show that \mathcal{A} is $\Pi_2^1(c_{\mathcal{A}})$. We first show:

Claim 4.25. *There is a $\Pi_2^1(c_{\mathcal{A}})$ formula $\Theta(x)$ such that $\Theta(\bar{c})$ holds if and only if \bar{c} is a minimal local witness.*

Proof. By induction on k it is easily seen that for each $k \in \omega \setminus 3$ the set

$$\{(c, c') \in \mathcal{P}(\omega) \times \mathcal{P}(\omega)^k : c \text{ is a } k\text{-localizer for } c'\}$$

is definable by a $\Pi_2^1(c_{\mathcal{A}})$ formula $\Theta_k(x, y)$, namely, the formula obtained by expressing $(*)_k$ in the language of set theory. In fact, $\langle \Theta_k(x, y) : k \in \omega \rangle$ is a recursive sequence of formulas, and so using a universal definable Π_2^1 truth predicate we can find a $\Pi_2^1(c_{\mathcal{A}})$ formula $\Theta(\bar{c})$ equivalent to

$$(\forall k \in \omega) \Theta_k(\bar{c}(k+3), \bar{c} \upharpoonright (k+3)). \quad \square$$

Let now $\Psi(a)$ be defined as follows:

$$\Psi(a) \stackrel{\text{def}}{\iff} [(\exists n \in \omega) a = a_n] \vee \left(\forall c \in (\mathcal{P}(\omega)) \left[c = \text{dc}(a) \Rightarrow \left(a = \text{cd}((c)_0, (c)_1, c) \wedge \Theta(\text{Seq}(c)) \right) \right] \right).$$

Clearly this formula is $\Pi_2^1(c_{\mathcal{A}})$. We will show that $\Psi(a) \iff a \in \mathcal{A}$. The non-trivial direction is “ \Rightarrow ,” which we show first.

Lemma 4.26. $(\forall a \in [\omega]^\omega) \Psi(a) \Rightarrow a \in \mathcal{A}$.

Proof. Suppose $\Psi(a)$ and to avoid trivialities let us suppose $a \notin \{a_n \mid n \in \omega\}$. Then $\bar{c} = \text{Seq}(\text{dc}(a))$ is a minimal local witness and so $\vec{b}(\bar{c})$ is defined, namely as the unique sequence coded by $\bar{c}(2)$ as in $(*)_3(a)$. Let us write $\vec{b}(\bar{c}) = \langle b_\xi : \xi < \alpha \rangle$. We need the following claim:

Claim 4.27. *The sequence $\vec{b}(\bar{c}) = \langle b_\xi : \xi < \alpha \rangle$ is an initial segment of $\langle a_\nu : \nu < \omega_2 \rangle$.*

Proof. Suppose not. Let $\nu < \alpha$ be least such that $b_\nu \neq a_\nu$. Write $c_\nu^* = \text{dc}(b_\nu)$ and $\bar{c}_\nu^* = \text{Seq}(c_\nu^*)$. Since \bar{c} is a minimal local witness, $\vec{b}(\bar{c})$ is a coherent candidate, and so $(c_\nu^*)_2$ codes almost disjointly via \mathcal{F} the sequence $\vec{b}(\bar{c}) \upharpoonright \nu$, which by assumption is $\langle a_\xi : \xi < \nu \rangle$.

We verify that \bar{c}_ν^* , too, is a minimal local witness: Firstly, $\bar{c}(3)$ is a 3-localizer for $\bar{c} \upharpoonright 3$. Then by (b) in $(*)_4$ and by Lemma 4.18 it holds that $\bar{c}_\nu^* \upharpoonright 3$ is a minimal 3-witness. More generally, since $\bar{c}(k)$ is a k -localizer for $c \upharpoonright k$, by (b) in $(*)_k$ we see that $\bar{c}_\nu^* \upharpoonright k$ is a minimal k -witness (cf. Lemma 4.21). Since this holds for each $k \in \omega$, \bar{c}_ν^* is a minimal local witness.

But then since \bar{c}_ν^* and \bar{c}_ν are both minimal local witnesses for the sequence $\langle a_\xi : \xi < \nu \rangle$, we must have $\bar{c}_\nu^* = \bar{c}_\nu$ by the definition of minimal local witness (see also Lemma 4.22). It follows that $a_\nu = \text{cd}(\bar{c}_\nu(0), \bar{c}_\nu(1), c_\nu) = \text{cd}(\bar{c}_\nu^*(0), \bar{c}_\nu^*(1), c_\nu^*) = b_\nu$, contradicting the choice of ν . Claim 4.25. \square

By the claim we can fix $\nu < \omega_2$ such that $\vec{b}(\bar{c}) = \langle a_\xi : \xi < \nu \rangle$. By the same argument as in the previous paragraph, $a = a_\nu$. Lemma 4.26. \square

Finally, for any ξ such that $\omega \leq \xi < \omega_2$ it is clear by construction that $a_\xi = \text{cd}(\bar{c}_\xi)$ and \bar{c}_ξ is a minimal local witness. Therefore $\Psi(a_\xi)$ holds. So $a \in \mathcal{A} \Rightarrow \Psi(a)$. Theorems 1.2& 4.1. \square

5. QUESTIONS

In [19], Schindler showed the consistency of BPFA+“Every projective set of reals is Lebesgue-measurable”, assuming that there is a Σ_2 -correct regular (i.e., a reflecting) cardinal above a remarkable cardinal. In light of this and Theorem 1.1 the following is a natural question:

Question 5.1. *Can one prove the consistency of BPFA+“All Π_2^1 MAD families are finite” from a Σ_2 -correct regular cardinal above a remarkable cardinal?*

Question 5.2. *Can BPFA be replaced by the Bounded Forcing Axiom for Axiom A in Theorem 1.2?*

Question 5.3. *Can the anti-large cardinal assumption be weakened? Can we assume a forcing axiom stronger than BPFA, but still compatible with such an anti-large cardinal assumption, and derive a form of Theorem 1.2?*

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