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DIPLOMA THESIS

On the existence of p-points and other ultrafilters in the Stone-Čech-compactification of $\mathbb{N}$

carried out at the

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by

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Preface

In this diploma thesis, we consider different kinds of ultrafilters on the natural numbers, in particular *p-points*.

The set of all ultrafilters on $\mathbb{N}$ (equipped with a natural topology) is a compact space containing $\mathbb{N}$ as a dense subset (the so-called Stone-Čech-compactification $\beta\mathbb{N}$). That's why p-points are called "points" because they can be viewed as a special type of points in this compact space.

It turns out that the existence of p-points is neither provable nor refutable from ZFC. It can be shown that the continuum hypothesis (i.e., the size of the continuum equals $\aleph_1$) implies the existence of p-points. Furthermore, p-points can also exist for arbitrarily large continuum.

In 1982, Saharon Shelah was able to construct a model of ZFC in which there is no p-point (using the technique of "iterated forcing"). The original proof can be found in Wimmers' article "The Shelah *P*-point independence theorem" (see [16]). There is a new and simpler proof (also due to Shelah), which uses the standard technique of a countable support iteration of proper forcings; it can be found in Shelah's book "Proper and improper forcing" ([13]) as well as in the book by Bartoszyński and Judah ([2]). In Shelah's model the size of the continuum is $\aleph_2$. It seems to be unknown if it is possible to construct a model of ZFC without p-points and larger continuum, e.g., $2^{\aleph_0} = \aleph_3$.

In Chapter 1, we consider models of ZFC in which there are p-points.

In Chapter 2, we investigate the Stone-Čech-compactification $\beta\mathbb{N}$ and examine the concept of a p-point from the topological point of view.

In Chapter 3, we construct a model of ZFC without p-points; we essentially follow Shelah's new proof.

Contents

Chapter 1

The existence of p-points

In this chapter, we show that the existence of p-points is consistent with ZFC: if the *continuum hypothesis* holds (i.e., $2^{\aleph_0} = \aleph_1$), then it is straightforward to construct a p-point; moreover, Martin's Axiom (a combinatorial principle which is consistent with arbitrarily large continuum) implies the existence of p-points as well. We also mention the notion of a Ramsey ultrafilter and prove its existence under Martin's Axiom. Finally, we show that $\mathfrak{d} = 2^{\aleph_0}$ still implies the existence of a p-point.

All notions and results of this chapter are well-known and can be found, e.g., in [8] or [2]; for details on Martin's Axiom, cf. the paper by Martin and Solovay ([11]).

1.1 Definitions and notation

In this section, we define what a p-point is and clarify our notation concerning forcing partial orders. For standard set theoretic notions which are not mentioned here see, e.g., Jech ([8]) or Kunen ([9]).

Filters and ultrafilters on ω

We recall the notion of a filter and briefly review basic facts about filters and ultrafilters on ω. For details, cf. [8, Ch.7].

Definition 1.1. A set $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a *filter on* ω if

1. $\omega \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$
2. if $X \in \mathcal{F}$ and $Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$
3. if $X \in \mathcal{F}$ and $X \subseteq Y$, then $Y \in \mathcal{F}$

So a filter on ω is a (non-empty) collection of subsets of ω closed under finite intersections and supersets (and not containing all subsets of ω).

If $\mathcal{F}$ is a filter on ω, the set $\mathcal{F}^\star = \{\omega \setminus X : \ X \in \mathcal{F}\}$ is an *ideal* on ω (the so-called "dual ideal"). The sets in a filter $\mathcal{F}$ (the "filter sets") can be viewed as the "large sets" (then the sets in the dual ideal $\mathcal{F}^\star$ are the "small" ones).

A family $\mathcal{G} \subseteq \mathcal{P}(\omega)$ has the *finite intersection property* if each finite subfamily has a non-empty intersection. (Every filter has the finite intersection property.)

Lemma 1.2. *If $\mathcal{G} \subseteq \mathcal{P}(\omega)$ has the finite intersection property, then there is a filter $\mathcal{F}$ extending $\mathcal{G}$.*

Proof. Let $\mathcal{F}$ be the set of all supersets of finite intersections of sets from $\mathcal{G}$; then $\mathcal{F}$ is a filter and $\mathcal{F} \supseteq \mathcal{G}$. □

$\mathcal{F}$ is the smallest filter which extends $\mathcal{G}$; we say that the family $\mathcal{G}$ *generates* the filter $\mathcal{F}$.

A filter is called *principal* if the intersection of all filter sets is non-empty. (The principal filters are those filters which are generated by a single non-empty set.) Otherwise the filter is called *non-principal* or *free*.

Definition 1.3. The Frechet filter $\mathfrak{Fr}$ is the set of all co-finite subsets of ω:

$$\mathfrak{Fr} := \{X \subseteq \omega : \ |\omega \setminus X| < \aleph_0\}$$

Obviously the Frechet filter is non-principal. Note that $\mathfrak{Fr}^\star = [\omega]^{<\omega}$.

Definition 1.4. A filter $\mathcal{F}$ on ω is an *ultrafilter* if for each $X \subseteq \omega$, either $X \in \mathcal{F}$ or $\omega \setminus X \in \mathcal{F}$.

An ultrafilter $\mathcal{F}$ can be thought of as a filter "deciding each subset of ω", in the sense that each subset of ω is either "large" or "small" with respect to $\mathcal{F}$.

Lemma 1.5. *A filter $\mathcal{F}$ is an ultrafilter if and only if it is maximal.* □

Lemma 1.6. *Every filter $\mathcal{F}$ can be extended to an ultrafilter.*

Proof. By Zorn's Lemma (which is equivalent to the axiom of choice), we get a maximal filter extending $\mathcal{F}$ which is an ultrafilter by Lemma 1.5. □

Note that an ultrafilter is non-principal if and only if it contains the Frechet filter. So there are non-principal ultrafilters on ω (just extend the Frechet filter). However, the axiom of choice (AC) is necessary here: there are models of ZF (= ZFC \ AC) in which each ultrafilter on ω is principal (see Jech's book [8, Example 15.59 on page 260] for a proof).

Of course, there are at most $2^{(2^{\aleph_0})}$ ultrafilters on ω, since each filter $\mathcal{F}$ belongs to $\mathcal{P}(\mathcal{P}(\omega))$. By a theorem of Pospíšil, the number of (non-principal) ultrafilters on ω is exactly $2^{(2^{\aleph_0})}$:

Theorem 1.7. *There are exactly $2^{(2^{\aleph_0})}$ non-principal ultrafilters on ω.*

Proof. See [8, Theorem 7.6 on page 75]. □

Pseudo-intersections, p-filters and p-points

Now we are going to define the notion of a p-point.

We say that X is *almost contained* in Y (denoted by $X \subseteq^\star Y$) if $X \setminus Y$ is finite:

$$X \subseteq^\star Y \iff |X \setminus Y| < \aleph_0.$$

Definition 1.8. Let $\{Y_i : i \in I\} \subseteq \mathcal{P}(\omega)$ be a family of infinite subsets of ω. An infinite set $X \subseteq \omega$ is called a *pseudo-intersection* of the family $\{Y_i : i \in I\}$ if it is almost contained in each of the Y_i:

$$\forall i \in I : \quad X \subseteq^\star Y_i.$$

In the following we assume that $\mathcal{F}$ extends the Frechet filter $\mathfrak{Fr}$ (so in particular, each filter set is infinite).

Definition 1.9. A filter $\mathcal{F}$ on ω is called a *p-filter* if for each countable collection $\{Y_n : n \in \omega\} \subseteq \mathcal{F}$ of filter sets there is a filter set $X \in \mathcal{F}$ such that $X \subseteq^\star Y_n$ for each $n \in \omega$.

In other words: a filter is a p-filter if each countable collection of filter sets has a pseudo-intersection *within* the filter. We will refer to this as the "p-filter property".

Definition 1.10. A non-principal ultrafilter $\mathcal{F}$ on ω is called a *p-point* if it is a p-filter.

The following equivalent characterization is easy to show:

Lemma 1.11. *A non-principal ultrafilter $\mathcal{F}$ is a p-point if and only if for every partition $\{X_n : n \in \omega\}$ of ω into infinitely many "small parts with respect to $\mathcal{F}$", i.e.,*

$$\forall n \in \omega \quad X_n \notin \mathcal{F},$$

there exists a filter set $X \in \mathcal{F}$ such that

$$\text{for each } n \in \omega : \quad |X \cap X_n| < \aleph_0.$$

Using this characterization, we can always find a non-principal ultrafilter on ω that is not a p-point. Fix some partition $\{X_n : n \in \omega\}$ of ω into infinitely many infinite pieces, and let $\mathcal{F}$ be the following filter: $Z \in \mathcal{F}$ if and only if except for finitely many n, $Z \cap X_n$ contains all but finitely many elements of X_n; if $\widetilde{\mathcal{F}}$ is an ultrafilter extending $\mathcal{F}$, then $\widetilde{\mathcal{F}}$ is no p-point (as can be easily seen).

Forcing

Forcing is a technique to generate new models of ZFC with prescribed properties. It was developed by Paul Cohen, who used it in his 1963 proof of the independence of the continuum hypothesis (and the axiom of choice). For details on forcing we refer to Kunen's "Introduction to Independence Proofs" ([9]) and Jech's book ([8]); for "iterated forcing" (used in 3.8) we additionally refer to Goldstern's "Tools for Your Forcing Construction" ([6]). Here we only give a notational remark.

Traditionally, there are two (contradictory) notations for interpreting a partial order as a forcing notion. We use the "Boolean" or "downwards" notation: if $(\mathbb{P}, \leq)$ is a forcing partial order, $q \leq p$ means "q extends p", "q is stronger than p" or "q has more information than p".

To avoid confusion, we employ the **alphabet convention** (see also [7]):

Whenever two conditions are comparable, the notation is chosen so that the variable used for the stronger condition comes "lexicographically" later.

So we write, e.g., $q \leq p$ (for q stronger than p), but try to avoid expressions like $p \leq r$ (for p stronger than r).

1.2 CH implies the existence of p-points

We are going to show how to construct a p-point if the continuum hypothesis (CH) holds. It follows that the existence of p-points is consistent with ZFC.

First of all, let's give a preliminary lemma:

Lemma 1.12. *Assume $\langle X_\alpha : \alpha < \delta \rangle$ is an almost decreasing sequence of infinite sets, i.e.,*

$$\forall \alpha < \beta < \delta : \; X_\alpha \supseteq^\star X_\beta$$

and $|X_\alpha| = \aleph_0$ for each $\alpha < \delta$. Then the set

$$\mathcal{F} := \{X \subseteq \omega : \; X \supseteq^\star X_\alpha \textit{ for some } \alpha < \delta\}$$

is a (proper) filter containing the Frechet filter.

Proof. Clearly, ω is in $\mathcal{F}$, and $\mathcal{F}$ is closed under supersets. Given two sets $Y, Z \in \mathcal{F}$, there are $\alpha, \beta < \delta$ such that $Y \supseteq^\star X_\alpha$ and $Z \supseteq^\star X_\beta$. But then $Y, Z \supseteq^\star X_{\max(\alpha,\beta)}$, consequently also $Y \cap Z \supseteq^\star X_{\max(\alpha,\beta)}$, and therefore $Y \cap Z$ is in $\mathcal{F}$. It's also clear, that $\mathcal{F}$ contains each co-finite set Y: for any α, $Y \supseteq^\star X_\alpha$. Finally, $\mathcal{F}$ does not contain the empty set (or any finite set), since all the X_α's are infinite. So $\mathcal{F}$ is a (proper) filter extending the Frechet filter. □

Remark. Note that the filter $\mathcal{F}$ in the lemma above can be thought of as the filter generated by the X_α's together with the Frechet filter; in other words, $\mathcal{F}$ is the set of all supersets of the X_α's without considering "finite changes".

Theorem 1.13. *Assume CH holds. Then there exists a p-point.*

Proof. We are going to construct a p-point $\mathcal{F}$ by building an almost decreasing sequence of infinite sets of length ω_1, i.e., a sequence $\langle X_\alpha : \alpha < \omega_1 \rangle \subseteq \mathcal{P}(\omega)$ with

$$\forall \alpha < \beta < \omega_1 : \; X_\alpha \supseteq^\star X_\beta$$

and $|X_\alpha| = \aleph_0$ for each $\alpha < \omega_1$. Then we define $\mathcal{F}$ to be the filter "generated" by the X_α's in the sense of Lemma 1.12. To ensure that $\mathcal{F}$ is an ultrafilter (as demanded), we go through all possible subsets of ω (due to CH, there are just $\aleph_1$ many), in each step "deciding" one of them, i.e., each time we consider some set A, we are going to put either the set A itself or its complement into the filter. The p-filter property of $\mathcal{F}$ will easily follow from the fact that the filter is constructed by an almost decreasing sequence.

More precisely, proceed as follows: using CH, we can fix an enumeration of $\mathcal{P}(\omega)$ of length ω_1, i.e., let

$$\mathcal{P}(\omega) = \{A_\alpha : \alpha < \omega_1\}.$$

Now start with $X_0 := \omega$. At a successor ordinal $\alpha + 1$, consider the sets X_α (in other words, the principal filter generated by X_α) and A_α, the respective set in our enumeration of $\mathcal{P}(\omega)$. Since (by induction) X_α is infinite, either $X_\alpha \cap A_\alpha$ or $X_\alpha \cap (\omega \setminus A_\alpha)$ is infinite (or both). So we can define

$$X_{\alpha+1} := \begin{cases} X_\alpha \cap A_\alpha & \text{if } |X_\alpha \cap A_\alpha| = \aleph_0 \\ X_\alpha \cap (\omega \setminus A_\alpha) & \text{otherwise} \end{cases} \tag{1.1}$$

and $X_{\alpha+1} \subseteq X_\alpha$ will again be infinite (and since $X_\alpha \subseteq^\star X_\gamma$ for all $\gamma < \alpha$ holds by induction, $X_{\alpha+1} \subseteq^\star X_\gamma$ for all $\gamma < \alpha + 1$ will be true as well).

The following remains to be checked: are we able to extend our almost decreasing sequence at every limit ordinal δ ($< \omega_1$)? This case can be handled by the following lemma:

Lemma 1.14. *Every* countable *almost decreasing sequence of infinite subsets of ω can be extended by a still infinite set.*

Proof. Let $\langle X_\alpha : \alpha < \delta \rangle \subseteq \mathcal{P}(\omega)$, $\delta < \omega_1$ limit, be an almost decreasing sequence of infinite sets, i.e.,

$$\forall \alpha < \beta < \delta : \; X_\alpha \supseteq^\star X_\beta$$

and $|X_\alpha| = \aleph_0$ for each $\alpha < \delta$. We are going to find an infinite set almost contained in all the X_α's, i.e., a set X_δ with $|X_\delta| = \aleph_0$ and

$$\forall \alpha < \delta : \; X_\alpha \supseteq^\star X_\delta.$$

In fact we *claim* the following: Whenever $\{Y_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ is a countable collection of infinite sets with the property that the intersection of finitely many of them is always infinite, the collection has an infinite pseudo-intersection, i.e., there is a set $Y \subseteq \omega$ with $|Y| = \aleph_0$ and

$$\forall n < \omega : \; Y \subseteq^\star Y_n. \tag{1.2}$$

How do we get such a set Y? First of all, we can assume (without loss of generality) that the Y_n's form a decreasing sequence $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \dots$ of infinite sets: just replace each Y_n by the finite intersection $\bigcap_{k \leq n} Y_k$; by assumption, they are still infinite. But now it's easy to find an infinite set Y satisfying (1.2): pick a strictly increasing sequence $\langle y_n : n \in \omega \rangle$ with $y_n \in Y_n$ for each n (this is possible since all the Y_n's are infinite) and define

$$Y := \{y_n : n \in \omega\};$$

clearly, $|Y| = \aleph_0$ and $Y \subseteq^\star Y_n$ for each $n \in \omega$, because $\{y_k : k \geq n\} \subseteq Y_n$ for each n, so our *claim* was true.

But this is sufficient to proof the lemma; the given set $\{X_\alpha : \alpha < \delta\}$ indeed is a countable ($|\delta| = \omega$) collection of infinite sets with the "intersection property" demanded in the claim: whenever $E \subseteq \delta$ is finite, we have

$$\bigcap_{\alpha \in E} X_\alpha \supseteq^\star X_{\max E},$$

which is infinite (compare with Lemma 1.12: the filter "generated" by the X_α's is closed under finite intersections and contains only infinite sets). So we get an infinite pseudo-intersection X_δ ($X_\delta \subseteq^\star X_\alpha$ for each $\alpha < \delta$), which finishes the proof of the lemma. □

Remark. In the above version of the proof, we essentially used the fact that $|\delta| = \omega$: we re-ordered the sequence $\langle X_\alpha : \alpha < \delta \rangle$ ($\delta < \omega_1$) to get a sequence of order-type ω; this enabled us (after making it really decreasing) to choose an appropriate set Y by picking an ω-sequence of elements. Instead of this, we also could have concentrated on the fact that $\mathrm{cf}(\delta) = \omega$: there is a sequence of ordinals $\langle \alpha_n : n \in \omega \rangle$ such that $\sup_{n\in\omega} \alpha_n = \delta$; considering the respective (thinned-out) ω-sequence $\langle X_{\alpha_n} : n \in \omega \rangle$, it's possible to find a pseudo-intersection X_δ as well.

Using this lemma, we can carry out the construction up to ω_1, obtaining an almost decreasing sequence of infinite sets, $\langle X_\alpha : \alpha < \omega_1 \rangle$. We define

$$\mathcal{F} := \{X \subseteq \omega : X \supseteq^\star X_\alpha \text{ for some } \alpha < \omega_1\} \tag{1.3}$$

and claim that $\mathcal{F}$ is a p-point.

First of all, by Lemma 1.12, $\mathcal{F}$ is a (proper) filter extending the Frechet filter. It is an ultrafilter, since our construction aimed at obtaining an ultrafilter: for any $A \subseteq \omega$, there is an $\alpha < \omega_1$ with $A = A_\alpha$; but then $X_{\alpha+1}$ is a subset of either A or $\omega \setminus A$, hence either A or $\omega \setminus A$ is in the filter $\mathcal{F}$.

It remains to show that $\mathcal{F}$ has the p-filter property; this is because the filter is generated by an almost decreasing sequence of uncountable cofinality: Let's assume, a countable collection of filter sets $\{Y_n : n \in \omega\} \subseteq \mathcal{F}$ is given. By the definition of $\mathcal{F}$, there is a corresponding countable collection of countable ordinals, say $\{\alpha_n : n \in \omega\} \subseteq \omega_1$, such that $Y_n \supseteq^\star X_{\alpha_n}$ for each $n \in \omega$. Now let

$$\alpha := \sup_{n\in\omega} \alpha_n < \omega_1. \tag{1.4}$$

Then $X_\alpha \in \mathcal{F}$ and (since $X_{\alpha_n} \supseteq^\star X_\alpha$ for each n) $X_\alpha \subseteq^\star Y_n$ for each $n \in \omega$, which finishes the proof of the theorem. □

Remark. Note, that in (1.3), $X \supseteq^\star X_\alpha$ could actually be replaced by $X \supseteq X_\alpha$, yielding the very same filter $\mathcal{F}$. This is because $\mathcal{F}$ is maximal, in the sense that the following is still true: for each set $A \subseteq \omega$, either A or $\omega \setminus A$ is in $\mathcal{F}$. (In contrast to Lemma 1.12, where it's not possible to remove the * in general, otherwise the set need not to be closed under intersections any more.)

1.3 Martin's Axiom and the tower number

Now we would like to think about where we actually used CH in the proof of Theorem 1.13. Is it possible to construct a p-point in a similar way without assuming CH?

The only reason why we had to assume CH in the proof of Theorem 1.13 is the fact that – in general – Lemma 1.14 only works for almost decreasing sequences of *countable* length: it is not clear how to extend sequences of length ω_1, for instance. Apart from this, the same proof would perfectly work: going through all the $2^{\aleph_0}$ many subsets of ω, we can build an almost decreasing sequence of length $2^{\aleph_0}$, yielding a p-point. (To establish the p-filter property, we use the fact that $\mathrm{cf}(2^{\aleph_0}) > \omega$; otherwise the countable sequence of ordinals in (1.4) could happen to be cofinal in $2^{\aleph_0}$.)

The tower number $\mathfrak{t}$

To overcome the limitations of Lemma 1.14, we introduce the following notions:

Definition 1.15 (Tower number)**.** A *tower* is a *maximal* almost decreasing sequence of infinite subsets of ω, i.e., a sequence $\langle X_\alpha : \alpha < \delta \rangle \subseteq \mathcal{P}(\omega)$ with

$$\forall \alpha < \beta < \delta : \; X_\alpha \supseteq^\star X_\beta$$

and $|X_\alpha| = \aleph_0$ for each $\alpha < \delta$, such that there is *no infinite set* X with

$$\forall \alpha < \delta : \; X_\alpha \supseteq^\star X.$$

The *tower number* $\mathfrak{t}$ is the least ordinal δ such that there exists a tower of length δ.

Remark. We should mention that the tower number $\mathfrak{t}$ is well-defined: in fact, there is always some tower (of length at most $2^{\aleph_0}$), as we will see in the last part of the proof of Lemma 1.16, i.e., it is impossible to have no tower at all.

The tower number $\mathfrak{t}$ is one of the so-called cardinal invariants (like the bounding number $\mathfrak{b}$, the dominating number $\mathfrak{d}$, etc.). Of course, $\mathfrak{t}$ is a limit ordinal; in fact, it is quite easy to show that it is a regular cardinal (see the remark after Lemma 1.14 on page 10: $\mathrm{cf}(\mathfrak{t}) = \mathfrak{t}$, since if $\mathrm{cf}(\mathfrak{t}) < \mathfrak{t}$ were true, there would be a thinned-out tower of order-type $\mathrm{cf}(\mathfrak{t})$, contradicting the fact that $\mathfrak{t}$ is the minimal length of a tower).

Using the proof of Theorem 1.13, we can easily show

Lemma 1.16. *The tower number* $\mathfrak{t}$ *is a regular cardinal satisfying*

$$\aleph_1 \leq \mathfrak{t} \leq 2^{\aleph_0}.$$

Proof. We have already discussed why $\mathfrak{t}$ is regular.

Now recall Lemma 1.14: every *countable* almost decreasing sequence of infinite subsets of ω can be extended by a still infinite set; in other words, there is no tower of countable length, which immediately tells us that $\aleph_1 \leq \mathfrak{t}$.

Finally, let's show that there is some tower of length at most $2^{\aleph_0}$. We just construct one, following the proof of Theorem 1.13. We don't assume CH or anything, so – instead – we consider a fixed enumeration of $\mathcal{P}(\omega)$ of length $2^{\aleph_0}$. We try to construct an almost decreasing sequence of infinite sets $\langle X_\alpha : \alpha < 2^{\aleph_0} \rangle$, at successor ordinals doing the very same as in Theorem 1.13. At limit ordinals, we try to find "pseudo-intersections", i.e., infinite sets almost contained in all the sets constructed so far. If this is not possible for some limit ordinal strictly below $2^{\aleph_0}$, we have found a tower, and $\mathfrak{t} < 2^{\aleph_0}$ (in this case, CH necessarily fails due to $\aleph_1 \leq \mathfrak{t}$). Otherwise we carry out the construction up to $2^{\aleph_0}$; we claim that the resulting sequence is a tower of length $2^{\aleph_0}$ (hence $\mathfrak{t} \leq 2^{\aleph_0}$). Define

$$\mathcal{F} := \{ Z \subseteq \omega : Z \supseteq^\star X_\alpha \text{ for some } \alpha < 2^{\aleph_0} \},$$

which is an ultrafilter. Assume towards a contradiction that there is an infinite X with $X \subseteq^\star X_\alpha$ for all $\alpha < 2^{\aleph_0}$; then $\mathcal{F} \subseteq \{Z \subseteq \omega : Z \supseteq^\star X\}$, so $\mathcal{F}$ fails to be an ultrafilter (split X into two infinite parts X_1 and X_2, then neither X_1 nor X_2 will be in $\mathcal{F}$), a contradiction. □

In terms of the tower number $\mathfrak{t}$, we are now able to state a generalization of Theorem 1.13 (without assuming CH); a large tower number allows us to go on at limits of uncountable cofinality:

Theorem 1.17. *Assume that the tower number* $\mathfrak{t} = 2^{\aleph_0}$. *Then there exists a p-point.*

Proof. We do almost the same as in the proof of Theorem 1.13, with ω_1 replaced by $2^{\aleph_0}$ throughout. Going through all the continuum many subsets of ω, we construct an almost decreasing sequence of length $2^{\aleph_0}$ (in fact, it's a tower). At successor ordinals, we do the very same as in Theorem 1.13, and at limit ordinals – instead of using Lemma 1.14 – we use the assumption $\mathfrak{t} = 2^{\aleph_0}$, which tells us that there is no tower of length less than $2^{\aleph_0}$. In the end, this gives us an ultrafilter $\mathcal{F}$; as already mentioned, we have to use $\mathrm{cf}(2^{\aleph_0}) > \omega$ (given by König's Theorem) to establish the p-filter property of $\mathcal{F}$. (So we do the same as in the last part of the proof of Lemma 1.16, knowing that the construction can be carried out up to $2^{\aleph_0}$.) □

Martin's Axiom

Now let's recall Martin's Axiom, a well-known combinatorial principle often used in forcing arguments:

Definition 1.18 (Martin's Axiom (MA))**.** Let $\mathbb{P}$ be a c.c.c. forcing (i.e., a forcing satisfying the countable chain condition) and let $\mathcal{D} = \{D_\alpha : \alpha < \kappa\}$ ($\kappa < 2^{\aleph_0}$) be a collection of fewer than continuum many dense subsets of $\mathbb{P}$. Then there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$, i.e., a filter G meeting all the D_α's:

$$\forall \alpha < \kappa : \; G \cap D_\alpha \neq \emptyset.$$

Of course, CH implies MA, but MA is also known to be consistent with arbitrarily large continuum:

Theorem 1.19 (Solovay and Tennenbaum)**.** *Assume* GCH *and let* κ *be a regular cardinal greater than* $\aleph_1$*. There exists a c.c.c. forcing notion* $\mathbb{P}$ *such that the generic extension* $\mathbf{V}[G]$ *by* $\mathbb{P}$ *satisfies Martin's Axiom and* $2^{\aleph_0} = \kappa$*.*

Proof. See [8, Theorem 16.13 on page 272]. □

Intuitively, one can say the following about MA: if MA holds, cardinals below the continuum often behave like $\aleph_0$ in some sense. The following application of Martin's Axiom is of interest to us:

Theorem 1.20. *Assume MA. Then the tower number* $\mathfrak{t} = 2^{\aleph_0}$*, i.e., every almost decreasing sequence of infinite subsets of* ω *of length less than the continuum can be extended (by a still infinite set).*

Proof. Assume MA and let $\langle X_\alpha : \alpha < \delta \rangle$, $\delta < 2^{\aleph_0}$, be an almost decreasing sequence of infinite sets, i.e.,

$$\forall \alpha < \beta < \delta : \; X_\alpha \supseteq^\star X_\beta$$

and $|X_\alpha| = \aleph_0$ for each $\alpha < \delta$. We shall find a still infinite set almost contained in all the X_α's, i.e., a set X with $|X| = \aleph_0$ and

$$\forall \alpha < \delta : \; X_\alpha \supseteq^\star X.$$

For this purpose, we will define an appropriate c.c.c. forcing $\mathbb{P}$; then we can derive the desired set X from the "generic object" given by MA.

Define the set

$$\mathcal{F} := \{X \subseteq \omega : \; X \supseteq^\star X_\alpha \text{ for some } \alpha < \delta\},$$

which is (by Lemma 1.12) a filter containing the Frechet filter. Now define the following forcing notion $(\mathbb{P}, \leq)$:

$$\mathbb{P} := \big\{(s, A) : \; s \in [\omega]^{<\omega}, \; A \in \mathcal{F}, \; \max s < \min A\big\},$$

and $(t, B) \leq (s, A)$, meaning (t, B) stronger than (s, A), if and only if the following holds:

$$\begin{array}{rl} \text{(i)} & s \subseteq t \\ \text{(ii)} & A \supseteq B \\ \text{(iii)} & t \setminus s \subseteq A \end{array} \tag{1.5}$$

Note that in (1.5), t is in fact an end-extension of s, since $t \setminus s \subseteq A$, which is (by $\max s < \min A$) completely above s.

Now, provided that G is any filter on this forcing (intended to be given by MA), we can define a set $X \subseteq \omega$ by

$$X := \bigcup \{s : \ (s, A) \in G \text{ for some } A\}. \tag{1.6}$$

So a condition (s, A) can be seen as follows: the first component s is a finite approximation of X (a finite initial segment of X) and the second component A determines what X is going to look like above $\max s$: (s, A) forces X to be an almost-subset of A, namely $X \setminus s$ will be contained in A. Using MA, we will be able to find a filter G such that the respective X is almost contained in each of the X_α's: for this, we just have to show that it is dense for a condition to have a set $A \subseteq X_\alpha$ as its second component.

But first of all, let's check that the forcing $\mathbb{P}$ defined above is indeed c.c.c. (otherwise we cannot apply MA):

Claim. *The forcing $\mathbb{P}$ defined above is* σ-centered*, i.e., $\mathbb{P}$ can be partitioned into countably many parts, say $\mathbb{P} = \bigcup_{i \in \omega} \mathbb{P}_i$, such that each part $\mathbb{P}_i$ is centered (meaning every finite collection of conditions taken from $\mathbb{P}_i$ has a lower bound). So, in particular, $\mathbb{P}$ has the* countable chain condition*.*

Proof. Let's define a partition of our forcing $\mathbb{P}$ as follows: two conditions belong to the same part if and only if their first components are the same:

$$(s_1, A_1) \sim (s_2, A_2) \ :\Longleftrightarrow \ \ s_1 = s_2$$

Of course it is a partition into just countably many parts since there are only countably many finite sets $s \subseteq \omega$. But now each part is centered; given finitely many conditions $(s, A_0), (s, A_1), \ldots, (s, A_{n-1})$ with the same first component s, we get a lower bound of these conditions by intersecting all the filter sets $A_0, A_1, \ldots, A_{n-1}$:

$$\text{if } A := \bigcap_{i \in n} A_i, \text{ then } \ \forall i < n \ \ (s, A) \leq (s, A_i);$$

in particular, all conditions within a single part are pairwise compatible.

Of course, $\mathbb{P}$ has the countable chain condition. In fact, any σ-centered forcing is also c.c.c.: if there were an uncountable antichain, there would be a single part of the partition containing still uncountably many elements of this antichain, but each two of them are compatible, a contradiction. $\square$

For each $\alpha < \delta$, define the following set:

$$D_\alpha := \{(s, A) \in \mathbb{P} : \; A \subseteq X_\alpha\}. \tag{1.7}$$

As mentioned above, we claim that each D_α is dense. Fix $(s, A) \in \mathbb{P}$. There is a condition stronger than (s, A) within D_α, namely $(s, A \cap X_\alpha)$: firstly, $(s, A \cap X_\alpha)$ is a condition, since both A and X_α are in the filter $\mathcal{F}$ and so is the intersection; clearly, $(s, A \cap X_\alpha) \leq (s, A)$ and $(s, A \cap X_\alpha) \in D_\alpha$, so D_α is dense. Similarly, the set

$$\widetilde{D}_n := \{(s, A) \in \mathbb{P} : \; \max s > n\} \tag{1.8}$$

is dense for each $n \in \omega$ (using the fact that $\mathcal{F}$ contains the Frechet filter). Now let

$$\mathcal{D} := \left\{D_\alpha : \; \alpha < \delta\right\} \cup \left\{\widetilde{D}_n : \; n \in \omega\right\}$$

be the collection of all these dense sets. Our forcing is c.c.c. and $\mathcal{D}$ is a collection of size less then the continuum (note that $\delta < 2^{\aleph_0}$ and the $\widetilde{D}_n$'s are just countably many). So we can apply MA to get a ($\mathcal{D}$-generic) filter G which intersects all of the D_α's and each $\widetilde{D}_n$. We claim that the respective X (defined in (1.6)) meets our requirements.

First of all, it's clear that X is infinite: given $n \in \omega$, G meets $\widetilde{D}_n$, hence there is an $s \subseteq X$ with $\max s > n$. It remains to prove that for each $\alpha < \delta$, $X \subseteq^\star X_\alpha$. Fix an α; $G \cap D_\alpha \neq \emptyset$, so pick $(s, A) \in G$ with $A \subseteq X_\alpha$. But this implies $X \setminus s \subseteq X_\alpha$, for the following reason: given an $n \in X \setminus s$, there is a condition $(t, B) \in G$ with $n \in t$; clearly we can assume that (t, B) is stronger than (s, A) – just take a common extension of (s, A) and (t, B) within G –, so $t \setminus s \subseteq A$ (see (1.5),(iii)), yielding $n \in t \setminus s \subseteq A \subseteq X_\alpha$. Hence X is indeed an infinite pseudo-intersection of the sequence $\langle X_\alpha : \; \alpha < \delta\rangle$, which finishes the proof of the theorem. $\square$

Corollary 1.21. *If MA holds, then there exists a p-point. Therefore, the existence of p-points is consistent with arbitrarily large continuum.*

Proof. Martin's Axiom implies $\mathfrak{t} = 2^{\aleph_0}$ (Theorem 1.20), which implies the existence of p-points (Theorem 1.17). The second statement follows from Theorem 1.19. $\square$

Ramsey ultrafilters

Let's define the notion of a Ramsey ultrafilter which is stronger than the notion of a p-point (by Lemma 1.11). Nevertheless, Martin's Axiom implies even the existence of a Ramsey ultrafilter.

Definition 1.22. A non-principal ultrafilter $\mathcal{F}$ is called a *Ramsey ultrafilter* if for every partition $\{X_n : n \in \omega\}$ of ω into infinitely many "small parts with respect to $\mathcal{F}$", i.e.,

$$\forall n \in \omega \quad X_n \notin \mathcal{F},$$

there exists a filter set $X \in \mathcal{F}$ such that

$$\text{for each } n \in \omega : \quad |X \cap X_n| \leq 1. \tag{1.9}$$

The term *Ramsey* ultrafilter reflects the fact that an ultrafilter $\mathcal{F}$ on ω is a Ramsey ultrafilter if and only if it fulfills the following "Ramsey" property: for every set $A \subseteq [\omega]^2$, there exists an $X \in \mathcal{F}$ such that $[X]^2 \subseteq A$ or $[X]^2 \cap A = \emptyset$, i.e., X is a "homogeneous set" (see [2, Theorem 4.5.2] for a proof).

We modify our construction of a p-point (assuming MA or CH) a little bit to obtain a Ramsey ultrafilter:

Theorem 1.23. *If the tower number* $\mathfrak{t} = 2^{\aleph_0}$*, then there exists a Ramsey ultrafilter. In particular, MA (or CH) implies the existence of a Ramsey ultrafilter (hence its existence is consistent with arbitrarily large continuum).*

Proof. The second statement will follow from the first: recall Theorem 1.20 which asserts that MA implies $\mathfrak{t} = 2^{\aleph_0}$; additionally, use Theorem 1.19.

Like in the proof of Theorem 1.13 (and Theorem 1.17 respectively), we construct a maximal almost decreasing sequence (i.e., a tower) of length $2^{\aleph_0}$, $\langle X_\alpha : \alpha < 2^{\aleph_0} \rangle$ (X_α infinite), to get our Ramsey ultrafilter, which will be defined as

$$\mathcal{F} := \{ Z \subseteq \omega : Z \supseteq^\star X_\alpha \text{ for some } \alpha < 2^{\aleph_0} \}.$$

To be able to continue the sequence at limit ordinals, we use our assumption $\mathfrak{t} = 2^{\aleph_0}$ (like in the proof of Theorem 1.17). But at successor ordinals, we do not only aim at obtaining an ultrafilter (like in (1.1) on page 8), but we even try to establish the "Ramsey property" right there (the ultrafilter property will follow anyway).

To deal with the "Ramsey property", we have to go through all possible partitions $\{Y_n : n \in \omega\}$ of ω into infinitely many pieces. We claim that there are exactly $2^{\aleph_0}$ of them; on the one hand, it's clear that there are at least $2^{\aleph_0}$

such partitions; on the other hand, the set of all such partitions is contained in the set of all ω-sequences of subsets of ω, $\mathcal{P}(\omega)^\omega$, which has size continuum:

$$|\mathcal{P}(\omega)^\omega| = |\mathcal{P}(\omega)|^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}.$$

So we can fix an enumeration of all these partitions of length $2^{\aleph_0}$, say

$$\{\mathcal{A}_\alpha : \alpha < 2^{\aleph_0}\},$$

where – for every $\alpha < 2^{\aleph_0}$ – $\mathcal{A}_\alpha$ equals some partition $\{Y_n : n \in \omega\}$. At each successor, we now try to establish the "Ramsey property" for the respective partition $\{Y_n : n \in \omega\}$: either we add some Y_n to the filter constructed so far (then this partition need not to be considered any more according to the definition of a Ramsey ultrafilter), or we try to add a set Y to the filter which obeys property (1.9) in the definition.

More precisely, the successor step (at the ordinal $\alpha + 1$) is handled as follows: consider the filter constructed so far (i.e., the set X_α) and $\mathcal{A}_\alpha$, the respective partition given by the enumeration. Now we try to find a set $X_{\alpha+1} \subseteq X_\alpha$, such that the filter generated by $X_{\alpha+1}$ already settles the "Ramsey property" for the partition $\mathcal{A}_\alpha = \{Y_n : n \in \omega\}$. There are two possible cases. *In case that* there is a $k \in \omega$ such that $X_\alpha \cap Y_k$ is infinite, we define

$$X_{\alpha+1} := X_\alpha \cap Y_k;$$

then $X_{\alpha+1} \subseteq X_\alpha$ and $|X_{\alpha+1}| = \aleph_0$; since $Y_k \supseteq X_{\alpha+1}$, the set Y_k will be in the final filter $\mathcal{F}$, therefore there is no "Ramsey property" to establish for this partition. *Otherwise*, $|X_\alpha \cap Y_n| < \aleph_0$ for each $n \in \omega$, hence – since X_α itself is infinite – $X_\alpha \cap Y_n$ will be non-empty for infinitely many $n \in \omega$; pick exactly one element out of each non-empty $X_\alpha \cap Y_n$, yielding an infinite set $X_{\alpha+1} \subseteq X_\alpha$, which will be in the final filter $\mathcal{F}$ and therefore establishes the "Ramsey property" for the partition $\mathcal{A}_\alpha$ (see (1.9)):

$$|X_{\alpha+1} \cap Y_n| \leq 1 \text{ for each } n \in \omega.$$

After carrying out the construction up to $2^{\aleph_0}$, we finally get a non-principal filter $\mathcal{F}$ (see Lemma 1.12), which is a Ramsey ultrafilter due to its construction (provided it is an ultrafilter). But we get the ultrafilter property for free. Assume that some set $A \subseteq \omega$ is given; in case $A \in \mathcal{F}$, we are finished; otherwise, define a partition $\{Y_n : n \in \omega\}$ such that $Y_0 := A$ and all the other Y_n's are finite (e.g. singletons); then $\{Y_n : n \in \omega\}$ is a valid partition with respect to the "Ramsey property" (since $Y_0 = A \notin \mathcal{F}$ and all the other Y_n's are finite and hence not in $\mathcal{F}$), therefore (see (1.9)) there is a set $Y \in \mathcal{F}$ such that $|Y \cap A| \leq 1$; so also $Y \setminus (Y \cap A) \in \mathcal{F}$ (note that $\mathcal{F}$ contains the Frechet filter) which is a subset of $\omega \setminus A$, yielding $\omega \setminus A \in \mathcal{F}$, hence $\mathcal{F}$ is an ultrafilter and the proof of the theorem is finished. $\square$

1.4 Dominating families

Let $\leq^\star$ denote the *eventual domination* on ω^ω, i.e., for $f, g \in \omega^\omega$,

$$f \leq^\star g \iff \exists n \in \omega \;\; \forall k \geq n \;\; f(k) \leq g(k) \tag{1.10}$$

The *dominating number* $\mathfrak{d}$ is the least possible size of a dominating family; a dominating family $A \subseteq \omega^\omega$ is a set of functions which "dominates" every single function in ω^ω with respect to $\leq^\star$:

$$\mathfrak{d} = \min\{|A| : \;\forall f \in \omega^\omega \; \exists g \in A \; f \leq^\star g\} \tag{1.11}$$

Clearly, $|\omega^\omega|$ itself is in the above set, so $\mathfrak{d} \leq 2^{\aleph_0} = |\omega^\omega|$. On the other hand, one can show by a simple diagonal argument that a countable set $A \subseteq \omega^\omega$ cannot be a dominating family: Given $A = \{f_k : \; k \in \omega\}$, define a function f not being dominated by any of the f_k's:

$$f(k) := \max\{f_i(k) : \; i \leq k\} + 1 \;\text{ for each }\; k \in \omega.$$

So we get

$$\aleph_1 \leq \mathfrak{d} \leq 2^{\aleph_0}$$

but the exact value of $\mathfrak{d}$ cannot be determined within ZFC.

$\mathfrak{d} = 2^{\aleph_0}$ implies the existence of p-points

It can be shown that the tower number $\mathfrak{t}$ is less or equal than $\mathfrak{d}$. (Hence MA also implies $\mathfrak{d} = 2^{\aleph_0}$; cf. Theorem 1.20.) The following theorem is due to Ketonen and can be found in [2]; because of $\mathfrak{t} \leq \mathfrak{d}$, it is a stronger result than Theorem 1.17:

Theorem 1.24. *Assume that every dominating family has the size of the continuum:*

$$\mathfrak{d} = 2^{\aleph_0}.$$

Then there exists a p-point.

Proof. We are going to construct the p-point by induction. In each step we consider a countable collection of subsets of ω: if possible, we add these sets to the filter constructed so far; but then we better have to find a pseudo-intersection of these sets which can be put into the filter as well, because we would like to have a p-point in the end.

The following lemma enables us to find such a pseudo-intersection, provided that the filter constructed so far is generated by not too many sets:

Lemma 1.25. *Let $\mathcal{F}$ be a filter (extending the Frechet filter) generated by fewer than $\mathfrak{d}$ many sets, and let $\{X_n : n \in \omega\} \subseteq \mathcal{F}$ be a countable collection of filter sets. Then there exists a set $X \subseteq \omega$ such that $X \subseteq^\star X_n$ for each $n \in \omega$ and $X \cap Y \neq \emptyset$ for every $Y \in \mathcal{F}$.*

Remark. In fact, even $|X \cap Y| = \aleph_0$ for every $Y \in \mathcal{F}$ holds in the lemma: assume there would be a $Y \in \mathcal{F}$ with $|X \cap Y| < \aleph_0$; then $Y \cap (\omega \setminus (X \cap Y))$ would be in $\mathcal{F}$ (since $\mathcal{F}$ contains the Frechet filter), but

$$X \cap Y \cap (\omega \setminus (X \cap Y)) = \emptyset$$

contradicting the lemma.

Proof. The filter $\mathcal{F}$ is generated by fewer than $\mathfrak{d}$ many sets, so there are sets $Y_\alpha \in \mathcal{F}$, $\alpha \in I$ with $|I| < \mathfrak{d}$ such that $\mathcal{F} = \{Y \subseteq \omega : \exists \alpha \in I \;\; Y \supseteq Y_\alpha\}$.

Without loss of generality we can assume that $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$; to achieve this, we just replace each X_n by $\bigcap_{k \leq n} X_k \subseteq X_n$.

For each $\alpha \in I$, define a function $f_\alpha \colon \omega \to \omega$ by

$$f_\alpha(n) := \min(X_n \cap Y_\alpha) \tag{1.12}$$

for $n \in \omega$. (Note that $X_n \cap Y_\alpha \in \mathcal{F}$, so it cannot happen to be empty.) Since $|I| < \mathfrak{d}$, $\{f_\alpha : \alpha \in I\}$ is *no* dominating family, so there is an $f \in \omega^\omega$ such that $\forall \alpha \in I$, $f \not\leq^\star f_\alpha$; fix such an f. It satisfies

$$\forall \alpha \in I \quad (\exists^\infty n \in \omega \quad f(n) > f_\alpha(n)) \tag{1.13}$$

where "$\exists^\infty n \in \omega$" denotes "there are infinitely many $n \in \omega$". (In particular, for each $\alpha \in I$, there is an $n \in \omega$ with $f(n) > f_\alpha(n)$.)

Now define

$$X := \bigcup_{n \in \omega} (X_n \cap f(n))$$

We claim that this X meets our requirements. First of all it's quite clear that $X \subseteq^\star X_n$ (i.e., $X \setminus X_n$ is finite) for each $n \in \omega$, since $X_k \cap f(k) \subseteq X_n$ for each $k \geq n$, hence

$$X \setminus \left(\bigcup_{k \in n} (X_k \cap f(k)) \right) \subseteq \bigcup_{k \geq n} (X_k \cap f(k)) \subseteq X_n$$

where the set $\bigcup_{k \in n} (X_k \cap f(k))$ is finite.

It remains to prove that $X \cap Y \neq \emptyset$ for every $Y \in \mathcal{F}$; it's sufficient to show that $X \cap Y_\alpha \neq \emptyset$ for all $\alpha \in I$ since every $Y \in \mathcal{F}$ contains one of the

Y_α's. Fix an $\alpha \in I$. There is an $n \in \omega$ with $f(n) > f_\alpha(n) = \min(X_n \cap Y_\alpha)$ (see (1.12) and (1.13)). So

$$\min(X_n \cap Y_\alpha) \in f(n) \cap X_n \cap Y_\alpha \subseteq X \cap Y_\alpha,$$

where the last inclusion holds according to the definition of the set X; therefore we have found an element within $X \cap Y_\alpha$ and the proof of the lemma is finished. □

To construct our p-point, we are going to build an ascending chain of filters $\mathcal{F}_\alpha$ ($\alpha \leq 2^{\aleph_0}$), in each step establishing the p-filter property for one countable collection of subsets of ω; since there are $|\mathcal{P}(\omega)^\omega| = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0}$ such collections, we can fix $\left\{\mathcal{A}_\alpha :\ \alpha \in 2^{\aleph_0}\right\}$ enumerating them. Again, let's assume that all these collections are decreasing sequences, i.e., for each $\alpha < 2^{\aleph_0}$ there are sets $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ such that $\mathcal{A}_\alpha = \{X_n :\ n \in \omega\}$.

Start with the Frechet filter, i.e., let $\mathcal{F}_0 = \mathfrak{Fr}$, the set of all co-finite sets. For each $\alpha < 2^{\aleph_0}$, we consider the filter $\mathcal{F}_\alpha$ constructed so far and $\mathcal{A}_\alpha$, a certain decreasing sequence of subsets of ω; if it is possible to put all the sets from $\mathcal{A}_\alpha$ into the filter $\mathcal{F}_\alpha$, we do so, and apply our lemma to this extended filter and $\mathcal{A}_\alpha$: this gives us an appropriate pseudo-intersection, which we put into the filter as well. Otherwise (in case $\mathcal{A}_\alpha$ is incompatible with the filter) we do nothing. At limits, we just take the union of the filters constructed so far.

More precisely, proceed as follows: at a successor ordinal $\alpha + 1$, consider the filter $\mathcal{F}_\alpha$ and the collection $\mathcal{A}_\alpha = \{X_n :\ n \in \omega\}$. Now there are two possibilities: firstly, it could happen that some of the X_n's in $\mathcal{A}_\alpha$ are disjoint from a set in $\mathcal{F}_\alpha$, i.e.,

$$\exists n \in \omega\ \ \exists Y \in \mathcal{F}_\alpha :\ X_n \cap Y = \emptyset.$$

Here we just set $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha$. (Note that the collection $\mathcal{A}_\alpha$ will not be a subset of the final filter $\mathcal{F}_{2^{\aleph_0}}$ in this case, since $\mathcal{F}_\alpha \subseteq \mathcal{F}_{2^{\aleph_0}}$; so there is no p-filter property to establish.)

Otherwise, there is a filter $\mathcal{F}_\alpha{}'$ generated by the filter $\mathcal{F}_\alpha$ and the sets from $\mathcal{A}_\alpha$, i.e., let

$$\mathcal{F}_\alpha{}' = \{Z \subseteq \omega :\ Z \supseteq Y \cap X_n \text{ for some } Y \in \mathcal{F}_\alpha \text{ and } X_n \in \mathcal{A}_\alpha\}$$

(Note that $\mathcal{A}_\alpha$ is already closed under finite intersections since it is a decreasing sequence, and $\mathcal{F}_\alpha$ is closed anyway.) By induction, $\mathcal{F}_\alpha$ (and hence $\mathcal{F}_\alpha{}'$) is generated by fewer than continuum many sets, and – according to the assumption of the theorem – $\mathfrak{d} = 2^{\aleph_0}$. So we can apply our lemma for $\mathcal{F}_\alpha{}'$

and $\mathcal{A}_\alpha \subseteq \mathcal{F}_\alpha{}'$ to get a set $X \subseteq \omega$ such that $X \subseteq^\star X_n$ for all the $X_n \in \mathcal{A}_\alpha$ and $X \cap Y \neq \emptyset$ for each $Y \in \mathcal{F}_\alpha{}'$. Therefore it's possible to further extend the filter by the set X, i.e., let

$$\mathcal{F}_{\alpha+1} = \{Z \subseteq \omega :\ Z \supseteq Y \cap X \text{ for some } Y \in \mathcal{F}_\alpha{}'\}$$

Of course, the filter $\mathcal{F}_{\alpha+1}$ is still generated by fewer than continuum many sets.

At a limit ordinal $\beta \leq 2^{\aleph_0}$, just let

$$\mathcal{F}_\beta = \bigcup_{\alpha<\beta} \mathcal{F}_\alpha.$$

By induction, all the $\mathcal{F}_\alpha$'s involved are generated by at most $|\beta|$ many sets, and so is the union. Hence for $\beta < 2^{\aleph_0}$, $\mathcal{F}_\beta$ is still generated by fewer than continuum many sets. This allows us to carry on the construction.

Now we claim that the resulting filter $\mathcal{F} = \mathcal{F}_{2^{\aleph_0}}$ is a p-point. Clearly, it is a filter containing the Frechet filter, hence non-principal. To prove that $\mathcal{F}$ is a p-filter, fix some decreasing sequence of filter sets, $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$, $X_n \in \mathcal{F}$ for each $n \in \omega$; there is an $\alpha < 2^{\aleph_0}$ such that $\mathcal{A}_\alpha = \{X_n :\ n \in \omega\}$. At step $\alpha + 1$, when $\mathcal{A}_\alpha$ was used in the construction, the "first possibility" described above could not have occurred: a $Y \in \mathcal{F}_\alpha \subseteq \mathcal{F}$ with $X_n \cap Y = \emptyset$ for some n would contradict $X_n \in \mathcal{F}$. So there will be an $X \in \mathcal{F}_{\alpha+1} \subseteq \mathcal{F}$ with $X \subseteq^\star X_n$ for each $n \in \omega$, establishing the p-filter property.

It remains to show that $\mathcal{F}$ is an ultrafilter, but we get this for free: if $X \subseteq \omega$ is any set of natural numbers, there must be an $\alpha < 2^{\aleph_0}$ such that $\mathcal{A}_\alpha = \{X_n :\ n \in \omega\}$ with $X_n = X$ for each $n \in \omega$; either the "first possibility" happened, then there is a $Y \in \mathcal{F}_\alpha \subseteq \mathcal{F}$ with $Y \cap X = \emptyset$ (and $Y \subseteq \omega \setminus X \in \mathcal{F}$), or we have put the set X itself into the filter $\mathcal{F}_\alpha{}' \subseteq \mathcal{F}_{\alpha+1} \subseteq \mathcal{F}$. So $\mathcal{F}$ is a p-point, which finishes the proof of the theorem. □

Chapter 2

Stone-Čech-compactification $\beta\omega$

In this chapter we examine the concept of a p-point (and ultrafilters on ω in general) from the topological point of view. This will also explain why "p-ultrafilters" are commonly called p-*points*. We shall show that the *Stone-Čech-compactification of* ω can be constructed as the set of all ultrafilters on ω, together with a natural topology known as Stone topology; in this context we will be able to give a topological characterization of p-points.

Everything in this chapter is part of the classical theory of $\beta\omega$ (cf. [5]). For a general introduction to topology, see e.g. [15].

2.1 Basic definitions

Let's recall the definition of a topological space:

Definition 2.1 (Topology)**.** Let X be a non-empty set. A set $\mathcal{O} \subseteq \mathcal{P}(X)$ is called a *topology* on X if the following holds:

1. $\emptyset \in \mathcal{O}$, $X \in \mathcal{O}$

2. $\mathcal{O}$ is closed under finite intersections:

$$O_0, \dots, O_{n-1} \in \mathcal{O} \implies \bigcap_{i<n} O_i \in \mathcal{O}.$$

3. $\mathcal{O}$ is closed under arbitrary unions:

$$(\forall i \in I \quad O_i \in \mathcal{O}) \implies \bigcup_{i \in I} O_i \in \mathcal{O}.$$

A *topological space* is a pair $(X, \mathcal{O})$, where $\mathcal{O}$ is a topology on X. The elements of $\mathcal{O}$ are called *open* sets, i.e., $\mathcal{O} = \{O \subseteq X : O \text{ open}\}$; a set is called *closed* if its complement is open.

In case $\mathcal{O} = \mathcal{P}(X)$, we call $\mathcal{O}$ the *discrete topology* on X, i.e., a space $(X, \mathcal{O})$ is discrete if all subsets of X are open.

Of course, the collection of closed sets satisfies the properties dual to (2) and (3) above: they are closed under finite unions and arbitrary intersections. Alternatively, one can define a topological space by specifying a "collection of closed sets", having properties dual to the three properties of a topology.

Recall that $\mathcal{B} \subseteq \mathcal{O}$ is a *base* for the topology $\mathcal{O}$ on X if every open set $O \in \mathcal{O}$ can be written as a union of sets in $\mathcal{B}$; in other words, for each $x \in O \in \mathcal{O}$ there is a $B \in \mathcal{B}$ such that $x \in B \subseteq O$. Each base $\mathcal{B}$ satisfies the following:

1. $\bigcup \mathcal{B} = X$
2. if $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ is the union of sets in $\mathcal{B}$.

Conversely, if a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfies these two properties, there is a unique topology $\mathcal{O}$ on X such that $\mathcal{B}$ is a base of $\mathcal{O}$ (in fact, $\mathcal{O}$ is the collection of unions of sets from $\mathcal{B}$); we say that the topology $\mathcal{O}$ is generated by $\mathcal{B}$. In particular, the two above properties are satisfied whenever $X \in \mathcal{B}$ and $\mathcal{B}$ is closed under finite intersections. This provides a convenient way to define a topology on X.

2.2 The topological space $\beta\omega$

We are now going to define the topological space $\beta\omega$. Later on we will justify our notation by showing that the space $\beta\omega$ is indeed the so-called Stone-Čech-compactification of ω (where ω is equipped with the discrete topology). Let

$$\beta\omega := \{p \subseteq \mathcal{P}(\omega) : p \text{ is an ultrafilter on } \omega\}$$

the set of all ultrafilters on ω (the principal ones included). Now define for each set $A \subseteq \omega$ a corresponding "base set" $\overline{A}$ by

$$\overline{A} := \{p \in \beta\omega : A \in p\},$$

in other words, $\overline{A} \subseteq \beta\omega$ is the collection of exactly those ultrafilters on ω which contain the set A:

$$p \in \overline{A} \iff A \in p. \tag{2.1}$$

Remark. Note that on the one hand each $p \in \beta\omega$ is an ultrafilter on ω and hence a collection of sets of natural numbers, but on the other hand it acts as a single point in the topological space $\beta\omega$. That is why ultrafilters are denoted by lowercase letters in the context of $\beta\omega$ and why ultrafilters with the p-filter property are called p-*points*.

Now let
$$\mathcal{B} := \big\{\overline{A} : \ A \subseteq \omega\big\},$$
which we claim to be a base for a topology on $\beta\omega$, known as the *Stone topology*. In fact, we even claim that $\mathcal{B}$ is closed under finite intersections (and $\beta\omega \in \mathcal{B}$). As mentioned above, this is sufficient for $\mathcal{B}$ being a base. We use the following lemma:

Lemma 2.2. *For all sets $A, B \subseteq \omega$, the following holds:*

1. $\overline{A} \cap \overline{B} = \overline{A \cap B}$
2. $\beta\omega \setminus \overline{A} = \overline{(\omega \setminus A)}$
3. $\overline{\omega} = \beta\omega$

Proof. (1) We have to show that for each ultrafilter $p \in \beta\omega$
$$\big(p \in \overline{A} \ \wedge \ p \in \overline{B}\big) \iff p \in \overline{A \cap B}.$$
By definition (see (2.1)), this is equivalent to
$$(A \in p \ \wedge \ B \in p) \iff A \cap B \in p,$$
which is clearly true, since p is a filter (the ultrafilter property isn't used in this case).

(2) Here we have to show that for each ultrafilter $p \in \beta\omega$
$$p \notin \overline{A} \iff p \in \overline{(\omega \setminus A)}.$$
By definition (see (2.1)), this is equivalent to
$$A \notin p \iff (\omega \setminus A) \in p,$$
which is true, since p is an ultrafilter (for the direction from left to right, the ultrafilter property is needed).

(3) For an ultrafilter $p \in \beta\omega$, $p \in \overline{\omega}$ is equivalent to $\omega \in p$ (by (2.1)), which is true for every filter on ω. □

So our set $\mathcal{B} = \{\overline{A} : A \subseteq \omega\}$ is closed under finite intersections (see Lemma 2.2 (1)) and contains the whole space $\beta\omega$ (see Lemma 2.2 (3)). Therefore $\mathcal{B}$ is the base of a topology on $\beta\omega$, as explained above:

Definition 2.3 ($\beta\omega$)**.** The topological space $\beta\omega$ is defined to be the set of all ultrafilters on ω equipped with the Stone topology, which is generated by its base $\mathcal{B} = \{\overline{A} : A \subseteq \omega\}$, where $\overline{A}$ is the set of all ultrafilters containing A.

From now on, when we write $\beta\omega$, we always mean the topological space, i.e., the set $\beta\omega$ together with this topology.

A set is called *clopen* if it is both closed and open:

Lemma 2.4. *For every $A \subseteq \omega$, the set $\overline{A} \subseteq \beta\omega$ is closed (and open), i.e., the collection $\mathcal{B} = \{\overline{A} : A \subseteq \omega\}$ is a* clopen base *of $\beta\omega$.*

In particular, $\mathcal{B}$ is also a "base for the closed sets": each closed subset of $\beta\omega$ can be written as an intersection of sets from $\mathcal{B}$.

Proof. Let $A \subseteq \omega$. By definition, $\overline{A}$ is open. But $\overline{A}$ is also closed since its complement is open (use Lemma 2.2 (2)):

$$\beta\omega \setminus \overline{A} = \overline{(\omega \setminus A)} \in \mathcal{B}.$$

Because every open set can be written as a union of sets from $\mathcal{B}$ and $\mathcal{B}$ is closed under complements, each closed set is an intersection of sets of $\mathcal{B}$. $\square$

2.3 Compact spaces

Before we further investigate $\beta\omega$, let's recall some topological notions:

Definition 2.5 (neighborhood)**.** Let $(X, \mathcal{O})$ be a topological space. For a given point $x \in X$, a set $U \subseteq X$ is called a *neighborhood of x* if there exists an open set $O \in \mathcal{O}$ such that $x \in O \subseteq U$. Let $\mathcal{U}(x)$ denote the set of all neighborhoods of x.

A collection $\mathcal{B}(x) \subseteq \mathcal{U}(x)$ is called a *neighborhood base of x* if for each $U \in \mathcal{U}(x)$ there is a $B \in \mathcal{B}(x)$ such that $B \subseteq U$.

Note that $\mathcal{U}(x)$ is closed under supersets and finite intersections, and for each $U \in \mathcal{U}(x)$, $x \in U$ (hence $U \neq \emptyset$), so $\mathcal{U}(x)$ is a filter; a neighborhood base $\mathcal{B}(x)$ is a filter base of the filter $\mathcal{U}(x)$. According to the definition, it's always possible to find a neighborhood base $\mathcal{B}(x)$ consisting of open sets, or even sets from $\mathcal{B}$ for any given base $\mathcal{B}$ of the topology $\mathcal{O}$: just take all elements of $\mathcal{B}$ which contain the point x.

Definition 2.6 (Hausdorff)**.** A topological space $(X, \mathcal{O})$ is called a *Hausdorff* space if distinct points can be separated by disjoint neighborhoods:

$$\forall x, y \in X,\ x \neq y \quad \exists U \in \mathcal{U}(x), \exists V \in \mathcal{U}(y): \quad U \cap V = \emptyset$$

Again, the neighborhoods in the definition above can be replaced by disjoint open sets (or sets from a base $\mathcal{B}$) containing x and y respectively.

Definition 2.7 (compactness)**.** A topological space $(X, \mathcal{O})$ is *compact* if every open cover of X has a finite subcover, i.e.,

$$\forall i \in I : O_i \in \mathcal{O}, \bigcup_{i \in I} O_i = X \implies \exists E \subseteq I : |E| < \aleph_0, \bigcup_{i \in E} O_i = X.$$

We are going to use the dual version of the definition later on:

Lemma 2.8. *A topological space $(X, \mathcal{O})$ is compact if and only if for every collection $\{A_i :\ i \in I\}$ of closed subsets of X the following holds:*

$$(\textit{for each finite } E \subseteq I : \bigcap_{i \in E} A_i \neq \emptyset) \implies \bigcap_{i \in I} A_i \neq \emptyset; \tag{2.2}$$

in other words, every collection of closed sets with the finite intersection property has a non-empty intersection.

Moreover, it is sufficient to consider "basic closed sets", i.e., only collections $\{A_i :\ i \in I\}$ where the A_i's are taken from a given "base for the closed sets".

Proof. Dualize the definition of compactness to get the first statement.

Now assume, that (2.2) holds for each collection of basic closed sets, and let $\{A_i :\ i \in I\}$ be a collection of (arbitrary) closed sets with the finite intersection property; we shall show that $\bigcap_{i \in I} A_i \neq \emptyset$. Each A_i can be written as the intersection of basic closed sets:

$$\forall i \in I :\ A_i = \bigcap_{j \in J_i} B_{ij};$$

the A_i's have the finite intersection property, and so the same is true for the collection of all these basic closed sets $\{B_{ij} :\ i \in I, j \in J_i\}$. By assumption, (2.2) can be applied to the latter collection, so

$$\emptyset \neq \bigcap_{i \in I, j \in J_i} B_{ij} = \bigcap_{i \in I} A_i,$$

which finishes the proof. □

2.4 $\beta\omega$ is a compact Hausdorff space

Let's return to our space $\beta\omega$ now. We claim that $\beta\omega$ is compact and has the Hausdorff property. For the latter, let us first investigate the neighborhoods of a "point" $p \in \beta\omega$: $\mathcal{B} = \{\overline{A} : A \subseteq \omega\}$ is a base of the topology of $\beta\omega$, therefore – see the discussion following Definition 2.5 – the set

$$\mathcal{B}(p) := \{\overline{A} : p \in \overline{A}\} = \{\overline{A} : A \in p\}$$

is a (clopen) neighborhood base of the point p. As mentioned, it's sufficient to consider such "basic neighborhoods"; so we can think of neighborhoods of p as follows: take some set $A \in p$, and the respective (clopen) neighborhood will be the set of ultrafilters containing A, i.e., the set of ultrafilters which "share" the set A with the ultrafilter p.

Furthermore, there is a natural embedding of the discrete topological space ω into the compact space $\beta\omega$. (When we talk about ω in a topological context within this chapter, we always mean ω equipped with the discrete topology.) There are two main types of ultrafilters on ω: on the one hand, the ultrafilters extending the Frechet filter (i.e., the non-principal ones), and on the other hand the principal ultrafilters; in fact, for each natural number $n \in \omega$, there is exactly one principal ultrafilter containing the singleton $\{n\}$. This gives rise to the natural embedding

$$\beta : \omega \to \beta\omega, \quad n \mapsto \beta(n) := \{X \subseteq \omega : n \in X\}. \tag{2.3}$$

Then the image $\beta[\omega] \subseteq \beta\omega$ of ω under the embedding β is exactly the set of all principal ultrafilters on ω. We will show that this image is dense in $\beta\omega$.

Now we are prepared for the following

Theorem 2.9. *The topological space $\beta\omega$ is a compact Hausdorff space containing a homeomorphic copy of ω (equipped with the discrete topology) as a dense subset.*

Proof. To show that $\beta\omega$ is a Hausdorff space, assume $p \neq q$ are two distinct points in $\beta\omega$; we will show that they can be separated by disjoint (basic) neighborhoods. Since $p \neq q$, w.l.o.g there is a set $A \subseteq \omega$ contained in the ultrafilter p but not in q:

$$A \in p \ \wedge \ A \notin q.$$

Equivalently, $p \in \overline{A}$ and $q \in \beta\omega \setminus \overline{A}$, so p and q are separated by the disjoint (clopen) neighborhoods $\overline{A}$ and $\beta\omega \setminus \overline{A}$ (the set $\beta\omega \setminus \overline{A}$ equals $\overline{(\omega \setminus A)}$ by Lemma 2.2 (2)). Note that the "w.l.o.g" above is not really necessary: in case A is in q but not in p, A can be replaced by its complement $\omega \setminus A$, which will be in p but not in q.

Remark. A topological space $(X, \mathcal{O})$ is called *connected* if it cannot be partitioned into two non-empty open sets, i.e., if $\emptyset$ and X are the only clopen sets. It is called *totally disconnected* if there is no subset of X with more than one element which is connected (with respect to the induced topology). Our proof of the Hausdorff property of $\beta\omega$ actually showed that each two distinct points can be separated by a clopen set (in which case the space is called *totally separated*, a property stronger than Hausdorff). Of course, this shows that $\beta\omega$ is totally disconnected.

Now we are going to show that $\beta\omega$ is compact, using the characterization of compactness in terms of "basic closed sets" (see Lemma 2.8). According to Lemma 2.4, $\mathcal{B} = \{\overline{A} : A \subseteq \omega\}$ is a base for the closed sets in $\beta\omega$. So let's assume that $\{\overline{A_i} : i \in I\}$ (with $A_i \subseteq \omega$ for each $i \in I$) is the given collection of basic closed sets with the finite intersection property:

$$\text{for each finite } E \subseteq I : \bigcap_{i \in E} \overline{A_i} \neq \emptyset;$$

we shall show that $\bigcap_{i \in I} \overline{A_i} \neq \emptyset$. Luckily, the mapping $A \mapsto \overline{A}$ does not commute with arbitrary intersections in general, otherwise we would run into problems, since $\bigcap_{i \in I} A_i$ can easily be empty and $\overline{\emptyset} = \emptyset$. But we can use Lemma 2.2 (1) for finite intersections: for each finite $E \subseteq I$

$$\emptyset \neq \bigcap_{i \in E} \overline{A_i} = \overline{\bigcap_{i \in E} A_i}\ , \text{ hence } \bigcap_{i \in E} A_i \neq \emptyset$$

(note that $A \neq \emptyset$ if and only if $\overline{A} \neq \emptyset$). Therefore the collection $\{A_i : i \in I\}$ has the finite intersection property, hence it generates a filter $\mathcal{F}$ on ω:

$$\mathcal{F} := \left\{ X \subseteq \omega : X \supseteq \bigcap_{i \in E} A_i \text{ for some finite } E \subseteq I \right\}.$$

By AC (e.g. Zorn's Lemma), it can be extended to an ultrafilter $p \supseteq \mathcal{F}$ on ω. Since $\mathcal{F}$ contains all the A_i's, also $A_i \in p$ for each $i \in I$; equivalently, $p \in \overline{A_i}$ for each $i \in I$, thereby establishing $p \in \bigcap_{i \in I} \overline{A_i} \neq \emptyset$, which finishes the argument concerning compactness.

Now we would like to show that $\beta\omega$ contains a (dense) homeomorphic copy of the discrete space ω. In other words, we have to show the following: the natural embedding $\beta : \omega \to \beta\omega$ defined by (2.3), which sends each $n \in \omega$ to the (unique) principal ultrafilter containing $\{n\}$, is a homeomorphism between ω and its image, i.e., the mapping β is injective and continuous in both directions. (A function from one topological space into another is *continuous* iff the preimage of every open set is open.)

Obviously, β is injective ($\{n\} \cap \{m\} = \emptyset$ for all $n \neq m$). The function $\beta : \omega \to \beta\omega$ is clearly continuous (in the forward direction), since the domain ω carries the discrete topology, which is sufficient: the preimage (under β) of any subset of $\beta\omega$ is in $\mathcal{P}(\omega)$, hence open. To prove the continuity of the inverse function it's enough to show that the function β itself is *open*, i.e., the image of any open set is open (since then the preimage of any open set under the inverse function will be open in $\beta\omega$, hence also open in $\beta[\omega]$ with the induced topology). Since ω carries the discrete topology, we have to show that the image of every singleton in ω is open in $\beta\omega$, i.e., $\{\beta(n)\}$ is open in $\beta\omega$ for each $n \in \omega$. But

$$\{\beta(n)\} = \{p \in \beta\omega : \ \{n\} \in p\} = \overline{\{n\}} \in \mathcal{B},$$

so $\{\beta(n)\}$ is (basic) open, and $\beta[\omega]$ is a homeomorphic copy of the discrete space ω.

Finally, let's explain why ω is *densely* embedded by β, i.e., why $\beta[\omega]$ is a dense subset of $\beta\omega$. (A subset D of a topological space is called *dense* if its closure is the whole space, i.e., if for each point p, any neighborhood of p contains some element from D. Equivalently, D is dense if every non-empty open set contains some element from D. Again, it's obviously sufficient to consider basic open sets.) Let $\overline{A} \in \mathcal{B}$ be some non-empty basic open set; we shall find some element of $\beta[\omega]$ in $\overline{A}$. Clearly $A \neq \emptyset$ (no filter contains the empty set), so we can pick some $n \in A$; but then the principal ultrafilter $\beta(n)$ is in $\overline{A}$: by definition, $\{n\} \in \beta(n)$, so $\{n\} \subseteq A \in \beta(n)$, yielding $\beta(n) \in \overline{A}$. In fact, the following holds:

$$\overline{A} \cap \beta[\omega] = \{\beta(n) : \ n \in A\}. \qquad \square$$

Remark. Note that $\beta\omega$ is not just Hausdorff but also satisfies a stronger separation axiom: in fact, $\beta\omega$ is normal. (A space is called *normal* if it is Hausdorff and, given any two disjoint closed sets F_1 and F_2, it is possible to separate them by neighborhoods, i.e., there are open sets O_1 and O_2 such that $F_i \subseteq O_i$ ($i = 1, 2$) and $O_1 \cap O_2 = \emptyset$. It should be mentioned that there is no agreement on the definition of normality: sometimes a normal space is not required to be Hausdorff; our space $\beta\omega$ is Hausdorff anyway, so it doesn't make any difference in this context.)

It can be shown that each compact Hausdorff space is normal, hence $\beta\omega$ is normal. To gain an insight into the correspondence of (arbitrary) closed sets in $\beta\omega$ and filters on ω, we nevertheless give a direct proof of the normality. Recall that our clopen base $\mathcal{B} = \left\{\overline{A} : \ A \subseteq \omega\right\}$ is a base for the closed sets.

So each closed set $F \subseteq \beta\omega$ can be written as the intersection of sets from $\mathcal{B}$: there is a family $\left\{\overline{A_i} : i \in I\right\} \subseteq \mathcal{B}$ with

$$F = \bigcap_{i \in I} \overline{A_i}.$$

In case F is non-empty, the corresponding family $\{A_i : i \in I\} \subseteq \mathcal{P}(\omega)$ has the finite intersection property, so it generates a filter $\mathcal{F} \supseteq \{A_i : i \in I\}$ on ω. An ultrafilter p lies in F if and only if p contains all the A_i's if and only if p extends the filter $\mathcal{F}$, i.e.,

$$F = \{p \in \beta\omega : p \supseteq \mathcal{F}\}. \tag{2.4}$$

Conversely, if $\mathcal{F}$ is a filter on ω, the set F of all ultrafilters extending $\mathcal{F}$ is a non-empty closed set. In fact, there is a one-to-one correspondence of (proper) filters on ω and (non-empty) closed subsets of $\beta\omega$. Whenever $\mathcal{F}_1 \subseteq \mathcal{F}_2$ are two filters on ω, the corresponding closed sets F_1 and F_2 will satisfy $F_1 \supseteq F_2$ and vice versa, since for an ultrafilter p, it's "more restrictive" to extend a bigger filter. (Using this, one can explicitly define the topological closure of an arbitrary set $X \subseteq \beta\omega$: take the intersection of all ultrafilters in X, which is the biggest filter $\mathcal{F}$ on ω contained in all these ultrafilters; the corresponding closed set F will be the closure of X, since it's the smallest closed set containing X.) In particular, a filter generated by just one set $A \subseteq \omega$ corresponds to the basic clopen set $\overline{A} \in \mathcal{B}$; when the filter grows, the corresponding closed set shrinks, and when the filter finally grows into an ultrafilter p, there is only one ultrafilter left "extending p", namely p itself, hence the corresponding closed set is the singleton $\{p\}$; if p is "further extended", it's no (proper) filter anymore but the whole powerset of ω while the corresponding closed set becomes empty.

Keeping this correspondence in mind, it's very easy to see that $\beta\omega$ is normal. Let F_1 and F_2 be two disjoint closed subsets of $\beta\omega$ and consider the corresponding filters $\mathcal{F}_1$ and $\mathcal{F}_2$; since there is no ultrafilter $p \in F_1 \cap F_2$, no ultrafilter can extend both $\mathcal{F}_1$ and $\mathcal{F}_2$, so there must be a reason for that: $\mathcal{F}_1 \cup \mathcal{F}_2$ does not have the finite intersection property, hence there are sets $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$ such that $A_1 \cap A_2 = \emptyset$; but now $\overline{A_1}$ and $\overline{A_2}$ act as disjoint (cl)open neighborhoods of F_1 and F_2 respectively:

$$\overline{A_1} \cap \overline{A_2} = \overline{A_1 \cap A_2} = \emptyset$$

and $\overline{A_1} \supseteq F_1$ and $\overline{A_2} \supseteq F_2$.

(In a similar fashion, one can also go through the proof of compactness with arbitrary closed subsets of $\beta\omega$, i.e., filters on ω; doing so, one can avoid the reduction to the case in which all the sets are basic closed sets.)

In the theorem above, we have found $\beta\omega$ to be a so-called *compactification* of the discrete topological space ω, i.e., a compact (Hausdorff) space with ω lying dense in it. (Due to the homeomorphic embedding β we can identify the natural numbers with their respective principal ultrafilters, so we can view ω as a subspace of $\beta\omega$, writing n instead of $\beta(n)$, ω instead of $\beta[\omega]\ldots$) Moreover, the space $\beta\omega$ is actually the so-called *Stone-Čech-compactification* of ω, i.e., $\beta\omega$ is – among all compactifications with the Hausdorff property – the "most general" one. We will investigate this "universal property" at the end of the chapter. Before we do that, we would like to find out what is special about p-points in $\beta\omega$.

2.5 Further properties of $\beta\omega$

Isolated points and non-principal ultrafilters

Let's try to further explore the structure of $\beta\omega$. There are two very different types of points in $\beta\omega$: the "natural numbers" itself (the principal ultrafilters) and very many "new points" (the non-principal ultrafilters) which were added to ω to make it compact, so to speak. By the theorem of Pospíšil (see Theorem 1.7 on page 6), there are exactly $2^{(2^{\aleph_0})}$ many (non-principal) ultrafilters on ω, which tells us the size of $\beta\omega$:

$$|\beta\omega| = \aleph_0 + 2^{(2^{\aleph_0})} = 2^{(2^{\aleph_0})}.$$

As seen in the proof above, each natural number n is an *isolated point* of $\beta\omega$: there is a (basic clopen) neighborhood of n just containing n itself, namely $\overline{\{n\}}$. In general, if A is a finite set, $\overline{A}$ contains all natural numbers in A and nothing else. But if A is infinite, we have the following situation: the points $p \in \overline{A}$ (i.e., the ultrafilters *containing* A) can essentially be viewed as the ultrafilters *on* the infinite set A, so $\overline{A}$ is homeomorphic to the whole space $\beta\omega$ and has cardinality $2^{(2^{\aleph_0})}$.

In contrast to the fact that all points from ω are isolated in $\beta\omega$, each (basic) neighborhood $\overline{A}$ of a point $p \in \beta\omega \setminus \omega$ contains very many points: since $p \in \overline{A}$ (i.e., $A \in p$) and p is non-principal, A is infinite, so there are infinitely many natural numbers in $\overline{A}$ (namely those which are in A) and $2^{(2^{\aleph_0})}$ points from $\beta\omega\setminus\omega$, among them p itself. To get rid of the isolated points, one often ignores all principal ultrafilters and just studies the subspace $\beta\omega \setminus \omega$, which is compact as well. (In general, a closed subset of a compact space is compact; since ω is open in $\beta\omega$ as the union of (cl)open singletons, its complement $\beta\omega \setminus \omega$ is closed.)

$\beta\omega$ is not metrizable

So how many (basic) neighborhoods do we need to uniquely determine an ultrafilter p in $\beta\omega \setminus \omega$? In other words: what is the least possible size of a family $\{\overline{A_i} : i \in I\}$ of neighborhoods of $p \in \beta\omega \setminus \omega$ such that

$$\bigcap_{i \in I} \overline{A_i} = \{p\}. \tag{2.5}$$

(Obviously finitely many are not enough, for they could be replaced by a single one, still containing $2^{(2^{\aleph_0})}$ points.)

We claim that it is essentially the same as asking for a neighborhood base of least possible size (see Definition 2.5). This is related to the question if the topological space $\beta\omega$ is metrizable or not:

Definition 2.10. A topological space $(X, \mathcal{O})$ is called *metrizable* if there is a metric $d : X \times X \to [0, \infty)$ on X such that the topology induced by the metric d equals the given topology $\mathcal{O}$.

A topological space is called *first-countable* if each point has a countable neighborhood base.

Lemma 2.11. *Every metric space (X, d) is first-countable. (So if a topological space is not first-countable, it is not metrizable.)*

Proof. Let $x \in X$. Then $\{B(x, \frac{1}{n}) : n \geq 1\}$ clearly is a countable (open) neighborhood base of the point x, where $B(x, r) = \{y \in X : d(x, y) < r\}$ denotes the open ball centered at x with radius $r > 0$. $\square$

We will show that no point in $\beta\omega \setminus \omega$ has a countable neighborhood base, implying that $\beta\omega$ is not metrizable.

So far, we only used $\mathcal{B}(p) = \{\overline{A} : p \in \overline{A}\}$ as a neighborhood base of p; the set $\mathcal{B}(p)$ is of size continuum. Since each neighborhood of p contains a neighborhood from the set $\mathcal{B}(p)$, we can concentrate on subsystems of $\mathcal{B}(p)$ when looking for possibly smaller bases. Whenever such a set

$$\{\overline{A_i} : i \in I\} \subseteq \mathcal{B}(p)$$

forms a neighborhood base for the point p, (2.5) will hold, i.e., only p itself will be in the intersection of the $\overline{A_i}$'s. (This is true for every Hausdorff space: given $y \neq x$, there is a neighborhood of x not containing y; so if a neighborhood base of x is given, the intersection of all elements from this base will be the singleton $\{x\}$; in fact, this is equivalent to T_1, a separation axiom strictly weaker than the Hausdorff property.)

Conversely, let $\{\overline{A_i} : i \in I\}$ be an infinite family of neighborhoods of $p \in \beta\omega\backslash\omega$ such that $\bigcap_{i\in I} \overline{A_i} = \{p\}$. Without loss of generality, we can assume that this family is closed under finite intersections: $\mathcal{B}(p)$ is closed under finite intersections, and replacing the above family by the set of all finite intersections of elements from the family does not increase its cardinality; after all, we are only interested in the size of such families. Let $\mathcal{F}$ be the filter generated by the A_i's (see also the remark following Theorem 2.9), i.e.,

$$\mathcal{F} := \{X \subseteq \omega : X \supseteq A_i \text{ for some } i \in I\}$$

(the A_i's form a filter base). Since $\bigcap_{i\in I} \overline{A_i} = \{p\}$, p is the only ultrafilter extending $\mathcal{F}$ (cf. (2.4)). But $\mathcal{F}$ itself is already an ultrafilter: if there would be a set $X \subseteq \omega$ such that neither X nor $\omega\backslash X$ belongs to $\mathcal{F}$, both $\mathcal{F}\cup\{\omega \setminus X\}$ and $\mathcal{F} \cup \{X\}$ could be extended to (different) ultrafilters, contradicting the assumption. So $\mathcal{F}$ actually equals p, hence $\{A_i : i \in I\}$ is a filter base for p, which is equivalent to $\{\overline{A_i} : i \in I\}$ being a neighborhood base for p. (Use the fact that $A \subseteq B$ is equivalent to $\overline{A} \subseteq \overline{B}$, see Lemma 2.12 (1).)

To sum up things: for any family $\{\overline{A_i} : i \in I\}$ of neighborhoods of a non-principal ultrafilter p, uniquely determining p is essentially the same as being a neighborhood base for p (apart from adding finite intersections), which is in turn the case if and only if the corresponding family $\{A_i : i \in I\}$ is a filter base for the ultrafilter p.

It can be shown that no non-principal ultrafilter on ω is generated by just countably many sets (in other words, it has no countable filter base), so no point in $p \in \beta\omega \setminus \omega$ has a countable neighborhood base. Therefore $\beta\omega$ is not metrizable.

Countable intersections of neighborhoods have non-empty interior

In fact, the following holds. Let $\{\overline{A_i} : i < \omega\}$ be a countable collection of neighborhoods of $p \in \beta\omega \setminus \omega$. Then this family will not uniquely determine p, i.e.,

$$\bigcap_{i<\omega} \overline{A_i} \supsetneqq \{p\};$$

but even more is true: provided that one simply ignores the principal ultrafilters and works within the compact space $\beta\omega\setminus\omega$, we claim that the (closed) set $\bigcap_{i<\omega} \overline{A_i}$ has a non-empty interior, i.e., there is a non-empty (basic) open set $\overline{A}$ such that

$$\bigcap_{i<\omega} \overline{A_i} \supseteq \overline{A}. \tag{2.6}$$

Note that A has to be infinite in this case, since each finite A yields an $\overline{A}$ only containing natural numbers (hence appearing empty within $\beta\omega\setminus\omega$). So our claim implies that the intersection of countably many neighborhoods of a non-principal ultrafilter still has $2^{(2^{\aleph_0})}$ elements.

Also note that this becomes untrue if viewed within the whole space $\beta\omega$; for instance, take $A_i = \omega \setminus i$ for each $i < \omega$ as a counterexample (i.e., the A_i's generate the Frechet filter); then $\bigcap_{i<\omega} \overline{A_i}$ contains each $p \in \beta\omega \setminus \omega$, but no natural number; since ω is dense in $\beta\omega$, the intersection $\bigcap_{i<\omega} \overline{A_i}$ has empty interior within $\beta\omega$; otherwise $\bigcap_{i<\omega} \overline{A_i}$ would contain some natural number, a contradiction.

Pseudo-intersections revisited: $A \subseteq^\star B \iff \overline{A} \subseteq \overline{B}$ **in** $\beta\omega \setminus \omega$

How can we find a set $A \subseteq \omega$ such that (2.6) holds within $\beta\omega \setminus \omega$? The following lemma will relate (2.6) to the notion of pseudo-intersection; recall that $A \subseteq^\star B$ denotes "A is almost contained in B", i.e.,

$$A \subseteq^\star B \iff |A \setminus B| < \aleph_0.$$

Lemma 2.12. *For all sets $A, B \subseteq \omega$, the following holds:*

1. $A \subseteq B \iff \overline{A} \subseteq \overline{B}$
2. $A \subseteq^\star B \iff \overline{A} \cap (\beta\omega \setminus \omega) \subseteq \overline{B}$

Proof. (1) We have to show that

$$A \subseteq B \iff \forall p \in \beta\omega \ \ (A \in p \implies B \in p).$$

Clearly, if $A \subseteq B$ and $A \in p$ for some ultrafilter p, also $B \in p$.

Conversely, if $A \not\subseteq B$, $A \setminus B \neq \emptyset$, so there is an ultrafilter $p \ni (A \setminus B)$; then $(A \setminus B) \subseteq A \in p$, but $B \notin p$ since $B \cap (A \setminus B) = \emptyset$. Note that in case $A \setminus B$ happens to be finite, the witnessing ultrafilter p is necessarily principal.

(2) Here we have to show that

$$A \subseteq^\star B \iff \forall p \in \beta\omega \setminus \omega \ \ (A \in p \implies B \in p).$$

First of all, note that if $p \in \beta\omega \setminus \omega$, p contains the Frechet filter, i.e., all cofinite sets, hence $A \in p$ implies $A \cap (\omega \setminus n) \in p$ for each $n < \omega$. So if $A \subseteq^\star B$, pick some n such that $A \cap (\omega \setminus n) \subseteq B$; whenever $A \in p$ for some $p \in \beta\omega \setminus \omega$, also $A \cap (\omega \setminus n) \in p$, so its superset B will be in p too.

Conversely, if $A \not\subseteq^\star B$, by definition $A \setminus B$ is infinite, so we are able to find a *non-principal* ultrafilter $p \in \beta\omega \setminus \omega$ containing $A \setminus B$; like in (1), it follows that $A \in p$, but $B \notin p$. □

Remark. We decided to give an explicit proof of the lemma above. Alternatively, we could have derived the result purely algebraically from Lemma 2.2 (1) and (2):

$$\overline{A} \setminus \overline{B} = \overline{A} \cap (\beta\omega \setminus \overline{B}) \overset{(2)}{=} \overline{A} \cap \overline{(\omega \setminus B)} \overset{(1)}{=} \overline{A \cap (\omega \setminus B)} = \overline{A \setminus B}. \tag{2.7}$$

To get Lemma 2.12, e.g. (2), recall that $\overline{A}$ contains *non-principal* ultrafilters if and only if A is infinite. So $\overline{A} \setminus \overline{B}$ contains a point from $\beta\omega \setminus \omega$ if and only if $|A \setminus B| = \aleph_0$, i.e., $A \not\subseteq^\star B$, which is the same as claim (2) in the lemma.

In general, the mapping $A \mapsto \overline{A}$ is a Boolean homomorphism between $\mathcal{P}(\omega)$ and the clopen base $\mathcal{B}$ in $\beta\omega$, i.e., it commutes with all Boolean set operations like complement, intersection (as proved in Lemma 2.2), union, set difference, symmetric difference etc., as shown for set difference in (2.7). The case of symmetric difference

$$\overline{A} \triangle \overline{B} = \overline{A \triangle B} \tag{2.8}$$

again tells us the following. Provided an ultrafilter p is non-principal, it does not depend on finite changes of the set A if p lies in $\overline{A}$ or not: if B equals A apart from finite changes, i.e., $A \triangle B$ is finite, according to (2.8), $\overline{A} \triangle \overline{B}$ will only contain principal ultrafilters. So if we work within the compact space $\beta\omega \setminus \omega$, we can simply ignore finite changes of a set $A \subseteq \omega$ when considering the respective basic clopen set (or neighborhood) $\overline{A}$.

We work in the space $\beta\omega \setminus \omega$ now. So we are allowed to state claim (2) of Lemma 2.12 simply as follows:

$$A \subseteq^\star B \iff \overline{A} \subseteq \overline{B} \qquad \text{(within } \beta\omega \setminus \omega) \tag{2.9}$$

Here $\overline{A}$ actually means "the set of all *non-principal* ultrafilters containing A". (Now $\overline{A}$ is empty if and only if A is finite.)

Recall the notion of pseudo-intersection: a (infinite) set $A \subseteq \omega$ is called a *pseudo-intersection* of the family $\{A_i : i \in I\} \subseteq \mathcal{P}(\omega)$ if

$$A \subseteq^\star A_i \text{ for each } i \in I.$$

Due to (2.9), $A \subseteq \omega$ is a pseudo-intersection of the family $\{A_i : i \in I\}$ if and only if $\overline{A}$ is contained in each element of the corresponding family $\{\overline{A_i} : i \in I\}$, i.e.,

$$\overline{A} \subseteq \bigcap_{i \in I} \overline{A_i}.$$

Therefore our claim that the intersection of countably many neighborhoods $\{\overline{A_i} : i < \omega\}$ of a point $p \in \beta\omega \setminus \omega$ has a non-empty interior (see (2.6)) now directly follows from the *claim* about pseudo-intersections in the proof of Lemma 1.14 on page 9:

"Whenever a countable collection of infinite sets satisfies the so-called *strong finite intersection property* (i.e., the intersection of finitely many of them is infinite), the collection has an infinite pseudo-intersection."

Our collection $\{A_i : i < \omega\}$ indeed satisfies the required strong finite intersection property, since the ultrafilter p which contains all the A_i's is non-principal. So we can find an infinite $A \subseteq^\star A_i$ for each $i < \omega$, yielding a non-empty (cl)open set $\overline{A}$ contained in $\bigcap_{i<\omega} \overline{A_i}$.

p-points from the topological point of view

Now we would like to examine the specific feature of p-points from the topological point of view. Let's recall the definition of a p-point:

Definition 2.13. An ultrafilter $p \in \beta\omega \setminus \omega$ is a *p-point* if every countable collection of sets from p has a pseudo-intersection *within* p, i.e., for each countable family $\{A_i : i < \omega\} \subseteq p$ there is a set $A \in p$ such that

$$A \subseteq^\star A_i \text{ for each } i < \omega.$$

As we have seen above, given a countable collection $\left\{\overline{A_i} : i < \omega\right\}$ of neighborhoods of any point $p \in \beta\omega \setminus \omega$, we can find a non-empty open set $\overline{A}$ such that

$$\overline{A} \subseteq \bigcap_{i<\omega} \overline{A_i},$$

but we did not mind the question if $\overline{A}$ can be chosen in such a way that the ultrafilter p itself lies within $\overline{A}$. In fact, this is exactly the property characterizing p-points:

Lemma 2.14. *A point $p \in \beta\omega \setminus \omega$ is a* p-point *if and only if the intersection of any countable collection of neighborhoods of p is again a neighborhood of p, i.e., for each countable collection $\left\{\overline{A_i} : i < \omega\right\}$ of neighborhoods of p there is a basic open neighborhood $\overline{A}$ of p such that*

$$p \in \overline{A} \subseteq \bigcap_{i<\omega} \overline{A_i}.$$

Figuratively speaking, countably many neighborhoods of p can be replaced by a single one even closer to p.

Proof. Use the definition of a p-point and (2.9). □

In other words, p-points are those elements of $\beta\omega \setminus \omega$ which are "not approachable by countably many neighborhoods": the intersection of countably many neighborhoods of $p \in \beta\omega \setminus \omega$ has a non-empty interior for sure, but p is a p-point if and only if p itself always lies in the interior of such an intersection; if p fails to be a point, it can happen that p lies on the boundary of the (closed) intersection.

In terms of the corresponding closed set

$$F = \bigcap_{i<\omega} \overline{A_i},$$

we can also put it this way: p is a p-point if and only if each countably generated closed set containing p (i.e., a set which can be written as a countable intersection of basic clopen sets) is in fact a (closed) neighborhood of p.

Local and global cardinal characteristics in $\beta\omega \setminus \omega$

We return to the question how many neighborhoods are required to uniquely determine an ultrafilter $p \in \beta\omega \setminus \omega$ (i.e., to get a singleton as the intersection)? Countably many are never sufficient, as we have seen. Moreover, the intersection of countably many has always a non-empty interior (yielding $2^{(2^{\aleph_0})}$ points in the intersection), being a stronger result than the above. So there are two different questions: how many neighborhoods are required to get an intersection with empty interior, and how many are required to get a one-point intersection. Of course, the latter condition may require more neighborhoods, since it is more difficult to fulfill.

These two questions are closely related to the cardinal characteristics (or "cardinal invariants") $\mathfrak{p}$ and $\mathfrak{u}$, the pseudo-intersection number and the ultrafilter number:

Definition 2.15. The pseudo-intersection number $\mathfrak{p}$ is the least possible size of a family with the strong finite intersection property ("s.f.i.p.") which has no infinite pseudo-intersection, i.e.,

$$\mathfrak{p} := \min\{|\mathcal{A}| : \ \mathcal{A} \subseteq \mathcal{P}(\omega) \text{ s.f.i.p}, \ \nexists Y \ (\ |Y| = \aleph_0 \ \ \wedge \ \ \forall A \in \mathcal{A} \ (Y \subseteq^\star A) \ )\}$$

Definition 2.16. The ultrafilter number $\mathfrak{u}$ is the least possible size of a family generating a non-principal ultrafilter on ω, i.e.,

$$\mathfrak{u} := \min\{|\mathcal{A}| : \ \mathcal{A} \subseteq \mathcal{P}(\omega), \ |\{p \in \beta\omega \setminus \omega : \ \mathcal{A} \subseteq p\}| = 1\}$$

From the topological point of view, this can be viewed as follows. The pseudo-intersection number $\mathfrak{p}$ tells us the least possible size of a family

$\{\overline{A_i} : i \in I\}$ of basic clopen sets such that the intersection $\bigcap_{i\in I} \overline{A_i}$ is non-empty (this is the same as saying $\{A_i : i \in I\}$ has the strong finite intersection property), but has empty interior (corresponding to the non-existence of an infinite pseudo-intersection). The ultrafilter number $\mathfrak{u}$ is the least possible size of a family $\{\overline{A_i} : i \in I\}$ such that the intersection $\bigcap_{i\in I} \overline{A_i}$ is a singleton $\{p\}$ (corresponding to the fact that $\{A_i : i \in I\}$ generates this ultrafilter p). Obviously $\mathfrak{p} \leq \mathfrak{u}$, but in fact, $\mathfrak{p} < \mathfrak{u}$ is known to be consistent with ZFC.

In the previous chapter, we have already introduced another cardinal invariant, the so-called tower number $\mathfrak{t}$ (see Definition 1.15 on page 11): $\mathfrak{t}$ is the least length of a $\subseteq^\star$-decreasing sequence of infinite sets which has no infinite pseudo-intersection. In topological terms, this means the following: $\mathfrak{t}$ tells us the minimal length of a $\subseteq$-decreasing sequence

$$\overline{A_0} \supseteq \overline{A_1} \supseteq \overline{A_2} \ldots \supseteq \overline{A_\omega} \supseteq \ldots \supseteq \overline{A_i} \supseteq \ldots \qquad (i < \delta)$$

of non-empty basic clopen sets such that the intersection $\bigcap_{i<\delta} \overline{A_i}$ has empty interior.

Note that each sequence of the above kind clearly has non-empty intersection, therefore it is just a special kind of a family having an intersection with empty interior: it follows, that $\mathfrak{p} \leq \mathfrak{t}$. It can also be shown that $\mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{u}$, where $\mathfrak{b}$ is the so-called bounding number (cf. the remark following Lemma 3.6); so $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{u}$.

We have seen that all these cardinal invariants are at least $\aleph_1$ (and at most $2^{\aleph_0}$); so if CH holds, all of them equal $\aleph_1 = 2^{\aleph_0}$. Otherwise, there are various possibilities. If Martin's Axiom (MA) holds, we have $\mathfrak{t} = 2^{\aleph_0}$; we have proved this in the previous chapter (see Theorem 1.20 on page 13). But when we carefully analyze the proof, we recognize that the $\subseteq^\star$-ordering of the involved sequence is not used at all. So the proof of the theorem actually shows $\mathfrak{p} = 2^{\aleph_0}$ rather than $\mathfrak{t} = 2^{\aleph_0}$. Consequently, under MA all the cardinal invariants mentioned above are equal to the continuum.

Let MA(σ-centered) denote the principle "Martin's Axiom restricted to σ-centered forcings". The forcing in the proof of Theorem 1.20 is σ-centered (see the claim on page 14), so the following holds:

$$\text{MA}(\sigma\text{-centered}) \implies \mathfrak{p} = 2^{\aleph_0}.$$

In fact, the converse is also true: it was shown by Bell that $\mathfrak{p} = 2^{\aleph_0}$ if and only if MA(σ-centered) holds (see [3]).

As explained above, $\mathfrak{p} \leq \mathfrak{t}$; i.e., – figuratively speaking – finding a suitable family $\{A_i : i \in I\}$ without infinite pseudo-intersection is "potentially harder" if the $\overline{A_i}$'s are additionally required to be ordered by inclusion. The question arises, if $\mathfrak{p} < \mathfrak{t}$ is consistently true (or if $\mathfrak{p} = \mathfrak{t}$ is actually provable from ZFC). Interestingly, this seems to be unknown:

Open question. *Is there a model of ZFC satisfying* $\mathfrak{p} < \mathfrak{t}$*?*

In such a model (if it exists) the continuum has to be at least $\aleph_3$: under CH, $\aleph_1 = \mathfrak{p} = \mathfrak{t} = 2^{\aleph_0}$ anyway; but also $2^{\aleph_0} = \aleph_2$ yields $\mathfrak{p} = \mathfrak{t}$; otherwise, we would have $\aleph_1 = \mathfrak{p} < \mathfrak{t} = \aleph_2 = 2^{\aleph_0}$; however, this is ruled out by a theorem generally true in ZFC (also for larger continuum), which we state without proof (see [14]):

$$\mathfrak{p} = \aleph_1 \implies \mathfrak{t} = \aleph_1.$$

Remark. We have asked above how many neighborhoods are required to get an intersection with non-empty interior (and to get a one-point intersection, respectively). One can assign "local cardinal characteristics" to each given ultrafilter p, indicating the minimal required number of neighborhoods (satisfying a certain property). For instance, $\chi(p)$ (the *character* of p) is the least possible size of a subset of p generating p, i.e., of a family of neighborhoods uniquely determining p. Similarly, $\pi\mathfrak{p}(p)$ tells us how many neighborhoods of p are required to get an intersection with empty interior. By the arguments given earlier, for each $p \in \beta\omega \setminus \omega$ we have

$$\aleph_1 \leq \pi\mathfrak{p}(p) \leq \chi(p).$$

There is another (potentially even smaller) cardinal which can be assigned to a point p: let $\mathfrak{p}(p)$ denote the least possible size of a family of neighborhoods of p such that p itself is not within the interior of the intersection; in other words, $\mathfrak{p}(p)$ tells us how many neighborhoods of p are certainly needed such that their intersection fails to be a neighborhood of p. Again, it is quite clear that

$$\aleph_0 \leq \mathfrak{p}(p) \leq \pi\mathfrak{p}(p) \leq \chi(p);$$

however, note that $\aleph_1 \leq \mathfrak{p}(p)$ is not true in general: in fact (according to Lemma 2.14)

$$\aleph_1 \leq \mathfrak{p}(p) \iff p \text{ is a p-point.}$$

The cardinals $\pi\mathfrak{p}(p)$ and $\chi(p)$ are dependent on which ultrafilter p is considered, whereas the respective cardinal invariants $\mathfrak{p}$ and $\mathfrak{u}$ describe the "global situation" in $\beta\omega \setminus \omega$: for instance, the ultrafilter number $\mathfrak{u}$ simply expresses the least possible size of a family generating *some* non-principal ultrafilter:

$$\mathfrak{u} = min_p \, \chi(p).$$

More about these "local characteristics" can be found in [4].

2.6 The universal property of $\beta\omega$

As promised, we would like to conclude this chapter by showing that $\beta\omega$ is indeed the so-called Stone-Čech-compactification of ω; to get this result, we are going to prove that $\beta\omega$ satisfies a certain "universal property" which characterizes the Stone-Čech-compactification of ω (up to homeomorphism).

Let's start with a general definition of the Stone-Čech-compactification of a topological space in terms of this universal property (in the following, we will not always explicitly mention the topology $\mathcal{O}$ when talking about a topological space $(X, \mathcal{O})$):

Definition 2.17. A topological space X is called *Tychonoff space* if it is Hausdorff and satisfies the following property: whenever $F \subseteq X$ is closed and $x \in X \setminus F$, there is a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f[F] \subseteq \{1\}$.

Definition 2.18. Let X be a topological space. A space K is called a *Hausdorff compactification* of X if K is a compact Hausdorff space containing (a homeomorphic copy of) X as a dense subset.

Remark. It can be shown that a topological space has a Hausdorff compactification if and only if it is Tychonoff.

Definition 2.19. Let X be a Tychonoff space. A compact space is called *Stone-Čech-compactification* of X (denoted by βX) if it is a Hausdorff compactification of X satisfying the following universal property:

($\star$) For each compact Hausdorff space Y and each continuous mapping $f : X \to Y$, there is a uniquely determined continuous mapping $\widetilde{f} : \beta X \to Y$ such that

$$\widetilde{f} \restriction X = f.$$

In other words, ($\star$) says that any continuous function from X to a compact Hausdorff space Y can be extended to a continuous function from $\beta X (\supseteq X)$ to Y in a unique way.

Remark. To be more precise, there is an embedding $\beta : X \to \beta X$, i.e., β is a homeomorphism between X and $\beta[X] \subseteq \beta X$, and $\beta[X]$ is dense in βX. As we already did with $\beta\omega$ earlier in this chapter, we identify X with $\beta[X]$, so we can write $X \subseteq \beta X$ etc.

We state the following theorem without proof:

Theorem 2.20. *Let X be a Tychonoff space. Then X has exactly one Stone-Čech-compactification in the sense of Definition 2.19 (up to homeomorphism).* □

Now we are going to show that $\beta\omega$ (as defined in Lemma 2.3 and used throughout this chapter) is indeed the unique (due to Theorem 2.20) Stone-Čech-compactification of the discrete space ω in the sense of Definition 2.19 (note that ω equipped with the discrete topology is a Tychonoff space; in fact, ω can be viewed as a metric space, and each metric space is Tychonoff):

Theorem 2.21. *The topological space $\beta\omega$ is the Stone-Čech-compactification of the discrete space ω.*

Proof. We have already proved earlier in this chapter that $\beta\omega$ is a Hausdorff compactification of ω (see Theorem 2.9 and Definition 2.18). So what remains to show is that $\beta\omega$ satisfies the universal property $(\star)$ in Definition 2.19.

Let Y be an arbitrary compact Hausdorff space, and let $f : \omega \to Y$ be a function from the natural numbers to Y. (Note that the property $(\star)$ only requires to be checked for continuous functions f, but – in our case – each function is continuous since ω carries the discrete topology.) We shall define a continuous function $\widetilde{f} : \beta\omega \to Y$ extending f.

If there is such a continuous function $\widetilde{f} : \beta\omega \to Y$, it is uniquely determined because Y is Hausdorff and it has prescribed values on $\omega \subseteq \beta\omega$, which is a dense subset of its domain. (This is true in general: it is easy to prove that each continuous function from a topological space to a Hausdorff space is uniquely determined by its values on a dense subset of its domain.)

For each ultrafilter $p \in \beta\omega$, consider the expression

$$\bigcap_{A \in p} \overline{f[A]}, \tag{2.10}$$

where $f[A]$ denotes the (pointwise) image of A under f, and $\overline{f[A]}$ its topological closure within the space Y.

Remark. Here we use the same notation for the topological closure of a set as we did earlier to denote a basic clopen set $\overline{A}$; this might seem to be confusing at first sight, but in fact it is not: viewing each set $A \subseteq \omega$ as a subset of $\beta\omega$ (due to $\omega \subseteq \beta\omega$), the set $\overline{A} \subseteq \beta\omega$ can be easily seen to be nothing else than the topological closure of the set $A \subseteq \beta\omega$, which justifies the notation.

We claim that the set (2.10) uniquely determines a point in the space Y:

$$\forall p \in \beta\omega : \quad \left| \bigcap_{A \in p} \overline{f[A]} \right| = 1; \tag{2.11}$$

once we have shown that, we can define $\widetilde{f}(p)$ to be the unique member of $\bigcap_{A\in p} \overline{f[A]}$.

On the one hand, $\bigcap_{A\in p} \overline{f[A]}$ is non-empty since Y is compact. To see that, recall the characterization of compactness given by Lemma 2.8: each collection of closed sets with the finite intersection property has a non-empty intersection. The set $\{A : A \in p\} \subseteq \mathcal{P}(\omega)$ has the finite intersection property (since p is a filter), so the same is true for the family $\{f[A] : A \in p\} \subseteq \mathcal{P}(Y)$. Due to $f[A] \subseteq \overline{f[A]}$, also $\{\overline{f[A]} : A \in p\}$ has the finite intersection property (and is a collection of closed sets), which gives

$$\bigcap_{A\in p} \overline{f[A]} \neq \emptyset.$$

On the other hand, $\bigcap_{A\in p} \overline{f[A]}$ cannot contain more than one point since Y is Hausdorff. To see that, let's first show the following

Lemma 2.22. *Let $p \in \beta\omega$ and $y \in Y$, and let $\mathcal{U}(y)$ denote the collection of all neighborhoods of y. Then*

$$y \in \bigcap_{A\in p} \overline{f[A]} \iff \forall U \in \mathcal{U}(y)\ \ f^{-1}[U] \in p.$$

Proof. Note that

$$y \in \bigcap_{A\in p} \overline{f[A]} \iff \forall A \in p\ \ \forall U \in \mathcal{U}(y)\quad U \cap f[A] \neq \emptyset. \tag{2.12}$$

($\Longrightarrow$) Assume $y \in \bigcap_{A\in p} \overline{f[A]}$ and $U \in \mathcal{U}(y)$. If $f^{-1}[U]$ were not in p, $\omega \setminus f^{-1}[U] \in p$ since p is an ultrafilter; but then (2.12) would imply

$$U \cap f[\omega \setminus f^{-1}[U]] \neq \emptyset,$$

which is impossible.

($\Longleftarrow$) Assume $y \notin \bigcap_{A\in p} \overline{f[A]}$. According to (2.12) we can choose a $U \in \mathcal{U}(y)$ such that there is an $A \in p$ with $U \cap f[A] = \emptyset$; it follows that $f^{-1}[U] \cap A = \emptyset$, so $f^{-1}[U]$ cannot be in p (since A is in p). □

Now it's easy to see that $\bigcap_{A\in p} \overline{f[A]}$ cannot contain two distinct points from Y (we only use the implication from left to right in the above lemma): let y_1, y_2 be two different points both contained in $\bigcap_{A\in p} \overline{f[A]}$; since Y is Hausdorff, we can find neighborhoods $U_1 \in \mathcal{U}(y_1)$ and $U_2 \in \mathcal{U}(y_2)$ such that $U_1 \cap U_2 = \emptyset$; using the lemma, we get both $f^{-1}[U_1] \in p$ and $f^{-1}[U_2] \in p$, which is impossible because $U_1 \cap U_2 = \emptyset$ implies $f^{-1}[U_1] \cap f^{-1}[U_2] = \emptyset$.

So the proof of (2.11) is finished and the function $\widetilde{f} : \beta\omega \to Y$ can be defined by

$$\text{for each } p \in \beta\omega: \qquad p \mapsto \widetilde{f}(p) \in \bigcap_{A \in p} \overline{f[A]}.$$

It remains to show that the function $\widetilde{f} : \beta\omega \to Y$ is continuous and indeed agrees with $f : \omega \to Y$ on the common domain $\omega \subseteq \beta\omega$. The latter is immediate from the definition of $\widetilde{f}$: given a natural number $n \in \omega$ (i.e., the principal ultrafilter p containing the singleton $\{n\}$), we have

$$\widetilde{f}(n) \in \bigcap_{A \in p} \overline{f[A]} \subseteq \overline{f[\{n\}]} = \overline{\{f(n)\}} = \{f(n)\}$$

since $\{n\} \in p$ and each singleton in a Hausdorff space is a closed set; so $\widetilde{f}(n) = f(n)$ for each $n \in \omega$.

To show that $\widetilde{f}$ is continuous, recall that a function $g : X \to Y$ from a topological space X to a topological space Y is continuous if and only if for each point $x \in X$ the following holds: whenever V is a neighborhood of $g(x)$, there is a neighborhood U of x such that $g[U] \subseteq V$. It can be easily shown that each compact Hausdorff space Y is *regular* (i.e., a closed set and a point not contained in it can be separated by neighborhoods); as a consequence, the closed neighborhoods of a point $y \in Y$ form a neighborhood base of y, i.e., for each neighborhood $V \in \mathcal{U}(y)$ there is a closed neighborhood V' of y such that $V' \subseteq V$.

So let $p \in \beta\omega$ and $V \in \mathcal{U}(\widetilde{f}(p))$ a neighborhood of $\widetilde{f}(p) \in Y$. Since Y is compact Hausdorff, we can choose a closed neighborhood $V' \in \mathcal{U}(\widetilde{f}(p))$ with $V' \subseteq V$. We are going to find a neighborhood of p such that its image under $\widetilde{f}$ is contained in $V' \subseteq V$. By Lemma 2.22, $A_0 := f^{-1}[V'] \subseteq \omega$ is in p (note that $y \in \bigcap_{A \in p} \overline{f[A]}$, the left side in the lemma, is the same as $\widetilde{f}(p) = y$); so the corresponding basic clopen set $\overline{A_0}$ is a neighborhood of p (recall $A \in p$ iff $p \in \overline{A}$). We claim that $\overline{A_0}$ is the desired neighborhood. Let p be in $\overline{A_0}$ (i.e., $A_0 \in p$); we have to show that $\widetilde{f}(p) \in V'$:

$$\widetilde{f}(p) \in \bigcap_{A \in p} \overline{f[A]} \subseteq \overline{f[A_0]} = \overline{f[f^{-1}[V']]} \subseteq V',$$

since $f[f^{-1}[V']$ is contained in V' and V' is closed, which finishes the proof of our theorem. □

Let's return to Lemma 2.22 once again to see what it actually means. Since $y \in \bigcap_{A \in p} \overline{f[A]}$ is equivalent to $y = \widetilde{f}(p)$, it can be restated as follows:

For each $y \in Y$ and each $p \in \beta\omega$,

$$\widetilde{f}(p) = y \iff \forall U \in \mathcal{U}(y) \;\; f^{-1}[U] \in p. \tag{2.13}$$

As we have seen in the proof above, the Hausdorff property of Y is responsible for the fact that $\widetilde{f}$ – seen as a relation between p and y – is a function, i.e., for a given $p \in \beta\omega$, there can only be one y with $\widetilde{f}(p) = y$. In contrast, for a given $y \in Y$, there can be several $p \in \beta\omega$ such that $\widetilde{f}(p) = y$. In fact, (2.13) enables us to explicitly describe the preimage of a point y under the function $\widetilde{f}$: for each $y \in Y$,

$$\widetilde{f}^{-1}(y) = \big\{p \in \beta\omega : \; \forall U \in \mathcal{U}(y) \;\; f^{-1}[U] \in p\big\}.$$

Since $\mathcal{U}(y)$ is a filter on Y (the "neighborhood filter"), the set

$$\big\{f^{-1}[U] : \; U \in \mathcal{U}(y)\big\} \subseteq \mathcal{P}(\omega) \tag{2.14}$$

generates a filter $\mathcal{F}_y$ on ω, provided $f^{-1}[U]$ is non-empty for all neighborhoods $U \in \mathcal{U}(y)$ (in case f is injective, the above set itself is a filter on ω). Consequently, the preimage of y under $\widetilde{f}$ is nothing else than the set of all ultrafilters on ω extending the filter $\mathcal{F}_y$ generated by (2.14):

$$\widetilde{f}^{-1}(y) = \{p \in \beta\omega : \; p \supseteq \mathcal{F}_y\}. \tag{2.15}$$

(Also cf. (2.4) in the remark following Theorem 2.9: $\widetilde{f}^{-1}(y)$ is the closed subset of $\beta\omega$ which corresponds to the filter $\mathcal{F}_y$ on ω; to argue differently, it has to be closed since it is the preimage of the closed singleton $\{y\}$ under the continuous function $\widetilde{f}$.)

We are going to see now in which sense $\beta\omega$ is the "most general" among all compactifications of ω. First of all, note that not all elements of the compact Hausdorff space Y have to be "hit" by the function $\widetilde{f}$, i.e., $\widetilde{f}^{-1}(y)$ can be empty for some y. For a $y \in Y$, this is the case if and only if there is some neighborhood $U \in \mathcal{U}(y)$ such that $f^{-1}[U] = \emptyset$ (see (2.14) and (2.15)); in other words, the image of $\beta\omega$ under $\widetilde{f}$ is exactly the topological closure of $f[\omega]$ in Y:

$$\widetilde{f}[\beta\omega] = \overline{f[\omega]}.$$

Since the remaining points in Y play no role at all (they are "too far away" from where something is happening), we can remove them and assume that $Y = \widetilde{f}[\beta\omega] = \overline{f[\omega]}$, i.e., $f[\omega]$ is dense in Y. (Note that $\overline{f[\omega]}$ is a closed subset of a compact space and hence compact, so we do not lose compactness when removing the unnecessary points.)

According to Definition 2.18, Y is a Hausdorff compactification of ω if Y together with the embedding f satisfies the following (in addition to Y being a compact Hausdorff space with $f[\omega]$ as a dense subset):

- $f : \omega \to Y$ is injective
- the inverse mapping $f^{-1} : f[\omega] \to \omega$ is continuous.

If these two conditions are fulfilled, $f : \omega \leftrightarrow f[\omega]$ is a homeomorphism (f itself is continuous anyway). The second condition holds if and only if each "natural number" $f(n)$ is an isolated point in Y.

So each Hausdorff compactification Y of ω can be viewed as a "quotient of $\beta\omega$ preserving ω": it contains (a homeomorphic copy of) the natural numbers as a dense subset ("the principal ultrafilters in $\beta\omega$"), and each point y in the remaining part of Y corresponds to a closed set of non-principal ultrafilters in $\beta\omega \setminus \omega$ (see (2.15)); in other words, $\beta\omega \setminus \omega$ is partitioned into closed sets by the equivalence relation

$$p_1 \sim p_2 \;\; :\Longleftrightarrow \;\; \widetilde{f}(p_1) = \widetilde{f}(p_2). \tag{2.16}$$

That's why $\beta\omega$ is said to be the *"most general" Hausdorff compactification of* ω.

To illustrate all this, we give two examples:

Example 2.23. Let Y be the compact interval $[0, 1]$ (with the standard topology) and let $f : \omega \to [0, 1]$ be an injective enumeration of the rational numbers within $[0, 1]$, i.e.,

$$\{f(n) : \; n \in \omega\} = \mathbb{Q} \cap [0, 1].$$

Then $f[\omega]$ is dense in the compact Hausdorff space $Y = [0, 1]$, but Y is not a compactification of ω since the second of the required properties above fails: f is injective, but none of the $f(n)$'s are isolated points (f^{-1} is "very discontinuous").

Nevertheless, we can examine the continuous function $\widetilde{f} : \beta\omega \to Y$ which extends $f : \omega \to Y$. Of course, each principal ultrafilter in $\beta\omega$ is mapped to the respective rational number $f(n)$. Since all numbers in $[0, 1]$ are accumulation points of the sequence $\langle f(n) : \; n \in \omega\rangle$, each $y \in [0, 1]$ equals $\widetilde{f}(p)$ for some non-principal $p \in \beta\omega \setminus \omega$.

For instance, if $y = \frac{\pi}{4} \notin \mathbb{Q}$, $\widetilde{f}^{-1}(y)$ is the set of all ultrafilters extending the filter $\mathcal{F}_y$, which is generated by

$$\left\{ f^{-1}\left[\left(\frac{\pi}{4} - \frac{1}{n}, \frac{\pi}{4} + \frac{1}{n}\right)\right] : \; n \geq 1 \right\}.$$

Because no rational number lies in each of the $\left(\frac{\pi}{4} - \frac{1}{n}, \frac{\pi}{4} + \frac{1}{n}\right)$, $\mathcal{F}_y$ contains every co-finite set, so only non-principal ultrafilters are mapped to $\frac{\pi}{4}$. But there are very many of them; in fact, $\mathcal{F}_y$ corresponds to a closed subset of $\beta\omega \setminus \omega$ (namely $\widetilde{f}^{-1}(\frac{\pi}{4})$) with non-empty interior: since $\mathcal{F}_y$ is generated by only countably many sets, we can find a pseudo-intersection, i.e., a set $A \subseteq \omega$ almost contained in all the $f^{-1}\big[\big(\frac{\pi}{4} - \frac{1}{n}, \frac{\pi}{4} + \frac{1}{n}\big)\big]$'s; to be more explicit, each $A \subseteq \omega$ with the property that the corresponding sequence $\langle f(n) : n \in A \rangle$ converges to $\frac{\pi}{4}$ will be such a pseudo-intersection. It follows that $\widetilde{f}(p) = \frac{\pi}{4}$ for each non-principal ultrafilter $p \in \beta\omega \setminus \omega$ containing such an A (which is – for $p \in \beta\omega \setminus \omega$ – the same as $p \in \overline{A} \subseteq \widetilde{f}^{-1}(\frac{\pi}{4})$).

If y is some rational point, again very many non-principal ultrafilters are mapped to y; in addition, also one natural number is mapped to the same y. (This reflects the fact that f^{-1} is not continuous.) Therefore – with the equivalence relation (2.16) – the whole space $\beta\omega$ is partitioned like that: there are $2^{\aleph_0}$ many parts (corresponding to the points in $[0, 1]$), each of them containing $2^{(2^{\aleph_0})}$ ultrafilters; countably many parts (those which correspond to the rationals) additionally contain one principal ultrafilter, all the other parts merely contain non-principal ultrafilters.

Example 2.24. Let Y again be the compact Hausdorff space $[0, 1]$ and let $f : \omega \longrightarrow [0, 1]$ be the following injective mapping:

$$f(n) = \begin{cases} \frac{1}{2^{k+2}} & \text{for } n = 2k \\ \frac{1}{2} + \frac{1}{2^{k+2}} & \text{for } n = 2k+1 \end{cases}$$

The sequence $\langle f(n) : n \in \omega \rangle$ has the two accumulation points 0 and $\frac{1}{2}$, so

$$\widetilde{f}[\beta\omega] = \overline{f[\omega]} = f[\omega] \cup \{0\} \cup \{1/2\} \subsetneqq [0, 1];$$

to make $\widetilde{f}$ surjective, we simply redefine Y to be the set $\overline{f[\omega]}$. Then Y is a Hausdorff compactification of ω according to Definition 2.18: Y is a compact Hausdorff space containing a homeomorphic copy of the discrete space ω as a dense subset (note that f is injective and $f(n)$ is isolated in Y for each $n \in \omega$).

Now let's see what $\widetilde{f}$ looks like. Of course, each principal ultrafilter is mapped to the respective $f(n)$, and there is no $f(n)$ with a non-principal ultrafilter in its preimage (since each $f(n)$ is isolated). For every non-principal ultrafilter $p \in \beta\omega \setminus \omega$, $\widetilde{f}(p)$ is either 0 or $\frac{1}{2}$. We can say that each non-principal ultrafilter p selects an accumulation point of the sequence $\langle f(n) : n \in \omega \rangle$: in general, this is true for each bounded sequence of real numbers $\langle a_n : n \in \omega \rangle$. In this example, the situation is very simple: there is a single set deciding whether $p \in \beta\omega \setminus \omega$ is mapped to 0 or $\frac{1}{2}$; let $Even \subseteq \omega$ denote the set of the even numbers, and $Odd = \omega \setminus Even$; then $\widetilde{f}(p) = 0$ if and only if $Even \in p$ (iff $Odd \notin p$). Put differently,

$$\widetilde{f}^{-1}(0) = \overline{Even} \cap (\beta\omega \setminus \omega),$$

whereas $\widetilde{f}^{-1}\left(\frac{1}{2}\right) = \overline{Odd} \cap (\beta\omega \setminus \omega)$. So Y can be viewed as the quotient of $\beta\omega$ with $\beta\omega \setminus \omega$ partitioned into the two parts $\overline{Even}$ and $\overline{Odd}$.

Remark. If we replace the sequence in the example above by a converging sequence of real numbers, say $f(n) = \frac{1}{2^n}$ for each $n \in \omega$, then the situation becomes even simpler: all non-principal ultrafilters are mapped to the limit of the sequence: for each $p \in \beta\omega \setminus \omega$,

$$\widetilde{f}(p) = \lim_{n \to \infty} f(n) = 0.$$

The set $\{f(n) : n \in \omega\} \cup \{0\}$ is a Hausdorff compactification of ω, and in some sense it is the "smallest one", the so-called *Alexandroff compactification* of ω: all elements of $\beta\omega \setminus \omega$ are identified with each other. Since it can be thought of as the discrete space ω together with *one* single point ∞ (each neighborhood of ∞ contains all but finitely many natural numbers), it is also called the *one-point compactification* of ω.

Chapter 3

A model of ZFC without p-points

In this chapter we are going to construct a model of ZFC in which there is no p-point. Consequently, the existence of p-points is not provable from ZFC. We essentially follow Shelah's (new) proof of this theorem. It can be found in the books by Bartoszyński and Judah ([2],1995) and Shelah ([13]).

We define Gregorieff's forcing notion $\mathbb{P}(\mathcal{F})$ (for a filter $\mathcal{F}$ on ω). Provided that $\mathcal{F}$ is an unbounded p-filter, $\mathbb{P}(\mathcal{F})$ is a proper and ω^ω-bounding forcing notion; to show this, we use a characterization of unbounded p-filters via infinite games given in a 1996 article of Laflamme ([10]). The main step in the proof will be Lemma 3.27, which shows that $\mathbb{P}(\mathcal{F})^\omega$ "kills" $\mathcal{F}$, in the sense that $\mathcal{F}$ cannot be extended to a p-point anymore. The remaining part of the proof is a standard iterated forcing argument: we define a countable support iteration of length ω_2 which eventually kills all p-points.

For details on forcing which are not mentioned here see [9], [8], [6], [7], [13], [2] and [1].

Throughout this chapter we assume that $\mathcal{F}$ extends the Frechet filter $\mathfrak{Fr}$ without mentioning it all the time.

3.1 Gregorieff's forcing $\mathbb{P}(\mathcal{F})$

Let $\mathcal{F}^\star$ denote the dual ideal of the filter $\mathcal{F}$, i.e.,

$$\mathcal{F}^\star := \{Z \subseteq \omega : \omega \setminus Z \in \mathcal{F}\}$$

The Cohen forcing can be defined as the set of all finite partial functions from ω to 2. In a certain sense, the following forcing notion is a generalization of the Cohen forcing. Given a filter $\mathcal{F}$, we define the forcing notion $\mathbb{P}(\mathcal{F})$ as follows: the conditions of the forcing are those partial functions from ω to 2 which are still undefined on a filter set (in other words: whose domain is very small with respect to the filter $\mathcal{F}$, i.e., in the dual ideal $\mathcal{F}^\star$), and the ordering is – of course – the reverse inclusion, as with the Cohen forcing. More formally, let's give

Definition 3.1. Let $\mathcal{F}$ be a filter on ω (extending the Frechet filter). Then we define the forcing notion $\mathbb{P}(\mathcal{F})$ as follows:

$$\mathbb{P}(\mathcal{F}) := \{p : \ p \colon \mathrm{dom}(p) \to 2 \ \text{ is a function}, \ \mathrm{dom}(p) \in \mathcal{F}^\star\}$$

For two conditions $p, q \in \mathbb{P}(\mathcal{F})$, q is stronger than p, if the function q extends the function p:

$$q \leq p \ :\Longleftrightarrow \ q \supseteq p$$

Note that $\mathbb{P}(\mathcal{F})$ equals the Cohen forcing if the filter $\mathcal{F}$ happens to be the Frechet filter $\mathfrak{Fr}$ itself (since $\mathcal{F}^\star = \mathfrak{Fr}^\star = [\omega]^{<\omega}$ in this case).

Basic facts about $\mathbb{P}(\mathcal{F})$

Let's see what happens when we force with $\mathbb{P}(\mathcal{F})$. The Cohen forcing $\mathbb{P}(\mathfrak{Fr})$ adds a new real to the ground model $\mathbf{V}$, and so does every $\mathbb{P}(\mathcal{F})$; the proof is the same: Suppose G is a $\mathbb{P}(\mathcal{F})$-generic filter over $\mathbf{V}$. We define

$$x := \bigcup G, \tag{3.1}$$

which we claim to be a new "generic" real, i.e., a function $x \in 2^\omega$ from ω to 2 which is not in $\mathbf{V}$, and "contains as much information as G does".

First of all, x is a function: if – for some $n \in \omega$ – $(n, 0)$ and $(n, 1)$ were both in $x = \bigcup G$, i.e., there are $p_0, p_1 \in G$ with $(n, 0) \in p_0$ and $(n, 1) \in p_1$, there would be (since G is a filter) a stronger condition $q \supseteq p_0, p_1$ in G with $q \supseteq \{(n, 0), (n, 1)\}$, contradicting the fact that q is a function.

By the standard density argument, $\mathrm{dom}(x) = \omega$. For each $n \in \omega$, let

$$D_n := \{q \in \mathbb{P}(\mathcal{F}) : \ n \in \mathrm{dom}(q)\} \in \mathbf{V}.$$

Obviously, D_n is dense for each n: for each condition $p \in \mathbb{P}(\mathcal{F})$, we can find a stronger condition $q \supseteq p$ within D_n; in case $n \in \mathrm{dom}(p)$, p itself is in D_n and we are done; otherwise $q := p \cup \{(n, 0)\}$ will be a function, which is stronger than p and in D_n (note that $\mathrm{dom}(q) \in \mathcal{F}^\star$ since $\mathrm{dom}(p) \in \mathcal{F}^\star$ and $\mathcal{F} \supseteq \mathfrak{Fr}$). Because G is generic, it meets every dense set in $\mathbf{V}$: for each $n \in \omega$, there is a $q \in G \cap D_n$, i.e., $n \in \mathrm{dom}(q) \subseteq \mathrm{dom}(x)$, yielding $\mathrm{dom}(x) = \omega$.

Note that $x \in \mathbf{V}$ if and only if $G \in \mathbf{V}$, because x and G are definable from each other: x was defined by G in (3.1), and G can be derived from x since it can be shown that $G = \{p \in \mathbb{P}(\mathcal{F}) : p \subseteq x\}$ (i.e., the generic filter G is the collection of all "small approximations" of the real x). So it suffices to show $G \notin \mathbf{V}$ to see that the real x added by the forcing $\mathbb{P}(\mathcal{F})$ does not belong to the ground model $\mathbf{V}$. But this is true for every "non-atomic" forcing:

Fact 3.2. *Let $\mathbb{P} \in \mathbf{V}$ be any forcing notion satisfying*

$$\forall p \in \mathbb{P} \ \exists q_0, q_1 \leq p : \ q_0 \perp q_1. \tag{3.2}$$

If G is a $\mathbb{P}$-generic filter over $\mathbf{V}$, then $G \notin \mathbf{V}$.

Proof. If $G \in \mathbf{V}$, then also $D := \mathbb{P} \setminus G \in \mathbf{V}$. It is easy to see that D is dense: let $p \in \mathbb{P}$; by (3.2), there are two incompatible conditions q_0 and q_1 below p, so at least one of them has to be in $\mathbb{P} \setminus G = D$ (since G is a filter). Because G is generic, it meets every dense set in $\mathbf{V}$, but $G \cap D = \emptyset$, a contradiction. $\square$

The forcing $\mathbb{P}(\mathcal{F})$ satisfies property (3.2): given a $p \in \mathbb{P}(\mathcal{F})$, $\mathrm{dom}(p)$ will be in $\mathcal{F}^\star$, so we can pick a natural number $n \notin \mathrm{dom}(p)$; then the conditions $q_0 := p \cup \{(n, 0)\}$ and $q_1 := p \cup \{(n, 1)\}$ are incompatible and stronger than p, as desired.

What does the set $\mathcal{F} \in \mathbf{V}$ look like within the generic extension $\mathbf{V}[G]$? Unless $\mathcal{F}$ is the Frechet filter, $\mathcal{F}$ will not be closed under supersets any more: if $Y \in \mathcal{F}$ is some set with $|\omega \setminus Y| = \aleph_0$, there have to be supersets of Y (i.e., subsets of $\omega \setminus Y$), which are not in $\mathbf{V}$ and therefore not in the set $\mathcal{F}$; otherwise all subsets of ω would be in $\mathbf{V}$ (since there is a bijective mapping between ω and $\omega \setminus Y$ within $\mathbf{V}$), contradicting the fact that $\{n \in \omega : x(n) = 0\}$ ("the generic subset of ω") is not in $\mathbf{V}$.

Of course, we can consider the filter $\widetilde{\mathcal{F}} \in \mathbf{V}[G]$ which is generated by $\mathcal{F}$; the next lemma shows that this filter $\widetilde{\mathcal{F}}$ can never be an ultrafilter:

Lemma 3.3. *Let G be a $\mathbb{P}(\mathcal{F})$-generic filter over $\mathbf{V}$, and let*

$$\widetilde{\mathcal{F}} := \{Z \subseteq \omega : \ Z \supseteq Y \text{ for some } Y \in \mathcal{F}\}^{\mathbf{V}[G]}.$$

Then $\widetilde{\mathcal{F}} \in \mathbf{V}[G]$ fails to be an ultrafilter.

In fact, the generic set $X := \{n \in \omega : x(n) = 1\}$ is not decided by $\widetilde{\mathcal{F}}$, i.e., the filter $\widetilde{\mathcal{F}}$ contains neither X nor $\omega \setminus X$.

Proof. Assume towards a contradiction that X (or $\omega \setminus X$) is in $\widetilde{\mathcal{F}}$, i.e., there is a set $Y \in \mathcal{F}$ with $Y \subseteq X$. (We can concentrate on the case concerning

X since the situation is completely symmetric.) Everything which is true in $\mathbf{V}[G]$ is forced by some condition, so there is a $p \in \mathbb{P}(\mathcal{F})$ such that

$$p \Vdash_{\mathbb{P}(\mathcal{F})} Y \subseteq \{n \in \omega : \dot{x}(n) = 1\},$$

where $\dot{x}$ is a name for the real $x = \bigcup G$. However, p cannot force this because it is only defined on a set which is "too small": $\mathrm{dom}(p) \in \mathcal{F}^\star$, so

$$(\omega \setminus \mathrm{dom}(p)) \cap Y \in \mathcal{F}$$

and hence non-empty; so we can pick some $n \in Y$, $n \notin \mathrm{dom}(p)$, and define

$$q := p \cup \{(n, 0)\};$$

now q is a condition stronger than p which forces $\dot{x}(n) = 0$, i.e., $n \notin X$, which is a contradiction (since $n \in Y$).

We can express this argument also as follows: for each given set $Y \in \mathcal{F}$ (so Y is in $\mathbf{V}$), it is "dense to force that the function x is not constant on the set Y"; consequently, Y cannot be completely contained in $\{n \in \omega : x(n) = 0\}$ or $\{n \in \omega : x(n) = 1\}$. □

If $\mathcal{F} \in \mathbf{V}$ is not an ultrafilter, the result of the lemma is clear (each set which is not decided by $\mathcal{F}$ in $\mathbf{V}$ won't be decided by $\widetilde{\mathcal{F}}$ in $\mathbf{V}[G]$ either). But even if $\mathcal{F}$ is an ultrafilter in $\mathbf{V}$, $\widetilde{\mathcal{F}}$ (i.e., the filter generated by $\mathcal{F}$ in $\mathbf{V}[G]$) always fails to be an ultrafilter in $\mathbf{V}[G]$. So the forcing $\mathbb{P}(\mathcal{F})$ can be said to "*destroy the ultrafilter*" which it is based on (provided $\mathcal{F}$ was an ultrafilter).

Remark. Compare this lemma with the following general theorem by Bartoszyński et al. (see [2, Theorem 6.2.2 on page 286]): each forcing notion adding reals destroys some ultrafilter of the ground model; more precisely, if r is some real which does not belong to $\mathbf{V}$, then there exists an ultrafilter $\mathcal{F}$ in $\mathbf{V}$ such that

$$\mathbf{V}[r] \models \{Z \subseteq \omega : Z \supseteq Y \text{ for some } Y \in \mathcal{F}\} \text{ is not an ultrafilter,}$$

where $\mathbf{V}[r]$ denotes the "smallest model containing $\mathbf{V}$ and the real r".

3.2 Unbounded filters

For each infinite set $X \subseteq \omega$, let f_X denote the increasing enumeration of X, i.e., $f_X : \omega \to \omega$ is the unique strictly increasing function from ω to ω such that

$$X = \{f_X(n) : n \in \omega\}.$$

So with each filter $\mathcal{F}$, we can associate a set of functions in a natural way (note that $\mathcal{F}$ does not contain any finite set):

$$F = \{f_X : X \in \mathcal{F}\} \subseteq \omega^\omega.$$

For two functions $f, g \in \omega^\omega$, let $f <^\star g$ denote

$$\exists n \in \omega \quad \forall k \geq n : \quad f(k) < g(k).$$

(Note that it does not matter if we use $<^\star$ or $\leq^\star$ in the following: whenever a function $g \in \omega^\omega$ bounds a set $F \subseteq \omega^\omega$ with respect to $\leq^\star$, the function "$g + 1$" will bound it with respect to $<^\star$.)

Definition 3.4. A filter $\mathcal{F}$ is called *unbounded* if the associated set of enumerating functions is unbounded, i.e., there is **no** function $g \in \omega^\omega$ such that

$$\forall X \in \mathcal{F} : \quad f_X <^\star g.$$

In other words: $\mathcal{F}$ is unbounded, if for each $g \in \omega^\omega$ there is an $X \in \mathcal{F}$ such that $f_X(k) \geq g(k)$ for infinitely many $k \in \omega$.

Lemma 3.5. *Let $\mathcal{F}$ be a filter. Then the following are equivalent:*

1. *$\mathcal{F}$ is unbounded (according to Definition 3.4)*
2. *For each strictly increasing sequence $n_0 < n_1 < n_2 < \dots$ of natural numbers, there exists an $X \in \mathcal{F}$ such that $X \cap [n_k, n_{k+1}) = \emptyset$ for infinitely many $k \in \omega$.*
3. *$\mathcal{F}$ is* non-meager*, i.e., the associated set $F = \{f_X : X \in \mathcal{F}\} \subseteq \omega^\omega$ of enumerating functions is non-meager within the topological space ω^ω.*

Proof. First of all, note that $f_X(k) \geq g(k)$ is equivalent to $|X \cap g(k)| \leq k$.

So $\mathcal{F}$ is unbounded (according to the definition) if and only if for each function $g \in \omega^\omega$ there is an $X \in \mathcal{F}$ such that

$$\forall n \in \omega \quad \exists k \geq n : \quad |X \cap g(k)| \leq k. \tag{3.3}$$

Figuratively speaking, (3.3) says that the set X is "quite thin" compared to the (fast growing) function $g \in \omega^\omega$.

(2) $\to$ (1) Let $g \in \omega^\omega$; we shall find an $X \in \mathcal{F}$ such that (3.3) holds; w.l.o.g. we can assume that $g(k) > k$ for each $k \in \omega$.

By repeatedly applying g, define the strictly increasing sequence

$$0 < g(0) < g(g(0)) < g^{(3)}(0) < \ldots < g^{(k)}(0) < \ldots, \quad k \in \omega$$

of natural numbers; by (2), there is a filter set $X \in \mathcal{F}$ such that

$$X \cap [g^{(k)}(0), g^{(k+1)}(0)) = \emptyset \tag{3.4}$$

for infinitely many $k \in \omega$. We show (3.3): given $n \in \omega$, choose $k \in \omega$ such that $g^{(k)}(0) \geq n$ and (3.4) holds; then $X \cap g^{(k+1)}(0) = X \cap g^{(k)}(0)$, so

$$\left|X \cap g(g^{(k)}(0))\right| = \left|X \cap g^{(k+1)}(0)\right| \overset{(3.4)}{=} \left|X \cap g^{(k)}(0)\right| \leq g^{(k)}(0),$$

and we are done.

(1) $\to$ (2) Let $n_0 < n_1 < n_2 < \dots$ be a strictly increasing sequence of natural numbers; we shall find a set $X \in \mathcal{F}$ missing infinitely many of the intervals $[n_k, n_{k+1})$.

Define a function $g \in \omega^\omega$ by $g(k) := n_{2k}$ for each $k \in \omega$; by (1), there is a filter set $X \in \mathcal{F}$ such that (3.3) holds, i.e.,

$$|X \cap g(k)| = |X \cap n_{2k}| \leq k$$

for infinitely many $k \in \omega$. Obviously, X has to be disjoint from infinitely many of the $[n_k, n_{k+1})$'s, which finishes the proof.

(1,2) $\leftrightarrow$ (3) Since we won't need this characterization of unboundedness in the further development, we omit the proof and refer to Bartoszyński's book [2, Theorem 4.1.2]. □

Lemma 3.6. *Each ultrafilter $\mathcal{F}$ is unbounded.*

Proof. We use the characterization of Lemma 3.5 (2).

Let $n_0 < n_1 < n_2 < \dots$ be a strictly increasing sequence of natural numbers and define the set

$$X := \bigcup_{k \in \omega} [n_{2k}, n_{2k+1}).$$

Since $\mathcal{F}$ is an ultrafilter, it contains either X or its complement $\omega \setminus X$. In either case, the respective filter set avoids every other interval (and hence infinitely many of them). □

Remark. As a consequence, one can easily show the following: the bounding number $\mathfrak{b}$ (the least possible size of an unbounded family $F \subseteq \omega^\omega$) is less or equal than the ultrafilter number $\mathfrak{u}$ (the least possible size of a family generating an ultrafilter). We have already mentioned this fact on page 38 in Chapter 2.

3.3 The p-filter game

Recall the notion of a p-filter: a filter $\mathcal{F}$ is called a p-filter if each countable collection $\{Y_n : n \in \omega\} \subseteq \mathcal{F}$ of filter sets has a pseudo-intersection *within* the filter, i.e., there is an $X \in \mathcal{F}$ such that $X \subseteq^\star Y_n$ for each $n \in \omega$. (So a p-point is a p-filter which is an ultrafilter.)

Due to Lemma 3.6, every p-point is an *unbounded p-filter*. Therefore, unbounded p-filters are – in some sense – "approximations of p-points". We are going to give a nice equivalent characterization of unbounded p-filters now, using the following infinite game:

Definition 3.7 (The p-filter game)**.** Let $\mathcal{F}$ be a filter. Then we define the infinite game $\mathcal{G}(\mathcal{F})$ as follows:

In the n-th move ($n \in \omega$),

- Player I plays a filter set $X_n \in \mathcal{F}$
- Player II responds with a finite subset $s_n \subseteq X_n$

Player I	X_0		X_1		X_2		$\cdots$	
Player II		s_0		s_1		s_2		$\cdots$

After these ω many moves, Player II wins the game $\mathcal{G}(\mathcal{F})$ if

$$\bigcup_{n\in\omega} s_n \in \mathcal{F}.$$

Figuratively speaking, Player I is the "nasty player" who tries to make Player II's life hard by choosing a "fast decreasing" sequence of filter sets.

The following characterization of unbounded p-filters is due to Laflamme (see [10]).

Theorem 3.8. *Let $\mathcal{F}$ be a filter. Then $\mathcal{F}$ is an unbounded p-filter if and only if Player I has **no** winning strategy in the game $\mathcal{G}(\mathcal{F})$.*

Remark. Note that Player II *never* has a winning strategy in the game $\mathcal{G}(\mathcal{F})$, no matter whether $\mathcal{F}$ is an unbounded p-filter or not (only $\mathcal{F} \supseteq \mathfrak{Fr}$ is needed). This can be seen as follows: if two games are played simultaneously, Player I can always arrange to win at least one of them (he makes sure that all the finite sets played by Player II in either game are pairwise disjoint; then Player II cannot win both games, since otherwise there were two disjoint sets in $\mathcal{F}$); consequently, no strategy for Player II can be a winning strategy.

Proof. A strategy for Player I can be described as follows: after each move of Player II, the strategy tells Player I what to do next, depending on what has been played so far; more precisely, in the beginning, Player I is told by the strategy what filter set X_0 to play; then Player II freely chooses some finite set $s_0 \subseteq X_0$, then the strategy – depending on the s_0-move of Player II – tells Player I to play a set X_1, etc.

Therefore a strategy for Player I is a function, which assigns a filter set $X_{\bar{s}} \in \mathcal{F}$ to each finite sequence $\bar{s} = \langle s_0, \ldots, s_{n-1} \rangle$ of finite sets which could have been played by Player II within the first n moves of the game (of course the filter sets played by Player I need not to be considered as arguments of this function, since they can in turn be "computed" by the strategy).

Now consider a fixed strategy for Player I. Provided Player I follows this strategy, we can build a tree $\mathcal{T}$, which describes all possible courses of the game (dependent on Player II's choices). The nodes of this tree $\mathcal{T}$ are finite sequences $\bar{s}$ of finite subsets of ω, and for each node $\bar{s} \in \mathcal{T}$, there is a filter set $X_{\bar{s}} \in \mathcal{F}$ assigned to $\bar{s}$, such that $\bar{s} = \langle s_0, \ldots, s_{n-1} \rangle$ extended by s_n (denoted by $\bar{s}^\frown s_n$) is in $\mathcal{T}$ if and only if s_n is a finite subset of $X_{\bar{s}}$.

We call such a tree an $\mathcal{F}$-tree of finite sets:

Definition 3.9. A tree $\mathcal{T} \subseteq \left([\omega]^{<\omega}\right)^{<\omega}$ is called an *$\mathcal{F}$-tree of finite sets* if $\langle\rangle \in \mathcal{T}$ and for all $\bar{s} \in \mathcal{T}$ there is an $X_{\bar{s}} \in \mathcal{F}$ with

$$\bar{s}^\frown t \in \mathcal{T} \iff t \in [X_{\bar{s}}]^{<\omega}.$$

Note that each node $\bar{s}$ has $\aleph_0$ immediate successors, since every $X_{\bar{s}}$ is infinite (due to $\mathcal{F} \supseteq \mathfrak{Fr}$) and a set of size $\aleph_0$ has countably many finite subsets.

Each infinite branch through $\mathcal{T}$ (i.e., an ω-sequence $\langle s_n : n \in \omega \rangle$ of finite sets such that $\langle s_0, \ldots, s_{n-1} \rangle \in \mathcal{T}$ for all $n \in \omega$) corresponds to a single course of the game. We are interested in branches through $\mathcal{T}$ whose union is in $\mathcal{F}$, i.e.,

$$\bigcup_{n \in \omega} s_n \in \mathcal{F},$$

since these branches correspond to the games which Player II wins. So an $\mathcal{F}$-tree of finite sets has at least one such branch if and only if the strategy for Player I (which the tree was based on) is **no** winning strategy.

Therefore Player I has no winning strategy at all (in the game $\mathcal{G}(\mathcal{F})$) if and only if every $\mathcal{F}$-tree of finite sets has a branch whose union is in $\mathcal{F}$; so we have reduced the problem to the following combinatorial lemma about $\mathcal{F}$-trees of finite sets. $\square$

Lemma 3.10. *Let $\mathcal{F}$ be a filter. Then $\mathcal{F}$ is an unbounded p-filter if and only if every $\mathcal{F}$-tree of finite sets has a branch whose union is in $\mathcal{F}$.*

Proof. For the easy direction, assume that every $\mathcal{F}$-tree of finite sets has a branch whose union is in $\mathcal{F}$. We will show that $\mathcal{F}$ is an unbounded p-filter.

To show that $\mathcal{F}$ is a p-filter, let $A_0 \supseteq A_1 \supseteq \ldots$ (each $A_i \in \mathcal{F}$) be a decreasing sequence of filter sets (note that w.l.o.g. we can assume that the given countable collection of filter sets is actually a decreasing sequence). Define a tree $\mathcal{T} \subseteq ([\omega]^{<\omega})^{<\omega}$ such that $X_{\bar{s}} = A_i$ for each $\bar{s}$ of length i, i.e., let

$$\mathcal{T} := \{\langle s_0, \ldots, s_{i-1}\rangle : i \in \omega,\ \forall j \in i\ \ s_j \subseteq A_j\}$$

with A_i assigned to $\langle s_0, \ldots, s_{i-1}\rangle$ for each $i \in \omega$. It's clear that $\mathcal{T}$ is an $\mathcal{F}$-tree of finite sets. Our assumption gives us a branch $\langle s_j : j \in \omega\rangle$ through $\mathcal{T}$ whose union $X := \bigcup_{j\in\omega} s_j$ is in $\mathcal{F}$. Now it's easy to see that $X \subseteq^\star A_i$ for each $i \in \omega$: $\forall j \geq i,\ s_j \subseteq A_i$ (because $s_j \subseteq A_j$ and $A_j \subseteq A_i$). So $\bigcup_{j\geq i} s_j \subseteq A_i$, which implies $X \subseteq^\star A_i$.

To show that $\mathcal{F}$ is unbounded, we are going to use the characterization of Lemma 3.5 (2): let $n_0 < n_1 < n_2 < \ldots$ be a sequence of natural numbers; we shall find a set $X \in \mathcal{F}$ missing infinitely many of the intervals $[n_k, n_{k+1})$.

We build a tree $\mathcal{T}$ (together with the filter sets $X_{\bar{s}}$ assigned to its nodes) as follows. Start with the empty sequence $\bar{s} = \langle\ \rangle$, and let $X_{\langle\ \rangle} = \omega \in \mathcal{F}$ be assigned to it. Because we want $\mathcal{T}$ to be an $\mathcal{F}$-tree of finite sets, we put the sequence $\langle s_0\rangle$ into the tree $\mathcal{T}$ for each finite subset $s_0 \subseteq X_{\langle\ \rangle} = \omega$. In general, having already $\bar{s} = \langle s_0, \ldots, s_{i-1}\rangle \in \mathcal{T}$, choose the least k such that $s_j \subseteq n_k$ for each $j < i$, and let $X_{\bar{s}} := \omega \setminus n_{k+1}$. (Note that $X_{\bar{s}} \in \mathcal{F}$ because $\mathcal{F} \supseteq \mathfrak{Fr}$.) Then the tree $\mathcal{T}$ is extended by all the sequences $\langle s_0, \ldots, s_{i-1}, s_i\rangle$ with $s_i \subseteq X_{\bar{s}}$. This way $\mathcal{T}$ becomes an $\mathcal{F}$-tree of finite sets; so it has a branch $\langle s_j : j \in \omega\rangle$ whose union $X := \bigcup_{j\in\omega} s_j$ is in $\mathcal{F}$. The set X proves $\mathcal{F}$ to be unbounded, for it misses infinitely many of the intervals $[n_k, n_{k+1})$: fix $m \in \omega$; we will find an interval $[n_k, n_{k+1})$ with $m < n_k$ and $X \cap [n_k, n_{k+1}) = \emptyset$. Choose $i \in \omega$ big enough such that the sequence $\langle s_0, \ldots, s_{i-1}\rangle$ is not bounded by m, i.e., $\exists j < i$ with $s_j \not\subseteq m$. (This is possible because X would be finite otherwise.) Consequently, if k is the least number such that $s_j \subseteq n_k$ for each $j < i$, then $m < n_k$, and $s_i \subseteq X_{\langle s_0,\ldots,s_{i-1}\rangle} = \omega \setminus n_{k+1}$. Moreover, for each $j > i$, $s_j \subseteq X_{\langle s_0,\ldots,s_{j-1}\rangle} \subseteq \omega \setminus n_{k+1}$. Therefore, for each $j \in \omega$, either $s_j \subseteq n_k$ (if $j < i$) or $s_j \subseteq \omega \setminus n_{k+1}$ (if $j \geq i$), so $X \cap [n_k, n_{k+1}) = \emptyset$, which finishes the proof of $\mathcal{F}$ being unbounded.

To prove the other direction, let's assume that $\mathcal{F}$ is an unbounded p-filter and let $\mathcal{T}$ be an $\mathcal{F}$-tree of finite sets. The filter sets assigned to each node $\bar{s} \in \mathcal{T}$ are denoted by $X_{\bar{s}}$, as above. We will find a branch through $\mathcal{T}$ whose union is in $\mathcal{F}$; in fact, we will construct a set $Y \in \mathcal{F}$ and split it up into ω-many finite pieces. The sequence of these finite pieces will be the desired branch through $\mathcal{T}$.

If the intersection of all the (countably many) filter sets $X_{\bar{s}}$ ($\bar{s} \in \mathcal{T}$) would be in the filter again, we could take this intersection as our filter set Y, and the proof would be finished: $\forall \bar{s} \in \mathcal{T} \;\; Y \subseteq X_{\bar{s}}$, so Y – split up into finite parts – would serve as a branch through $\mathcal{T}$. However, this doesn't work, so we try to approximate it by intersecting more and more of the filter sets $X_{\bar{s}}$, but just finitely many of them each time. We get a decreasing sequence of filter sets $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$; to get such a set A_k, we only intersect filter sets assigned to nodes up to a certain level of the tree (i.e., sets $X_{\bar{s}}$ up to a certain length of $\bar{s}$). But the tree splits infinitely many times at each node, so we also have to put some bound on the components of $\bar{s} = \langle s_0, \ldots, s_{i-1} \rangle$ itself. More precisely, define for each $k \in \omega$

$$A_k := \bigcap \left\{ X_{\langle s_0, \ldots, s_{i-1} \rangle} : \langle s_0, \ldots, s_{i-1} \rangle \in \mathcal{T}, \;\; i \leq k, \;\; \forall j \in i \;\; s_j \subseteq k \right\} \tag{3.5}$$

Note that $A_k \in \mathcal{F}$ for each $k \in \omega$, because it's the intersection of finitely many filter sets; after all, there are just finitely many sequences $\langle s_0, \ldots, s_{i-1} \rangle$ with the above restrictions. It is clear that the sequence is decreasing, because more and more sets $X_{\bar{s}}$ contribute to the intersection when k is increasing. (Note also that $A_0 = X_{\langle \, \rangle}$, because for $i = 0$, $\langle s_0, \ldots, s_{i-1} \rangle$ becomes the empty sequence $\langle \, \rangle \in \mathcal{T}$, and therefore $A_0 = \bigcap \{ X_{\langle \, \rangle} \} = X_{\langle \, \rangle}$.)

For we have a decreasing sequence of filter sets $A_0 \supseteq A_1 \supseteq \ldots$, we can now use our assumption that $\mathcal{F}$ is a p-filter. So there exists a set $Y_1 \in \mathcal{F}$ such that $Y_1 \subseteq^{\star} A_k$ for each $k \in \omega$, i.e.,

$$\forall k \in \omega \;\; \exists n \in \omega : \;\; Y_1 \setminus n \subseteq A_k \tag{3.6}$$

This enables us to choose a strictly increasing sequence $\langle n_k : \; k \in \omega \rangle$ of natural numbers such that

$$\forall k \in \omega : \;\; Y_1 \setminus n_{k+1} \subseteq A_{n_k} \tag{3.7}$$

More precisely, construct the sequence $\langle n_k : \; k \in \omega \rangle$ as follows: Start with $n_0 = 0$, and define inductively (using (3.6))

$$n_{k+1} := \min\{ n > n_k : \;\; Y_1 \setminus n \subseteq A_{n_k} \}$$

Clearly, this sequence is strictly increasing and satisfies (3.7). Splitting up Y_1 (into ω-many pieces) at the points given by the sequence $\langle n_k : \; k \in \omega \rangle$ will "almost" give us a branch through $\mathcal{T}$ (whose union would be in $\mathcal{F}$ since it equals $Y_1 \in \mathcal{F}$):

$$\langle Y_1 \cap [n_k, n_{k+1}) : \; k \in \omega \rangle \tag{3.8}$$

The only additional requirement to finish the proof right now would be the following instead of (3.7):

$$\forall k \in \omega : \; Y_1 \setminus n_k \subseteq A_{n_k} \tag{3.9}$$

Note that this would imply $Y_1 \cap [n_k, n_{k+1}) \subseteq Y_1 \setminus n_k \subseteq A_{n_k}$, and since $A_{n_k} \subseteq X_{\langle Y_1 \cap [n_0,n_1),\ldots,Y_1 \cap [n_{k-1},n_k) \rangle}$ by the definition of A_{n_k} (see (3.5) and use the fact that $k \le n_k$ and $Y_1 \cap [n_j, n_{j+1}) \subseteq n_k \; \forall j \in k$), the sequence (3.8) would be a branch through $\mathcal{T}$ and the proof would be finished.

But (3.9) will not be true in general. So what to do now? We would like to thin out Y_1 in a way that it has no elements within the interval $[n_k, n_{k+1})$ any more, but continues to be in the filter $\mathcal{F}$. Of course, this is not possible for (almost) all $k \in \omega$ (the resulting set would be finite and therefore not in $\mathcal{F}$), just for infinitely many $k \in \omega$. Once we have modified Y_1 like that (resulting in a set $Y \in \mathcal{F}$), we get a branch through $\mathcal{T}$ by splitting it up into parts ranging over several intervals of the form $[n_k, n_{k+1})$, each of them having no elements within the first interval. For instance, if such a part happens to range over $[n_k, n_{k+4})$, then $Y \cap [n_k, n_{k+4}) = Y \cap [n_{k+1}, n_{k+4}) \subseteq Y \setminus n_{k+1} \subseteq A_{n_k}$, which enables us to finish the proof.

To thin out Y_1 appropriately, we use our assumption that $\mathcal{F}$ is unbounded (see Lemma 3.5 (2)); after all, we have not used it so far. From our increasing sequence $\langle n_k : \; k \in \omega \rangle$ we get a set $Y_2 \in \mathcal{F}$ such that $Y_2 \cap [n_k, n_{k+1}) = \emptyset$ for infinitely many $k \in \omega$, i.e., if $\langle k_\ell : \; \ell \in \omega \rangle$ is the sequence of these k's, we have

$$\forall \ell \in \omega \quad Y_2 \cap [n_{k_\ell}, n_{k_\ell + 1}) = \emptyset \tag{3.10}$$

Now let $Y := Y_1 \cap Y_2$. Then $Y \in \mathcal{F}$, and since $Y \subseteq Y_1$ and $Y \subseteq Y_2$, we can replace both Y_1 and Y_2 by Y in (3.7) and (3.10), respectively:

$$\forall k \in \omega : \quad Y \setminus n_{k+1} \subseteq A_{n_k} \tag{3.11}$$

$$\forall \ell \in \omega \quad Y \cap [n_{k_\ell}, n_{k_\ell + 1}) = \emptyset \tag{3.12}$$

Now let's split up Y to get a branch $\langle s_\ell : \; \ell \in \omega \rangle$ through $\mathcal{T}$. For each $\ell \in \omega$ define

$$s_\ell := Y \cap [n_{k_\ell}, n_{k_{\ell+1}}) \tag{3.13}$$

Note that the union of the s_ℓ's equals Y (except for finitely many elements below n_{k_0}), so the union of the branch is in the filter:

$$\bigcup_{\ell \in \omega} s_\ell = Y \setminus n_{k_0} = Y \cap \underbrace{(\omega \setminus n_{k_0})}_{\in \, \mathfrak{Fr} \subseteq \mathcal{F}} \in \mathcal{F} \tag{3.14}$$

It remains to show that $\langle s_\ell : \; \ell \in \omega \rangle$ is indeed a branch through $\mathcal{T}$.

We will show that $\langle s_0, \ldots, s_{\ell-1} \rangle \in \mathcal{T}$ for each $\ell \in \omega$, by induction on ℓ. For $\ell = 0$, $\langle s_0, \ldots, s_{\ell-1} \rangle = \langle\ \rangle$, which is in $\mathcal{T}$. So assume that $\langle s_0, \ldots, s_{\ell-1} \rangle \in \mathcal{T}$ for some $\ell \in \omega$. We have to show that $\langle s_0, \ldots, s_{\ell-1}, s_\ell \rangle \in \mathcal{T}$. Using the definition of an $\mathcal{F}$-tree of finite sets (see Definition 3.9), this is certainly true if

$$s_\ell \subseteq X_{\langle s_0, \ldots, s_{\ell-1} \rangle} \tag{3.15}$$

We claim that actually $s_\ell \subseteq A_{n_{k_\ell}}$ and $A_{n_{k_\ell}} \subseteq X_{\langle s_0, \ldots, s_{\ell-1} \rangle}$. Using (3.11) and the fact that Y contains no elements "in the lowermost part" $[n_{k_\ell}, n_{k_\ell}+1)$ of the interval $[n_{k_\ell}, n_{k_{\ell+1}})$ (see (3.12)), it's easy to see that $s_\ell \subseteq A_{n_{k_\ell}}$:

$$s_\ell \stackrel{(3.13)}{=} Y \cap [n_{k_\ell}, n_{k_{\ell+1}}) \stackrel{(3.12)}{=} Y \cap [n_{k_\ell}+1, n_{k_{\ell+1}}) \subseteq Y \setminus n_{k_\ell}+1 \stackrel{(3.11)}{\subseteq} A_{n_{k_\ell}}$$

Finally, $A_{n_{k_\ell}} \subseteq X_{\langle s_0, \ldots, s_{\ell-1} \rangle}$, which follows from the definition of the A_k's (see (3.5)):

$$A_{n_{k_\ell}} = \bigcap \left\{ X_{\langle s_0, \ldots, s_{i-1} \rangle} : \langle s_0, \ldots, s_{i-1} \rangle \in \mathcal{T},\ \ i \leq n_{k_\ell},\ \ \forall j \in i \ \ s_j \subseteq n_{k_\ell} \right\}$$

and $X_{\langle s_0, \ldots, s_{\ell-1} \rangle}$ contributes to this intersection, since $\langle s_0, \ldots, s_{\ell-1} \rangle \in \mathcal{T}$ by the induction hypothesis, $\ell \leq n_{k_\ell}$ (note that both $\langle n_k : k \in \omega \rangle$ and $\langle k_\ell : \ell \in \omega \rangle$ are strictly increasing) and $s_j \subseteq n_{k_\ell}$ for each $j \in \ell$ (see (3.13)). So the induction is complete and $\langle s_\ell : \ell \in \omega \rangle$ is indeed a branch through $\mathcal{T}$ (whose union is in $\mathcal{F}$), which finishes our proof. $\square$

3.4 Proper forcing

The notion of "proper forcing" is due to Shelah (see also his book on the topic, [13]). There are several equivalent definitions of properness, e.g., in terms of stationary sets, or (countable) elementary submodels, ...

We would like to define properness by a characterization which makes use of an infinite game. For details, see Goldstern's article "A Taste of Proper Forcing" ([7]):

Definition 3.11 (Proper game)**.** Let $\mathbb{P}$ be a forcing notion, and let $p \in \mathbb{P}$. The *proper game* (for $\mathbb{P}$, below p) is a countable game defined as follows:

In the n-th move ($n \in \omega$),

- Player I plays a $\mathbb{P}$-name $\dot{\alpha}_n$ for an ordinal (below p, i.e., $p \Vdash_{\mathbb{P}} \dot{\alpha}_n \in Ord$)
- Player II responds with a countable set of ordinals $B_n \subseteq Ord$.

After these ω many moves, Player II wins the game if there is a stronger condition $q \leq p$ such that

$$q \Vdash_{\mathbb{P}} \forall n \in \omega \ \exists k \in \omega : \ \dot{\alpha}_n \in B_k.$$

Definition 3.12 (Proper forcing)**.** A forcing notion $\mathbb{P}$ is called *proper* if for all $p \in \mathbb{P}$, Player II has a winning strategy in the proper game (below p).

It is quite easy to show that properness is a generalization of both the countable chain condition (c.c.c.) and of being σ-closed.

Fact 3.13. *Let $\mathbb{P}$ be a σ-closed forcing notion. Then $\mathbb{P}$ is proper.*

Proof. Let $p \in \mathbb{P}$. Note that for each $p' \leq p$ and each ordinal name $\dot{\alpha}$ below p, there is a condition $q \leq p'$ "deciding" $\dot{\alpha}$, i.e., there is $\beta \in Ord$ such that $q \Vdash \dot{\alpha} = \beta$.

Player II has the following winning strategy for the proper game: while Player I plays the ordinal names $\dot{\alpha}_n$ below p, she will construct a decreasing sequence $\langle p_n : n \in \omega \rangle$ of conditions ($p \geq p_0 \geq p_1 \geq \ldots$) together with a sequence of ordinals $\langle \beta_n : n \in \omega \rangle$ such that for each $n \in \omega$, $p_n \Vdash \dot{\alpha}_n = \beta_n$ (responding with the singleton $B_n := \{\beta_n\}$ in each move).

Since $\mathbb{P}$ is σ-closed, the above sequence of conditions has a lower bound, i.e., there is a condition q with $q \leq p_n$ for each $n \in \omega$; so

$$q \Vdash \forall n \in \omega : \ \dot{\alpha}_n = \beta_n \in B_n,$$

hence $\mathbb{P}$ is proper. □

Fact 3.14. *Let $\mathbb{P}$ be a forcing notion with the countable chain condition. Then $\mathbb{P}$ is proper.*

Proof. Let $p \in \mathbb{P}$. Since $\mathbb{P}$ is c.c.c., for each ordinal name $\dot{\alpha}_n$ (played by I), there is a countable set of ordinals B_n such that $p \Vdash \dot{\alpha}_n \in B_n$; this can be seen as follows: the set

$$D := \{q \leq p : \ \exists \beta_q \in Ord : \ \ q \Vdash \dot{\alpha}_n = \beta_q\} \tag{3.16}$$

is obviously open dense below p (see also the proof of the previous fact); if $A \subseteq D$ is a maximal antichain within D, then p forces $\dot{\alpha}_n$ to be in the set

$$B_n := \{\beta_q : \ q \in A\},$$

which is (by the c.c.c.) countable.

So $\mathbb{P}$ is proper, since Player II has the following winning strategy for the proper game: for each ordinal name $\dot{\alpha}_n$ played by Player I, he responds with the above set B_n; in the end, Player II has won the game, since p itself forces $\dot{\alpha}_n \in B_n$ for each $n \in \omega$. □

It is also well-known that neither c.c.c. forcings nor σ-closed forcings collapse ω_1. As a matter of fact, even the following holds:

Fact 3.15. *Let $\mathbb{P}$ be a σ-closed forcing and G a generic filter over $\mathbf{V}$. Then the following holds: If $f \in \mathbf{V}[G]$ is a function from ω to $\mathbf{V}$, then $f \in \mathbf{V}$.*

In particular, there are no new countable sets of ordinals in $\mathbf{V}[G]$.

Proof (Sketch). The proof is like in Fact 3.13. Let $\dot{f}$ be a name for f; if $f \notin \mathbf{V}$, then there is a condition p forcing it. In $\mathbf{V}$, construct a decreasing sequence of conditions $p \geq p_0 \geq p_1 \geq \ldots$, such that for each $n \in \omega$, p_n decides $\dot{f}(n)$, yielding a function $g \in \mathbf{V}$ with $p_n \Vdash \dot{f}(n) = g(n)$; now choose q below every p_n, then q will force $\dot{f} = g$, but $g \in \mathbf{V}$, a contradiction. $\square$

Therefore, σ-closed forcings do not collapse ω_1: if there were a function $f \in \mathbf{V}[G]$, $f : \omega \to \omega_1^{\mathbf{V}}$ cofinal, then f would be in $\mathbf{V}$, a contradiction.

Fact 3.16. *Let $\mathbb{P}$ be a c.c.c. forcing notion and G a generic filter over $\mathbf{V}$. If $f \in \mathbf{V}[G]$ is a function from ω to $\mathbf{V}$, then there is a function $F \in \mathbf{V}$ such that for each $n \in \omega$, $f(n) \in F(n)$, and every $F(n)$ is countable. (Figuratively speaking, the values of the function in $\mathbf{V}[G]$ can be "simultaneously captured" by countable sets in $\mathbf{V}$.)*

In particular, every countable set of ordinals in $\mathbf{V}[G]$ is covered by a countable set from $\mathbf{V}$.

Proof (Sketch). The proof is like in Fact 3.14. Let $\dot{f}$ be a name for f; working in $\mathbf{V}$, one can find, for each $\dot{f}(n)$, a countable maximal antichain within the dense set of conditions deciding the value of f at n (like in (3.16)), yielding a countable set $F(n)$ of possible values. The resulting function $F \in \mathbf{V}$ is as desired.

(Note that the proof also works if f is a function from some cardinal κ to $\mathbf{V}$.) $\square$

Therefore, a c.c.c forcing does not collapse ω_1: if there were a function $f \in \mathbf{V}[G]$, $f : \omega \to \omega_1^{\mathbf{V}}$ cofinal, the set $\bigcup_{n\in\omega} F(n) \in \mathbf{V}$ would be countable in $\mathbf{V}$, but unbounded in $\omega_1^{\mathbf{V}}$, a contradiction. (Of course, higher cardinals and cofinalities are preserved as well, for the same reason.)

Proper forcings preserve ω_1

Fact 3.17. *Let $\mathbb{P}$ be a proper forcing notion and G a generic filter over $\mathbf{V}$. If $f \in \mathbf{V}[G]$ is a function from ω to Ord, then there is a (in $\mathbf{V}$) countable set $B \in \mathbf{V}$ such that $\{f(n) : n \in \omega\} \subseteq B$.*

In other words: every countable set of ordinals in $\mathbf{V}[G]$ is covered by a countable set from $\mathbf{V}$.

Proof. Let $f \in \mathbf{V}[G]$, $f : \omega \to Ord$, and let $\dot{f}$ be a name for f. Assume towards a contradiction that there is no countable set in $\mathbf{V}$ containing $\{f(n) : n \in \omega\}$, and fix a condition $p \in \mathbb{P}$ forcing this.

Now work in $\mathbf{V}$. Since $p \Vdash \dot{f} : \omega \to Ord$, for each $n \in \omega$, there is an ordinal name $\dot{\alpha}_n$ with $p \Vdash \dot{f}(n) = \dot{\alpha}_n$. Since $\mathbb{P}$ is proper, Player II has a winning strategy for the proper game (below p). So let's play the proper game: Player I plays the ordinal names $\dot{\alpha}_n$, and Player II shall respond with countable sets of ordinals B_n according to her winning strategy; consequently, Player II wins the game, i.e., there is a condition $q \leq p$ stronger than p such that

$$q \Vdash \forall n \in \omega : \ \dot{\alpha}_n \in B,$$

where B is the countable union $\bigcup_{n \in \omega} B_n$. Hence $q \Vdash \{\dot{f}(n) : n \in \omega\} \subseteq B$, which is a contradiction. □

Corollary 3.18. *Assume $\mathbb{P}$ is a proper forcing notion. Then ω_1 is preserved. (More generally, the property "$\mathrm{cf}(\alpha) > \omega$" is preserved.)*

Proof. Assume ω_1 is collapsed, i.e., there is a function $f \in \mathbf{V}[G]$, $f : \omega \to \omega_1^{\mathbf{V}}$ cofinal in $\omega_1^{\mathbf{V}}$. By Fact 3.17, there is a set $B \in \mathbf{V}$ of ordinals, countable in $\mathbf{V}$, such that $\{f(n) : n \in \omega\} \subseteq B$; hence in $\mathbf{V}$, B is a countable unbounded subset of ω_1, a contradiction.

(The proof concerning cofinalities is similar.) □

Fact 3.19. *Assume $\mathbb{P}$ is a proper forcing notion, and G is a generic filter over $\mathbf{V}$.*

Let $A \in \mathbf{V}[G]$ such that $\mathbf{V}[G] \models |A| = \aleph_0$ and $A \subseteq \mathbf{V}$ (i.e., a countable set merely containing elements from the ground model $\mathbf{V}$). Then there is a set $B \in \mathbf{V}$, countable in $\mathbf{V}$, such that $B \supseteq A$ ("B covers A").

Proof (Sketch). The proof is a slight modification of the proof of Fact 3.17.

Let $A \subseteq \mathbf{V}$ be a countable set in $\mathbf{V}[G]$, with $\dot{A}$ being a name for A, and let $p \in \mathbb{P}$ be a condition forcing "there is no countable set in $\mathbf{V}$ covering $\dot{A}$".

Working in $\mathbf{V}$, we can fix a name $\dot{f}$ with $p \Vdash$ "$\dot{f} : \omega \to \dot{A}$, $\dot{f}$ bijective". So for each $n \in \omega$, $\dot{f}(n)$ is a name for a ground model element; like in (3.16), we can find a maximal antichain of conditions below p deciding $\dot{f}(n)$; we can translate such a name into an ordinal name $\dot{\alpha}_n$, using a bijection (within $\mathbf{V}$) between the set of possible values of $\dot{f}(n)$ and some set of ordinals.

These ordinal names $\dot{\alpha}_n$ are now thrown into the proper game: in the end, there is a countable set of ordinals and a condition $q \leq p$ forcing all the $\dot{\alpha}_n$ to be in this countable set; this countable set of ordinals can be retranslated (by the bijection mentioned above) into a countable set $B \in \mathbf{V}$ of ground model elements, such that $q \Vdash \{\dot{f}(n) : n \in \omega\} = \dot{A} \subseteq B$, a contradiction. □

3.5 ω^ω-bounding forcings

Definition 3.20 (ω^ω-bounding)**.** A forcing notion $\mathbb{P}$ is called *ω^ω-bounding* if (for each generic filter G) every function (in ω^ω) in the generic extension $\mathbf{V}[G]$ is dominated by some function in the ground model, i.e.,

$$\forall f \in \mathbf{V}[G] \cap \omega^\omega \;\; \exists g \in \mathbf{V} \cap \omega^\omega : \quad \forall n \in \omega \;\; f(n) \leq g(n).$$

Remark. Note that we could have written $f \leq^\star g$ instead of $f \leq g$ above: whenever there is a function $g \in \mathbf{V}$ which "almost dominates" f, we can alter the value of g at finitely many natural numbers, again obtaining a function in $\mathbf{V}$ which "really dominates" f.

Of course, every σ-closed forcing is ω^ω-bounding: each $f \in \mathbf{V}[G] \cap \omega^\omega$ is actually in the ground model $\mathbf{V}$ (see Fact 3.15). However, there are (c.c.c.) forcings which are not ω^ω-bounding (e.g. the Cohen forcing).

ω^ω-bounding proper forcings preserve unbounded p-filters

Lemma 3.21. *Let $\mathbb{P}$ be a forcing notion, which is proper and ω^ω-bounding, and let G be a generic filter over $\mathbf{V}$.*

If $\mathcal{F} \in \mathbf{V}$ is an unbounded p-filter, then in $\mathbf{V}[G]$, the filter generated by $\mathcal{F}$ is still an unbounded p-filter.

Remark. In fact, the properness of $\mathbb{P}$ is responsible for the preservation of the p-filter property of the filter $\mathcal{F}$, whereas $\mathcal{F}$ remains unbounded because $\mathbb{P}$ is ω^ω-bounding.

Proof. Let $\mathcal{F} \in \mathbf{V}$ be an unbounded p-filter. In $\mathbf{V}[G]$, define $\widetilde{\mathcal{F}}$ to be the filter generated by $\mathcal{F}$, i.e., let

$$\widetilde{\mathcal{F}} := \{Z \subseteq \omega : \;\exists Y \in \mathcal{F} \;\; Y \subseteq Z\}^{\mathbf{V}[G]}. \tag{3.17}$$

We claim that $\mathbf{V}[G] \models$ "$\widetilde{\mathcal{F}}$ is an unbounded p-filter".

To show that $\widetilde{\mathcal{F}}$ is unbounded (see Definition 3.4), assume that there is a function $\widetilde{g} \in \omega^\omega \cap \mathbf{V}[G]$ such that

$$\forall X \in \widetilde{\mathcal{F}} : \quad f_X <^\star \widetilde{g};$$

since $\mathbb{P}$ is ω^ω-bounding, there is a function $g \in \omega^\omega \cap \mathbf{V}$ with $\widetilde{g}(n) \leq g(n)$ for each $n \in \omega$, hence

$$\mathbf{V} \models \;\; \forall X \in \mathcal{F} : \;\; f_X <^\star g,$$

contradicting the fact that $\mathbf{V} \models$ "$\mathcal{F}$ is unbounded".

To show that $\widetilde{\mathcal{F}}$ is a p-filter, let $\{Z_n : n \in \omega\} \subseteq \widetilde{\mathcal{F}}$ be some countable collection of filter sets (in $\mathbf{V}[G]$); we shall find a pseudo-intersection within the filter, i.e., a set $X \in \widetilde{\mathcal{F}}$ with $X \subseteq^\star Z_n$ for each $n \in \omega$.

By (3.17), we can choose (still in $\mathbf{V}[G]$) a set $\{Y_n : n \in \omega\} \subseteq \mathcal{F}$ such that $Y_n \subseteq Z_n$ for each n (note that $\{Y_n : n \in \omega\} \subseteq \mathbf{V}$, but the set will not be an element of $\mathbf{V}$ in general). Now we can apply Fact 3.19 to cover this set by a countable set from $\mathbf{V}$: let $B \in \mathbf{V}$ be countable in $\mathbf{V}$ with $\{Y_n : n \in \omega\} \subseteq B$ (we can assume that $B \subseteq \mathcal{F}$: just replace B by $B \cap \mathcal{F}$). So in $\mathbf{V}$, B is a countable collection of filter sets; since $\mathcal{F}$ is a p-filter in $\mathbf{V}$, B has a pseudo-intersection within $\mathcal{F}$, i.e., there is a filter set $X \in \mathcal{F}$ such that $X \subseteq^\star Y$ for every $Y \in B$; so $X \in \widetilde{\mathcal{F}}$ and $X \subseteq^\star Y_n \subseteq Z_n$ for each $n \in \omega$, as desired. □

3.6 $\mathbb{P}(\mathcal{F})$ is ω^ω-bounding and proper

We are going to show now that our forcing $\mathbb{P}(\mathcal{F})$ is proper and ω^ω-bounding, provided that $\mathcal{F}$ is an unbounded p-filter.

Recall that whenever p is a condition which forces $\dot{n} \in \omega$, we can find a stronger condition $q \leq p$ deciding the value of $\dot{n}$ in the extension:

$$p \Vdash \dot{n} \in \omega \implies \exists q \leq p \ \exists m \in \omega : \ q \Vdash \dot{n} = m. \tag{3.18}$$

To show, e.g., that $\mathbb{P}(\mathcal{F})$ is ω^ω-bounding, we could try to proceed as follows: whenever $p \in \mathbb{P}(\mathcal{F})$ is a condition with

$$p \Vdash \dot{f} \in \omega^\omega \ \wedge \ \nexists g \in \mathbf{V} \cap \omega^\omega : \dot{f} \leq g,$$

we can build a decreasing sequence of conditions $p \geq q_0 \geq q_1 \geq q_2 \geq \ldots$, deciding each of the values $\dot{f}(0), \dot{f}(1), \dot{f}(2), \ldots$ (i.e., for some function $g \in \mathbf{V}$, $q_n \Vdash \dot{f}(n) = g(n)$); after these ω-many steps, we would appreciate to have a condition q stronger than all the q_n's, since then $q \Vdash \dot{f} = g$, contradiction. Of course, this does not work; the problem is that we cannot expect to get such a lower bound q in general, since $\mathrm{dom}(q) \supseteq \bigcup_{n\in\omega} \mathrm{dom}(q_n)$ is required to be in the ideal $\mathcal{F}^\star$ (i.e., q still undefined on a filter set), but we could even end up with $\mathrm{dom}(q) = \omega$. (Of course, σ-closed forcings are shown to be ω^ω-bounding and proper like that, see Fact 3.15 and Fact 3.13, but $\mathbb{P}(\mathcal{F})$ is not σ-closed.)

So we will try to prevent the domains of the q_n from growing to fast while constructing the decreasing sequence $p \geq q_0 \geq q_1 \geq q_2 \geq \ldots$; in fact, we were too ambitious above: we do not need to "precisely decide" each $\dot{f}(n)$ to find a ground model function dominating $\dot{f}$; given $\dot{f}(n)$, it's sufficient to look for a finite set $H \subseteq \omega$ – instead of a single number – and a condition q_n

with $q_n \Vdash \dot{f}(n) \in H$. With this, we are able to ensure at each step that the partial function $q_n \leq q_{n-1}$ remains undefined on any prescribed finite subset of ω (provided q_{n-1} was undefined there), as the following lemma shows; the game theoretical characterization of unbounded p-filters (see Theorem 3.8) will in turn guarantee that it is possible to choose these finite sets in such a way that we finally obtain a lower bound $q \in \mathbb{P}(\mathcal{F})$ with small domain.

Lemma 3.22. *Let $p \in \mathbb{P}(\mathcal{F})$. Suppose $p \Vdash \dot{n} \in \omega$, and $s \subseteq \omega$ is some finite set disjoint from* $\mathrm{dom}(p)$, *i.e., $s \in [\omega \setminus \mathrm{dom}(p)]^{<\omega}$.*

Then there exists a stronger condition $q \leq p$, $q \in \mathbb{P}(\mathcal{F})$, and a finite set $H \subseteq \omega$ such that $\mathrm{dom}(q) \cap s = \emptyset$ *and*

$$q \Vdash \dot{n} \in H.$$

Proof. Recall the definition of the forcing $\mathbb{P}(\mathcal{F})$ (see Definition 3.1); in particular, note that $q \leq p$ ("q is stronger than p") if $q \supseteq p$. Let $p \in \mathbb{P}(\mathcal{F})$ with $p \Vdash \dot{n} \in \omega$, and let $s \subseteq \omega$ be a finite set with $\mathrm{dom}(p) \cap s = \emptyset$. The size of the finite set s is denoted by $|s|$.

Let $\langle t_i : i < 2^{|s|} \rangle$ be an enumeration of ${}^{s}2$, i.e., an enumeration of the $2^{|s|}$-many functions from s to 2. We claim that it is possible to construct a sequence of conditions $p \geq q_0 \geq q_1 \geq q_2 \geq \ldots \geq q_{2^{|s|}-1}$ in $\mathbb{P}(\mathcal{F})$ (each of them undefined on s, i.e., $\mathrm{dom}(q_i) \cap s = \emptyset$ for each i), together with a sequence of natural numbers $n_0, n_1, n_2, \ldots, n_{2^{|s|}-1}$ such that for each $i < 2^{|s|}$

$$q_i \cup t_i \Vdash \dot{n} = n_i. \tag{3.19}$$

Note that $q_i \cup t_i$ is indeed a condition in $\mathbb{P}(\mathcal{F})$, because first of all it is a partial function from ω to 2 since both q_i and t_i are partial functions from ω to 2 and their domains are disjoint ($\mathrm{dom}(t_i) = s$ and $\mathrm{dom}(q_i) \cap s = \emptyset$). Secondly, $q_i \in \mathbb{P}(\mathcal{F})$, so $\mathrm{dom}(q_i) \in \mathcal{F}^\star$, and $\mathrm{dom}(q_i \cup t_i) = \mathrm{dom}(q_i) \cup s \in \mathcal{F}^\star$ since s is finite and $\mathcal{F} \supseteq \mathfrak{Fr}$.

How can we build such sequences? Start with $p \in \mathbb{P}(\mathcal{F})$; $\mathrm{dom}(p) \cap s = \emptyset$, so $p \cup t_0 \leq p$, hence $p \cup t_0 \Vdash \dot{n} \in \omega$. Using (3.18), we get a $n_0 \in \omega$ and a condition $r \leq p \cup t_0$ such that $r \Vdash \dot{n} = n_0$. Now let

$$q_0 := r \setminus t_0 = r \restriction (\omega \setminus s).$$

Then $\mathrm{dom}(q_0) \cap s = \emptyset$, and (since $r = q_0 \cup t_0 \leq p \cup t_0$) $q_0 \leq p$ and $q_0 \cup t_0 \Vdash \dot{n} = n_0$ (yielding (3.19) for $i = 0$).

In general (for $1 \leq i < 2^{|s|}$), having already constructed $q_{i-1} \leq p$, consider $q_{i-1} \cup t_i \leq p$ which forces $\dot{n} \in \omega$. Again using (3.18), choose $r \leq q_{i-1} \cup t_i$ and $n_i \in \omega$ with $r \Vdash \dot{n} = n_i$. Define

$$q_i := r \setminus t_i = r \restriction (\omega \setminus s).$$

Then $q_i \leq q_{i-1}$, $\mathrm{dom}(q_i) \cap s = \emptyset$ and $q_i \cup t_i \Vdash \dot{n} = n_i$, thereby establishing (3.19).

Now we can finish the proof of the lemma as follows: Let q be the strongest condition of the constructed sequence and let H be the set of all n_i:

$$q := q_{2^{|s|}-1} \qquad H := \{n_i :\ i < 2^{|s|}\}$$

It's clear that $q \in \mathbb{P}(\mathcal{F})$, $q \leq p$, $H \subseteq \omega$ is finite, and $\mathrm{dom}(q) \cap s = \emptyset$. It remains to show that $q \Vdash \dot{n} \in H$.

Let's assume G is a generic filter through $\mathbb{P}(\mathcal{F})$ with $q \in G$. We claim that $\dot{n}[G] \in H$. Recall that $\mathrm{dom}(\bigcup G) = \omega$ (see (3.1) on page 49), so $(\bigcup G \restriction s) \in G$ (since s is finite); fix $i < 2^{|s|}$ with $(\bigcup G \restriction s) = t_i$; because $q \in G$ and $q \leq q_i$, also $q_i \in G$, and together with $t_i \in G$ we get $q_i \cup t_i \in G$. But the condition $q_i \cup t_i$ forces $\dot{n} = n_i$ (see (3.19)), so $\dot{n}[G] = n_i \in H$, which finishes the proof of the lemma. □

We now use our game theoretical characterization of unbounded p-filters to show that $\mathbb{P}(\mathcal{F})$ is indeed ω^ω-bounding:

Lemma 3.23. *Let $\mathcal{F}$ be an unbounded p-filter. Then $\mathbb{P}(\mathcal{F})$ is ω^ω-bounding.*

Proof. First of all, let's restate Lemma 3.22 in the following way:

Whenever $\dot{n}$, p and s satisfy $p \Vdash \dot{n} \in \omega$ and $s \in [\omega \setminus \mathrm{dom}(p)]^{<\omega}$, the lemma gives us a condition $q = q(\dot{n}, p, s) \leq p$ and a finite set $H = H(\dot{n}, p, s) \subseteq \omega$ such that $\mathrm{dom}(q) \cap s = \emptyset$ and $q \Vdash \dot{n} \in H$.

Assume (towards a contradiction) there is a function $f \in \mathbf{V}[G] \cap \omega^\omega$ not being dominated by any ground model function, $\dot{f}$ a name for it, and $p \in \mathbb{P}(\mathcal{F})$ a condition forcing all this (in particular, $p \Vdash \dot{f}(n) \in \omega$ for each n). Recall that whenever p is a condition in $\mathbb{P}(\mathcal{F})$, the set $\omega \setminus \mathrm{dom}(p)$ is in $\mathcal{F}$.

We play the p-filter game (see Definition 3.7): Player II freely chooses finite subsets of the filter sets played by I, whereas Player I sticks to a strategy definable from the above operation $q(\cdot,\cdot,\cdot)$ together with $\dot{f}$.

Player I begins and plays $X_0 := \omega \setminus \mathrm{dom}(p) \in \mathcal{F}$. Player II responds with some finite $s_0 \subseteq X_0$. Let $q_0 := q(\dot{f}(0), p, s_0) \leq p$ and let $H_0 := H(\dot{f}(0), p, s_0)$; then Player I plays $X_1 := \omega \setminus \mathrm{dom}(q_0) \in \mathcal{F}$. Player II again chooses some finite $s_1 \subseteq X_1$. Note that $\mathrm{dom}(q_0) \cap s_0 = \emptyset$, i.e., $s_0 \subseteq X_1$, hence also $s_0 \cup s_1$ is a finite subset of $X_1 = \omega \setminus \mathrm{dom}(q_0)$; so we are allowed to define

$$q_1 := q(\dot{f}(1), q_0, s_0 \cup s_1) \leq q_0 \quad \text{and} \quad H_1 := H(\dot{f}(1), q_0, s_0 \cup s_1).$$

Again, Player I plays $X_2 := \omega \setminus \mathrm{dom}(q_1) \in \mathcal{F}$, and Player II chooses some finite $s_2 \subseteq X_2$. The finite set $s_0 \cup s_1 \cup s_2$ will again be a subset of $X_2 = \omega \setminus \mathrm{dom}(q_1)$, so we can go on in this manner ...

After ω many steps, we have obtained a decreasing sequence of conditions $p \geq q_0 \geq q_1 \geq q_2 \geq \ldots$ together with a sequence of finite sets $H_0, H_1, H_2, \ldots$, such that

$$\forall n \in \omega : \quad q_n \Vdash \dot{f}(n) \in H_n. \tag{3.20}$$

Note that all the q_n and H_n are actually dependent on the finite sets chosen by Player II, i.e., H_2, e.g., should rather be denoted by $H_{\langle s_0, s_1, s_2 \rangle}$, ...

Since $\mathcal{F}$ is an unbounded p-filter, by Theorem 3.8, there is *no* winning strategy for Player I. Consequently, Player II had a chance to win, i.e., the sequence of finite sets $s_0, s_1, s_2, \ldots$ can be chosen such that $\bigcup_{n \in \omega} s_n \in \mathcal{F}$. But then the corresponding sequence of conditions $p \subseteq q_0 \subseteq q_1 \subseteq q_2 \ldots$ has a "lower" bound *in* $\mathbb{P}(\mathcal{F})$: in fact, $q := \bigcup_{n \in \omega} q_n$ is such a condition since $\mathrm{dom}(q) \subseteq (\omega \setminus \bigcup_{n \in \omega} s_n) \in \mathcal{F}^\star$; this is because $(\bigcup_{i \leq n} s_i) \cap \mathrm{dom}(q_n) = \emptyset$ for each $n \in \omega$ due to the construction.

Now we can define a function $g \in \mathbf{V} \cap \omega^\omega$ by

$$\forall n \in \omega \quad g(n) := \max(H_n).$$

Then $q \leq q_n$ for each n together with (3.20) yields

$$q \Vdash \forall n \in \omega : \quad \dot{f}(n) \leq g(n),$$

which is a contradiction, thus finishing the proof of the lemma. □

Now we are going to show that the forcing $\mathbb{P}(\mathcal{F})$ is proper (provided $\mathcal{F}$ is an unbounded p-filter): in fact, we shall find a winning strategy for Player II in the proper game.

Similar to (3.18), each ordinal name below p can be decided by a stronger condition q:

$$p \Vdash \dot{\alpha} \in Ord \implies \exists q \leq p \ \exists \beta \in Ord : \quad q \Vdash \dot{\alpha} = \beta. \tag{3.21}$$

Therefore – analogous to Lemma 3.22 – we have the following

Lemma 3.24. *Suppose that $\dot{\alpha}$, p and s are given such that $p \Vdash \dot{\alpha} \in Ord$ and $s \in [\omega \setminus \mathrm{dom}(p)]^{<\omega}$; then we can find a condition $q = q(\dot{\alpha}, p, s) \leq p$ and a finite set $H = H(\dot{\alpha}, p, s) \subseteq Ord$ such that $\mathrm{dom}(q) \cap s = \emptyset$ and $q \Vdash \dot{\alpha} \in H$.*

Proof. Same as the proof of Lemma 3.22: just use (3.21) instead of (3.18). □

From this, we get

Lemma 3.25. *Let $\mathcal{F}$ be an unbounded p-filter. Then $\mathbb{P}(\mathcal{F})$ is proper.*

Proof. Recall that $\mathbb{P}(\mathcal{F})$ is proper if for each $p \in \mathbb{P}(\mathcal{F})$ Player II has a winning strategy in the proper game below p. So let $p \in \mathbb{P}(\mathcal{F})$. We shall define a winning strategy for Player II in the proper game: Player I will play ordinal names $\dot\alpha_n$ below p, and Player II will respond with countable sets $B_n \subseteq Ord$ according to her strategy. We will have to show that Player II wins the game, i.e., there is a condition $q \leq p$ such that

$$q \Vdash \forall n \in \omega \;\; \exists k \in \omega : \;\; \dot\alpha_n \in B_k. \tag{3.22}$$

As in the proof of Lemma 3.23, we want to make use of the game theoretical characterization of unbounded p-filters; so imagine, we are playing the p-filter game: using Lemma 3.24, we obtain a decreasing sequence of conditions $p \geq q_{\langle s_0 \rangle} \geq q_{\langle s_0, s_1 \rangle} \geq q_{\langle s_0, s_1, s_2 \rangle} \geq \dots$ and finite sets of ordinals $H_{\langle s_0 \rangle}, H_{\langle s_0, s_1 \rangle}, H_{\langle s_0, s_1, s_2 \rangle}, \dots$ such that

$$\forall n \in \omega : \quad q_{\langle s_0, \dots, s_n \rangle} \Vdash \dot\alpha_n \in H_{\langle s_0, \dots, s_n \rangle}.$$

(Proceed as in Lemma 3.23, just use $q(\dot\alpha_n, \cdot, \cdot)$ and $H(\dot\alpha_n, \cdot, \cdot)$ instead of $q(\dot f(n), \cdot, \cdot)$ and $H(\dot f(n), \cdot, \cdot)$ respectively.)

If there were a winning strategy for Player II in the p-filter game, we could directly derive a winning strategy for Player II in the proper game: in the n-th move (Player I has played $\dot\alpha_n$), Player II would respond with the (finite) set $H_{\langle s_0, \dots, s_n \rangle}$; in the end, there were a condition q forcing $\dot\alpha_n \in H_{\langle s_0, \dots, s_n \rangle}$ for each $n \in \omega$, witnessing that Player II had won the proper game.

But Player II does not have a winning strategy in the p-filter game (see also the remark concerning the p-filter game on page 54); we just know that Player I has no winning strategy (by the fact that $\mathcal{F}$ is an unbounded p-filter and Theorem 3.8). So we have to modify things to gain a winning strategy for Player II in the proper game: Player II "goes through all possible courses of the p-filter game" while playing the proper game, i.e., she actually builds the whole $\mathcal{F}$-tree $\mathcal{T}$ of finite sets, as in Lemma 3.10; in the n-th move, Player II plays the set

$$B_n := \bigcup \big\{ H_{\langle s_0, \dots, s_n \rangle} : \; \langle s_0, \dots, s_n \rangle \in \mathcal{T} \big\}$$

(note that each B_n is countable); after ω many moves, Player II has won the proper game, for the following reason: Player I has no winning strategy in the p-filter game, so we can fix a sequence $s_0, s_1, s_2, \dots$ such that $\bigcup_{n \in \omega} s_n \in \mathcal{F}$; consequently, the corresponding sequence $p \geq q_{\langle s_0 \rangle} \geq q_{\langle s_0, s_1 \rangle} \geq q_{\langle s_0, s_1, s_2 \rangle} \geq \dots$ has a lower bound $q \in \mathbb{P}(\mathcal{F})$ satisfying

$$q \Vdash \forall n \in \omega : \;\; \dot\alpha_n \in H_{\langle s_0, \dots, s_n \rangle} \subseteq B_n. \qquad \square$$

Remark. Note that we have obtained a little bit more than actually required in (3.22): for each n, $\dot{\alpha}_n$ is forced to be in exactly the set B_n, not only in B_k for some $k \in \omega$ (i.e., in the union $B = \bigcup_{k\in\omega} B_k$). The same is true for σ-closed forcings and c.c.c. forcings (see Fact 3.13 and Fact 3.14). There are also forcing notions which are proper but do not satisfy this stronger property (see Jech's book [8, Exercises 31.5 and 31.6]).

In fact, there is a concept due to J.Baumgartner called Axiom A, which is even stronger and still follows from being σ-closed or c.c.c. (see [8, Definition 31.10]). However, our forcing notion $\mathbb{P}(\mathcal{F})$ *does not* satisfy Axiom A (see [12] for a proof).

3.7 Forcing with $\mathbb{P}(\mathcal{F})^\omega$ "kills" $\mathcal{F}$

In this section we are going to show how to "kill an unbounded p-filter $\mathcal{F}$" (by a suitable forcing). This means that in the generic extension, $\mathcal{F}$ cannot be extended to a p-point (i.e., each ultrafilter containing $\mathcal{F}$ will fail to be a p-filter); moreover, this will remain true whenever such an extension is further extended by any ω^ω-bounding forcing notion.

For technical reasons, we consider the full-support ω-product $\mathbb{P}(\mathcal{F})^\omega$:

$$\mathbb{P}(\mathcal{F})^\omega = \prod_{j\in\omega} \mathbb{P}(\mathcal{F}).$$

In other words, a condition $p \in \mathbb{P}(\mathcal{F})^\omega$ is an ω-sequence $\langle p_j : j \in \omega\rangle$, where each component p_j is a condition in $\mathbb{P}(\mathcal{F})$; the ordering is also componentwise, i.e., $q \le p$ if and only if $q_j \le p_j$ for each $j \in \omega$. Note that this forcing will introduce an ω-sequence $\langle x_j : j \in \omega\rangle$ of new reals.

For $X \subseteq \omega \times \omega$, let $(X)_j = \{m \in \omega : \langle j, m\rangle \in X\}$. Now define the following filter on $\omega \times \omega$:

$$\overline{\mathcal{F}} := \{X \subseteq \omega \times \omega : \forall j \in \omega : (X)_j \in \mathcal{F}\}.$$

It can be easily seen that the forcing $\mathbb{P}(\mathcal{F})^\omega$ is isomorphic to the forcing $\mathbb{P}(\overline{\mathcal{F}})$: if $p \in \mathbb{P}(\overline{\mathcal{F}})$, then for each $j \in \omega$, $p \restriction (\{j\} \times \omega)$ can be viewed as a condition $p_j \in \mathbb{P}(\mathcal{F})$, so p is essentially the same as the corresponding ω-sequence $\langle p_j : j \in \omega\rangle \in \mathbb{P}(\mathcal{F})^\omega$.

Using this, $\mathbb{P}(\mathcal{F})^\omega$ can be shown to be quite similar to $\mathbb{P}(\mathcal{F})$:

Lemma 3.26. *Let $\mathcal{F}$ be an unbounded p-filter. Then the forcing $\mathbb{P}(\mathcal{F})^\omega$ is proper and ω^ω-bounding.*

Proof (Sketch). $\mathbb{P}(\mathcal{F})^\omega$ is isomorphic to $\mathbb{P}(\overline{\mathcal{F}})$; so – by Lemma 3.23 and Lemma 3.25 – it is sufficient to show that $\overline{\mathcal{F}}$ is an unbounded p-filter on $\omega \times \omega$ (provided $\mathcal{F}$ is one); note that for a filter on $\omega \times \omega$, the notion of unboundedness is well-defined because it can be shown to be independent of the ordering of ω.

To show that $\overline{\mathcal{F}}$ is a p-filter, suppose $\{A_k : k \in \omega\} \subseteq \overline{\mathcal{F}}$. For each $k \in \omega$, $A_k = \bigcup_{j\in\omega} \{j\} \times (A_k)_j$, where $(A_k)_j \in \mathcal{F}$. Since $\mathcal{F}$ is a p-filter, there is a $Y \in \mathcal{F}$ with $Y \subseteq^\star (A_k)_j$ for all $k, j \in \omega$. Let $Y_j := Y \cap \bigcap_{k\leq j} (A_k)_j \in \mathcal{F}$; then $A := \bigcup_{j\in\omega} \{j\} \times Y_j \in \overline{\mathcal{F}}$ will be the desired pseudo-intersection of the family $\{A_k : k \in \omega\}$.

To show that $\overline{\mathcal{F}}$ is unbounded, assume that $\omega \times \omega$ is ordered in such a way that for each $k \in \omega$, the square $k \times k$ consists of "the first k^2 elements of $\omega \times \omega$". It is quite easy to see that a filter is unbounded if and only if for each increasing surjective function $h \in \omega^\omega$ there is a filter set X such that there are infinitely many $k \in \omega$ with $|X \cap k| \leq h(k)$ (compare this to (3.3); here h is interesting if it "grows slowly"; such an h can be thought of as the "inverse" of a fast growing strictly increasing function g). Fix such a function h; let $\widetilde{h}$ be such that $\widetilde{h}^2(k) \leq h(k)$ for each k (i.e., $\widetilde{h}$ is even slower growing). Since $\mathcal{F}$ is unbounded, there is a $Y \in \mathcal{F}$ such that $|Y \cap k| \leq \widetilde{h}(k)$ for infinitely many k. Define

$$X := \bigcup_{j\in\omega} \{j\} \times (Y \setminus f_Y(j)) \in \overline{\mathcal{F}}$$

(i.e., $(X)_j$ is the set Y with the first j elements removed). We claim that X is the desired set, i.e.,

$$|X \cap (k \times k)| \leq h(k) \; (\leq h(k^2)\,) \tag{3.23}$$

for infinitely many $k \in \omega$. In fact, (3.23) holds whenever $|Y \cap k| \leq \widetilde{h}(k)$ (in this case $(Y \setminus f_Y(j)) \cap k = \emptyset$ for each $j \geq \widetilde{h}(k)$):

$$X \cap (k \times k) = \bigcup_{j<k} \{j\} \times ((Y \setminus f_Y(j)) \cap k) = \bigcup_{j<\widetilde{h}(k)} \{j\} \times ((Y \setminus f_Y(j)) \cap k),$$

so $|X \cap (k \times k)| \leq \widetilde{h}(k) \cdot \widetilde{h}(k) \leq h(k)$, which finishes the proof. □

Now we are going to show that "$\mathbb{P}(\mathcal{F})^\omega$ indeed kills $\mathcal{F}$":

Lemma 3.27. *Let $\mathcal{F}$ be an unbounded p-filter. Suppose G is a $\mathbb{P}(\mathcal{F})^\omega$-generic filter over $\mathbf{V}$. Let $\mathbb{P} \in \mathbf{V}[G]$ be any ω^ω-bounding notion of forcing, and let H be $\mathbb{P}$-generic over $\mathbf{V}[G]$. Then the following holds:*

$$\mathbf{V}[G][H] \models \text{"There is no p-point extending } \mathcal{F}.\text{"}$$

In other words: whenever the filter $\mathcal{F}$ is extended to an ultrafilter $\widetilde{\mathcal{F}}$ in an ω^ω-bounding forcing extension of $\mathbf{V}[G]$, $\widetilde{\mathcal{F}}$ will fail to be a p-filter.

Proof. Assume towards a contradiction that there is a p-point $\widetilde{\mathcal{F}} \in \mathbf{V}[G][H]$ with $\widetilde{\mathcal{F}} \supseteq \mathcal{F}$ (so $\widetilde{\mathcal{F}}$ is an *ultrafilter* satisfying the *p-filter* property).

Let $\langle x_j : j \in \omega \rangle \in \mathbf{V}[G]$ be the sequence of reals introduced by G. Because $\widetilde{\mathcal{F}}$ is an *ultrafilter*, for each $j \in \omega$, exactly one of the two sets

$$\{m : x_j(m) = 0\} \text{ and } \{m : x_j(m) = 1\} \tag{3.24}$$

will be in $\widetilde{\mathcal{F}}$. Let $\varepsilon \in \mathbf{V}[G][H] \cap 2^\omega$ be the (unique) function with

$$\{m : x_j(m) = \varepsilon(j)\} \in \widetilde{\mathcal{F}} \tag{3.25}$$

for each $j \in \omega$, i.e., ε tells us which of the two sets in (3.24) is selected by the ultrafilter (for a given j).

For the time being, let's assume that the function ε is not eventually constant, i.e.,

$$\forall k \in \omega \quad \exists j_0, j_1 > k : \quad \varepsilon(j_0) = 0 \text{ and } \varepsilon(j_1) = 1. \tag{3.26}$$

Define the function $f_2 \in \mathbf{V}[G][H] \cap \omega^\omega$ as follows:

$$f_2(k) := \min \{i \in \omega : \exists j \in (k, i) \ \ \varepsilon(k) = \varepsilon(j)\} \tag{3.27}$$

for each $k \in \omega$ (this is well-defined by the assumption (3.26)). Since the forcing $\mathbb{P} \in \mathbf{V}[G]$ is ω^ω-bounding, there is an $f_1 \in \mathbf{V}[G] \cap \omega^\omega$ with

$$\forall k \in \omega \quad f_2(k) \leq f_1(k),$$

and since the forcing $\mathbb{P}(\mathcal{F})^\omega \in \mathbf{V}$ is ω^ω-bounding as well (see Lemma 3.26), we can find some function $f_0 \in \mathbf{V} \cap \omega^\omega$ such that

$$\forall k \in \omega \quad k < f_2(k) \leq f_1(k) \leq f_0(k).$$

Now recursively define a strictly increasing sequence $\bar{k} = \langle k_n : n \in \omega \rangle \in \mathbf{V}$:

$$\begin{aligned} k_0 &:= 0 \\ k_{n+1} &:= f_0(k_n) \text{ for each } n \in \omega. \end{aligned}$$

Because of (3.27) and $(k_n, f_2(k_n)) \subseteq (k_n, f_0(k_n)) = (k_n, k_{n+1})$, the sequence $\bar{k} \in \mathbf{V}$ satisfies for each $n \in \omega$

$$\exists j \in (k_n, k_{n+1}) \quad \varepsilon(k_n) = \varepsilon(j). \tag{3.28}$$

In case ε happens to be eventually constant (i.e., (3.26) fails), we just pick some $k_0 \in \omega$ such that ε is constant on $\omega \setminus k_0$ and define the sequence $\bar{k} \in \mathbf{V}$ by

$$k_{n+1} := k_n + 2 \text{ for each } n \in \omega$$

instead; then (3.28) will again hold for each n.

Now define for each $n \in \omega$

$$A_n := \{m \in \omega : \exists j \in (k_n, k_{n+1}) \ x_{k_n}(m) = x_j(m)\} \in \mathbf{V}[G]. \tag{3.29}$$

Claim. *For each $n \in \omega$, the set A_n is in $\widetilde{\mathcal{F}}$.*

Proof. Let $n \in \omega$. Note that for each $j \in (k_n, k_{n+1})$,

$$A_n \supseteq \{m : x_{k_n}(m) = x_j(m)\}. \tag{3.30}$$

By (3.28), we can pick a $j \in (k_n, k_{n+1})$ with $\varepsilon(k_n) = \varepsilon(j)$. By (3.25), both $\{m : x_{k_n}(m) = \varepsilon(k_n)\}$ and $\{m : x_j(m) = \varepsilon(j)\}$ are in $\widetilde{\mathcal{F}}$, and so the same is true for its intersection

$$\{m : x_{k_n}(m) = x_j(m) = \varepsilon(j)\} \subseteq \{m : x_{k_n}(m) = x_j(m)\} \in \widetilde{\mathcal{F}},$$

which in turn implies $A_n \in \widetilde{\mathcal{F}}$ (see (3.30)). □

Because $\widetilde{\mathcal{F}}$ is a *p-filter*, the collection $\{A_n : n \in \omega\} \subseteq \widetilde{\mathcal{F}}$ has a pseudo-intersection within the filter $\widetilde{\mathcal{F}}$, i.e., there is an $X \in \widetilde{\mathcal{F}}$ with $X \subseteq^\star A_n$ for each $n \in \omega$. In other words, there is a function $\widetilde{g} \in \mathbf{V}[G][H] \cap \omega^\omega$ such that $X \subseteq A_n \cup \widetilde{g}(n)$ for each n, so

$$X \subseteq \bigcap_{n \in \omega} (A_n \cup \widetilde{g}(n)) \in \widetilde{\mathcal{F}}.$$

Like above, we can find a function $g \in \mathbf{V} \cap \omega^\omega$ which dominates $\widetilde{g}$, i.e., for each $n \in \omega$, $\widetilde{g}(n) \leq g(n)$ (since both $\mathbb{P} \in \mathbf{V}[G]$ and $\mathbb{P}(\mathcal{F})^\omega \in \mathbf{V}$ are ω^ω-bounding). Consequently, also

$$\bigcap_{n \in \omega} (A_n \cup g(n)) \in \widetilde{\mathcal{F}}. \tag{3.31}$$

From this, we are going to derive a contradiction: in fact, the following lemma tells us that it is "dense to force $\bigcap_{n \in \omega} (A_n \cup g(n))$ into the dual ideal":

Lemma 3.28. *Let $\mathcal{F} \in \mathbf{V}$ be an unbounded p-filter (and the forcing $\mathbb{P}(\mathcal{F})^\omega$ as above). Assume that*

$$\bar{k} = \langle k_n : n \in \omega \rangle \in \mathbf{V} \cap \omega^\omega$$

is a strictly increasing sequence of natural numbers and $g \in \mathbf{V} \cap \omega^\omega$ *is any function from* ω *to* ω *(both in the ground model).*

For each $n \in \omega$*, let* $\dot{A}_n$ *(depending on* $\bar{k}$*) be a name for the set* A_n *as defined in* (3.29)*, i.e.,*

$$\Vdash_{\mathbb{P}(\mathcal{F})^\omega} \dot{A}_n = \{m \in \omega : \ \exists j \in (k_n, k_{n+1}) \ \ \dot{x}_{k_n}(m) = \dot{x}_j(m)\}$$

(with $\dot{x}_j$ *being a name for* x_j*). Then the set*

$$D_{\bar{k},g} = \left\{ q \in \mathbb{P}(\mathcal{F})^\omega : \ \exists Y \in \mathcal{F}^\star \quad q \Vdash_{\mathbb{P}(\mathcal{F})^\omega} \bigcap_{n\in\omega} (\dot{A}_n \cup g(n)) \subseteq Y \right\} \tag{3.32}$$

is (open) dense and in $\mathbf{V}$*.*

Using this lemma (which we shall prove afterwards), it is easy to see how to finish the proof of Lemma 3.27: take the sequence $\bar{k} \in \mathbf{V}$ and the function $g \in \mathbf{V}$ from above, i.e., choose them in a way such that (3.31) holds (note that A_n is actually dependent on the sequence $\bar{k}$, and the generics G and H and the p-point $\widetilde{\mathcal{F}}$ are fixed right from the start); now consider the set $D_{\bar{k},g} \in \mathbf{V}$, which is dense in $\mathbb{P}(\mathcal{F})^\omega$ by Lemma 3.28; so there is a condition $q \in G \cap D_{\bar{k},g}$; by (3.32), we get a "small set" $Y \in \mathcal{F}^\star$ with

$$\bigcap_{n\in\omega} (A_n \cup g(n)) \subseteq Y,$$

contradicting (3.31) (since $\mathcal{F} \subseteq \widetilde{\mathcal{F}}$), and we are done. □

Proof of Lemma 3.28. Let $\bar{k} \in \mathbf{V} \cap \omega^\omega$ be strictly increasing and let $g \in \mathbf{V} \cap \omega^\omega$. Obviously $D_{\bar{k},g} \in \mathbf{V}$ since it is defined within $\mathbf{V}$. We are going to show that $D_{\bar{k},g}$ is dense. So fix an arbitrary condition $p \in \mathbb{P}(\mathcal{F})^\omega$; we shall find a stronger condition $q \leq p$ with $q \in D_{\bar{k},g}$.

We work in $\mathbf{V}$. By definition of $\mathbb{P}(\mathcal{F})^\omega$, $p = \langle p_j : \ j \in \omega \rangle$, and for each $j \in \omega$, $\mathrm{dom}(p_j) \in \mathcal{F}^\star$. Define

$$Y_n := \bigcup \{\mathrm{dom}(p_j) : \ j \in [k_n, k_{n+1})\}; \tag{3.33}$$

for each $n \in \omega$, $Y_n \in \mathcal{F}^\star$ because it is the finite union of sets in the ideal $\mathcal{F}^\star$. Since $\mathcal{F}$ is a p-filter, the collection $\{Y_n : \ n \in \omega\} \subseteq \mathcal{F}^\star$ has a "pseudo-union" in the ideal $\mathcal{F}^\star$, i.e., there is a $Y \in \mathcal{F}^\star$ with $Y_n \subseteq^\star Y$ for each $n \in \omega$; so there is an $h \in \omega^\omega$ such that

$$\forall n \in \omega \quad Y_n \setminus h(n) \subseteq Y. \tag{3.34}$$

Note that w.l.o.g. we can assume that $h \in \omega^\omega$ is strictly increasing and satisfies

$$\forall n \in \omega \quad g(n) \leq h(n); \tag{3.35}$$

moreover, let's assume that $[0, h(0)) \subseteq Y$ ($\mathcal{F}^\star$ contains each finite set, so just add this finite interval to Y).

Now let us define a condition $q = \langle q_j : j \in \omega \rangle$ as follows. If $j < k_0$, let $q_j := p_j$. Otherwise, there is some $n \in \omega$ with $j \in [k_n, k_{n+1})$: let

$$\mathrm{dom}(q_j) = \mathrm{dom}(p_j) \cup [h(n), h(n+1))$$

and for each $m \in \mathrm{dom}(q_j)$ let

$$q_j(m) = \begin{cases} p_j(m) & \text{if } m \in \mathrm{dom}(p_j) \\ 1 & \text{if } j = k_n \text{ and } m \in [h(n), h(n+1)) \setminus \mathrm{dom}(p_j) \\ 0 & \text{if } j > k_n \text{ and } m \in [h(n), h(n+1)) \setminus \mathrm{dom}(p_j) \end{cases} \tag{3.36}$$

Note that for each $j \in \omega$, $\mathrm{dom}(q_j) \in \mathcal{F}^\star$ and q_j is a function, so q is a condition in $\mathbb{P}(\mathcal{F})^\omega$. Since $q_j \supseteq p_j$ for each $j \in \omega$, q is stronger than p. It remains to show that q is indeed in $D_{\bar{k},g}$.

Let G be a generic filter for the forcing $\mathbb{P}(\mathcal{F})^\omega$ with $q \in G$. We have to show that

$$\bigcap_{n \in \omega} (A_n \cup g(n)) \subseteq Y.$$

So let $m \in \bigcap_{n \in \omega}(A_n \cup g(n))$. Either $m \in [0, h(0))$ – then we are done since $[0, h(0)) \subseteq Y$ – or we can fix $n \in \omega$ with $m \in [h(n), h(n+1))$. In case that $m \in Y_n$, the proof is finished: $m \in Y_n \setminus h(n)$, so $m \in Y$ by (3.34). Otherwise, for each $j \in [k_n, k_{n+1})$, $m \notin \mathrm{dom}(p_j)$ (see (3.33), the definition of the set Y_n). But then for each $j \in (k_n, k_{n+1})$

$$1 = q_{k_n}(m) \neq q_j(m) = 0$$

due to the definition of q (see (3.36)). Since $q \in G$, we have

$$\forall j \in (k_n, k_{n+1}) \;\; x_{k_n}(m) \neq x_j(m),$$

which implies $m \notin A_n$ (see (3.29)); moreover $m \in [h(n), h(n+1))$ together with (3.35) gives $m \notin g(n)$, contradicting $m \in \bigcap_{n \in \omega}(A_n \cup g(n))$. □

3.8 Killing all p-points by iterating $\mathbb{P}(\mathcal{F})^\omega$

Now we are going to iterate forcings of the form $\mathbb{P}(\mathcal{F})^\omega$ to obtain a model of ZFC with no p-point. The procedure is a typical application of a countable support iteration.

We start with a ground model satisfying GCH and define a countable support iteration of length ω_2 such that for every unbounded p-filter $\mathcal{F}$ which is in the ground model or turns up at some intermediate stage, the forcing $\mathbb{P}(\mathcal{F})^\omega$ (which is proper and ω^ω-bounding) is applied in the course of the iteration; by Lemma 3.27, the unbounded p-filter $\mathcal{F}$ is "killed" by $\mathbb{P}(\mathcal{F})^\omega$, in the sense that $\mathcal{F}$ cannot be extended to a p-point anymore. We will prove that each p-point in the final model contains an unbounded p-filter of some intermediate model; this shows that the existence of p-points in the final model is impossible.

Countable support iteration

First of all, we recall the notion of a countable support iteration and review some important facts:

Definition 3.29. Let δ be an ordinal. By induction on $\alpha < \delta$, we define the notion of a countable support iteration.

The sequence $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle \in \mathbf{V}$ is a *countable support iteration* of length δ if for all $\alpha < \delta$,

1. $\mathbb{P}_\alpha = \lim_{\text{count}} \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle$
2. $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is a forcing notion,

where $\lim_{\text{count}} \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ is the set of all partial functions p on α such that $\mathrm{dom}(p)$ is *countable* and

$$\forall \beta \in \mathrm{dom}(p) \subseteq \alpha : \quad p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \dot{\mathbb{Q}}_\beta;$$

for $p, q \in \mathbb{P}_\alpha$ we define

$$q \leq_{\mathbb{P}_\alpha} p \iff \forall \beta \in \mathrm{dom}(q) \cup \mathrm{dom}(p) : \quad q \restriction \beta \Vdash_{\mathbb{P}_\beta} q(\beta) \leq_{\dot{\mathbb{Q}}_\beta} p(\beta),$$

where we let $p(\beta)$ be the largest condition of $\dot{\mathbb{Q}}_\beta$ in case that $\beta \notin \mathrm{dom}(p)$ (and the same with q).

We define $\mathbb{P}_\delta = \lim_{\text{count}} \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$.

Fact 3.30. *The map which sends $p \in \mathbb{P}_{\alpha+1}$ to $\langle p \restriction \alpha, p(\alpha) \rangle$ is an isomorphism between $\mathbb{P}_{\alpha+1}$ and $\mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha$; so $\mathbb{P}_{\alpha+1}$ is actually a composition of two forcings:*

1. $p \in \mathbb{P}_{\alpha+1} \iff p \restriction \alpha \in \mathbb{P}_\alpha \;\wedge\; p \restriction \alpha \Vdash_{\mathbb{P}_\alpha} p(\alpha) \in \dot{\mathbb{Q}}_\alpha$
2. *for* $p, q \in \mathbb{P}_{\alpha+1}$,

$$q \leq_{\mathbb{P}_{\alpha+1}} p \iff q \restriction \alpha \leq_{\mathbb{P}_\alpha} p \restriction \alpha \;\wedge\; q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} q(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} p(\alpha).$$

If α is a limit ordinal, then

1. $p \in \mathbb{P}_\alpha \iff \mathrm{dom}(p) \subseteq \alpha$ *is countable* $\wedge \ \forall \beta < \alpha : \ p \restriction \beta \in \mathbb{P}_\beta$
2. *for* $p, q \in \mathbb{P}_\alpha$,

$$q \leq_{\mathbb{P}_\alpha} p \iff \forall \beta < \alpha : \ q \restriction \beta \leq_{\mathbb{P}_\beta} p \restriction \beta.$$

Proof. See [6, Fact 1.8 and 1.7]. □

So there are two kinds of limits in a countable support iteration:

- If $\mathrm{cf}(\alpha) > \omega$, then $\mathbb{P}_\alpha$ is just the union of all the $\mathbb{P}_\beta$ with $\beta < \alpha$ (since the domain of each condition in $\mathbb{P}_\alpha$ is bounded below α); such a limit is called a *direct limit*.
- If $\mathrm{cf}(\alpha) = \omega$, then new conditions appear at α: $p \in \mathbb{P}_\alpha$ if and only if for each $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$; such a limit is called an *inverse limit*.

Fact 3.31. *Assume $G \subseteq \mathbb{P}_\delta$ is a $\mathbb{P}_\delta$-generic filter over* $\mathbf{V}$*, and let*

$$G_\alpha := G \cap \mathbb{P}_\alpha$$

for each $\alpha \leq \delta$. Then $G_\alpha \subseteq \mathbb{P}_\alpha$ is $\mathbb{P}_\alpha$-generic over $\mathbf{V}$.

Moreover, the generic extensions form an increasing sequence of models, i.e.,

$$\mathbf{V} \subseteq \mathbf{V}[G_\beta] \subseteq \mathbf{V}[G_\alpha] \subseteq \mathbf{V}[G]$$

for each $\beta \leq \alpha \leq \delta$.

Proof. For a proof and further details see [6] (in particular Fact 1.15 and the section about quotient forcing). □

The reason why **properness** is such an interesting property is the fact that it is **preserved under countable support iterations** (of any length):

Theorem 3.32. *Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \ \beta < \alpha \rangle$ be a countable support iteration of proper forcing notions $\dot{\mathbb{Q}}_\beta$, i.e., for each $\beta < \alpha$,*

$$\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta \text{ is a proper forcing notion.}$$

Then the countable support limit $\mathbb{P}_\alpha = \lim_{count} \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \ \beta < \alpha \rangle$ is proper too.

Proof. See [6, Corollary 3.19 on page 327]. □

In Goldstern's "Tools for your forcing construction" [6] one can find a very general preservation theorem (concerning countable support iterations). As a special case of this theorem we get the fact that the property of being ω^ω-bounding is preserved under countable support iterations:

Theorem 3.33. *Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \ \beta < \alpha \rangle$ be a countable support iteration of proper ω^ω-bounding forcing notions $\dot{\mathbb{Q}}_\beta$, i.e., for each $\beta < \alpha$,*

$$\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta \text{ is a proper and } \omega^\omega\text{-bounding forcing notion.}$$

Then the countable support limit $\mathbb{P}_\alpha$ is (proper and) ω^ω-bounding.

Proof. See [6, Corollary 6.6 on page 343]. □

The next lemma shows that there are no new reals at limit stages of uncountable cofinality:

Lemma 3.34. *Assume $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \ \beta < \alpha \rangle$ is a countable support iteration with limit $\mathbb{P}_\alpha$, and $\gamma \leq \alpha$ is a limit ordinal with $\mathrm{cf}(\gamma) > \omega$. Then:*

$$\Vdash_{\mathbb{P}_\gamma} \ \text{If } \ \mathrm{cf}(\gamma) > \omega, \ \text{then } \ \omega^\omega \cap \mathbf{V}[G_\gamma] = \bigcup_{\beta<\gamma} \omega^\omega \cap \mathbf{V}[G_\beta].$$

Proof. The idea is the following: in $\mathbf{V}[G_\gamma]$, the values of any function in $\omega^\omega \cap \mathbf{V}[G_\gamma]$ are decided by countably many conditions in $\mathbf{V}[G_\gamma]$ whose domain is bounded in γ (by $\mathrm{cf}(\gamma) > \omega$ in $\mathbf{V}[G_\gamma]$). For the details, see [6, Lemma 1.20 on page 318]. □

It will be crucial to have not more than $\aleph_2$ many unbounded p-filters in each intermediate model of our iteration; moreover, the "forcing iterands" must not be too big:

Lemma 3.35. *Assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Then there are (at most) $\aleph_2$ many (unbounded p-)filters on ω; moreover, for each unbounded p-filter $\mathcal{F}$, the forcing notion $\mathbb{P}(\mathcal{F})^\omega$ is of size $\aleph_1$.*

Proof. For each filter $\mathcal{F}$ on ω, $\mathcal{F} \subseteq \mathcal{P}(\omega)$, i.e., $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\omega))$, so there are (at most) $|\mathcal{P}(\mathcal{P}(\omega))| = 2^{(2^{\aleph_0})} = 2^{\aleph_1} = \aleph_2$ many filters on ω.

Let $\mathcal{F}$ be an unbounded p-filter. Recall that

$$\mathbb{P}(\mathcal{F}) = \{p : \ p\colon \mathrm{dom}(p) \to 2 \ \text{ is a function, } \ \mathrm{dom}(p) \in \mathcal{F}^\star\}\,;$$

there are $|\mathcal{P}(\omega)| = 2^{\aleph_0}$ many possibilities for the domain of a function in $\mathbb{P}(\mathcal{F})$, and for each domain, there are (at most) $2^{\aleph_0}$ many possible functions; so $|\mathbb{P}(\mathcal{F})| = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$. The same is true for the forcing $\mathbb{P}(\mathcal{F})^\omega$, since $|\mathbb{P}(\mathcal{F})^\omega| = |\mathbb{P}(\mathcal{F})|^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \aleph_1$. (To argue differently, $\mathbb{P}(\mathcal{F})^\omega$ is isomorphic to $\mathbb{P}(\overline{\mathcal{F}})$, and $\overline{\mathcal{F}}$ is an unbounded p-filter, as we have seen in Lemma 3.26.) □

If the iterands are not too big, the resulting forcing will have the $\aleph_2$-c.c.:

Theorem 3.36. *Assume CH in the ground model* $\mathbf{V}$ *(i.e.,* $\mathbf{V} \models 2^{\aleph_0} = \aleph_1$*). Let* $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle \in \mathbf{V}$ *be a countable support iteration (with limit* $\mathbb{P}_\alpha$*) of length* $\alpha \leq \omega_2$ *of proper forcings* $\dot{\mathbb{Q}}_\beta$ *of size* $\aleph_1$*, i.e., for each* $\beta < \alpha$,

$$\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta \text{ is proper and } \left|\dot{\mathbb{Q}}_\beta\right| \leq \aleph_1.$$

Then $\mathbb{P}_\alpha$ *satisfies the* $\aleph_2$*-c.c. (i.e., each antichain has size at most* $\aleph_1$*).*

Proof. See [1, Theorem 2.9 on page 20]. □

Consequently, all cardinals (and cofinalities) will be preserved:

Lemma 3.37. *Let* $\mathbb{P}$ *be a proper forcing notion satisfying the* $\aleph_2$*-c.c.; then* $\mathbb{P}$ *preserves all cardinals and cofinalities.*

Proof. The forcing $\mathbb{P}$ is proper, so by Corollary 3.18, ω_1 is preserved (and the property "$\mathrm{cf}(\alpha) > \omega$").

Since $\mathbb{P}$ has the $\aleph_2$-c.c., all cardinals and cofinalities greater or equal ω_2 are preserved (the well-known proof is similar to Fact 3.16). □

For Lemma 3.35 to apply, we need CH in each intermediate model:

Theorem 3.38. *Like in Theorem 3.36, assume CH and let* $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ *be a countable support iteration of length* $\alpha < \omega_2$ *of proper forcings of size* $\aleph_1$*.*

Then CH holds in every generic extension by the forcing $\mathbb{P}_\alpha$*, i.e.,*

$$\Vdash_{\mathbb{P}_\alpha} 2^{\aleph_0} = \aleph_1.$$

(Note that this is not true any more for $\alpha = \omega_2$*.)*

Proof. See [1, Theorem 2.11 on page 22]. □

Using the concept of *nice names*, we can put an upper bound on the number of subsets (in the extension) of a cardinal (cf. also the books of Jech [8, Lemma 15.1 on page 225] and Kunen [9, Ch. 7, Lemma 5.12 and Lemma 5.13]):

Lemma 3.39. *Assume* $\mathbb{P} \in \mathbf{V}$ *is a forcing satisfying the* κ^+*-c.c., i.e., each antichain has size at most* κ*, and* λ *is some cardinal in* $\mathbf{V}$*. Let* $\mu = ((|\mathbb{P}|^\kappa)^\lambda)^{\mathbf{V}}$*. Then*

$$\Vdash_{\mathbb{P}} 2^\lambda \leq \mu.$$

Proof. It can be shown that each subset of λ in the extension has a *nice name*, i.e., a name of the form $\bigcup\{\{\xi\} \times A_\xi : \ \xi \in \lambda\}$, where $A_\xi \subseteq \mathbb{P}$ is an antichain for each $\xi \in \lambda$ (A_ξ is a subset of the maximal antichain within the open dense set of conditions deciding whether ξ is in the considered subset of λ or not, namely the set of those conditions which force ξ to be in this subset).

Consequently, we get an upper bound for the value of 2^λ in the generic extension by counting the number of nice names (within $\mathbf{V}$). Since $\mathbb{P}$ satisfies the κ^+-c.c., there are at most $|\mathbb{P}|^\kappa$ antichains in $\mathbb{P}$; therefore the number of nice names (for subsets of λ) is bounded by $(|\mathbb{P}|^\kappa)^\lambda = \mu$, which finishes the proof of the lemma. $\square$

Consistently there are no p-points

Now we are prepared to prove the main theorem of this chapter:

Theorem 3.40 (Shelah)**.** *It is consistent with ZFC that p-points do not exist (provided ZFC is consistent).*

Proof. We start with a ground model $\mathbf{V}$ satisfying the *generalized continuum hypothesis* (GCH), i.e., $\mathbf{V} \models 2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for each $\alpha \in Ord$. In particular,

$$\mathbf{V} \models 2^{\aleph_0} = \aleph_1 \ \wedge \ 2^{\aleph_1} = \aleph_2,$$

which is all we will need in the following.

We shall define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \ \alpha < \omega_2 \rangle \in \mathbf{V}$ of length $\delta = \omega_2$, with countable support limit $\mathbb{P}_{\omega_2}$. Within the ground model $\mathbf{V}$, we fix a bookkeeping device: let $\iota : \omega_2 \times \omega_2 \leftrightarrow \omega_2$ be a bijection with the property that $\alpha = \iota(\beta, \eta) \geq \beta$ for each $(\beta, \eta) \in \omega_2 \times \omega_2$.

Let $\beta < \omega_2$, and assume that $\mathbb{P}_\beta$ preserves all cardinals and cofinalities, and $\Vdash_{\mathbb{P}_\beta} 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2$. Then (see Lemma 3.35) there is a family of $\mathbb{P}_\beta$-names $\langle \dot{\mathcal{F}}_{\beta,\eta} : \ \eta \in \omega_2 \rangle \in \mathbf{V}$, such that

$$\Vdash_{\mathbb{P}_\beta} \text{“}\{\dot{\mathcal{F}}_{\beta,\eta} : \ \eta \in \omega_2\} \text{ is the set of all unbounded p-filters”}.$$

Note that $\dot{\mathcal{F}}_{\beta,\eta}$ can be viewed as an $\mathbb{P}_\alpha$-name for each $\alpha \geq \beta$. If so, we assume $\Vdash_{\mathbb{P}_\alpha}$ “$\dot{\mathcal{F}}_{\beta,\eta}$ is a filter”, i.e., without changing the notation, $\dot{\mathcal{F}}_{\beta,\eta}$ then denotes a name for the “filter generated by the set $\dot{\mathcal{F}}_{\beta,\eta}$”.

We are going to show now (by induction on $\alpha < \omega_2$) that it is possible to define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \ \alpha < \omega_2 \rangle \in \mathbf{V}$ such that for every $\alpha \in \omega_2$, the following conditions are satisfied:

1. $\mathbb{P}_\alpha$ is proper and ω^ω-bounding and satisfies the $\aleph_2$-c.c. (and hence preserves all cardinals and cofinalities), and

   $\Vdash_{\mathbb{P}_\alpha} 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2$.

   In fact: if $\mathbf{V} \models$ GCH, then $\Vdash_{\mathbb{P}_\alpha}$ GCH.

2. There is (for future use) a family of $\mathbb{P}_\alpha$-names $\langle \dot{\mathcal{F}}_{\alpha,\eta} : \eta \in \omega_2 \rangle$ satisfying

   $$\Vdash_{\mathbb{P}_\alpha} \text{“}\{\dot{\mathcal{F}}_{\alpha,\eta} : \eta \in \omega_2\} \text{ is the set of all unbounded p-filters”}.$$

3. Let $(\beta, \eta) \in \omega_2 \times \omega_2$ be such that $\iota(\beta, \eta) = \alpha \geq \beta$; then

   $$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = \mathbb{P}(\dot{\mathcal{F}}_{\beta,\eta})^\omega$$

4. $\Vdash_{\mathbb{P}_\alpha}$“$\dot{\mathcal{F}}_{\beta,\eta}$ is an unbounded p-filter, so $\dot{\mathbb{Q}}_\alpha$ is proper and ω^ω-bounding”

5. $\Vdash_{\mathbb{P}_\alpha} \left|\dot{\mathbb{Q}}_\alpha\right| \leq \aleph_1$

Let's assume that the conditions (1)–(5) are true for all $\beta < \alpha$.

(1) Since for each $\beta < \alpha$, $\dot{\mathbb{Q}}_\beta$ is forced to be proper and ω^ω-bounding (see condition (4)), $\mathbb{P}_\alpha$ is also proper and ω^ω-bounding by Theorem 3.32 and Theorem 3.33. Since $\mathbf{V} \models$ CH and condition (5) holds for each $\beta < \alpha$, $\mathbb{P}_\alpha$ has the $\aleph_2$-c.c. by Theorem 3.36. (So by Lemma 3.37, $\mathbb{P}_\alpha$ preserves all cardinals and cofinalities.) Similarly, Theorem 3.38 yields $\Vdash_{\mathbb{P}_\alpha} 2^{\aleph_0} = \aleph_1$. It can be shown (also by induction on α) that $\mathbb{P}_\alpha$ has (a dense subset of) size $2^{\aleph_1}$ (in $\mathbf{V}$); if λ is some cardinal in $\mathbf{V}$, let $\mu = ((|\mathbb{P}_\alpha|^{\aleph_1})^\lambda)^\mathbf{V} = (((2^{\aleph_1})^{\aleph_1})^\lambda)^\mathbf{V} = (2^\lambda)^\mathbf{V}$ (provided $\lambda \geq \aleph_1$); so, by Lemma 3.39, $\Vdash_{\mathbb{P}_\alpha} 2^\lambda \leq (2^\lambda)^\mathbf{V}$, hence $\Vdash_{\mathbb{P}_\alpha} 2^{\aleph_1} = \aleph_2$ (by $\mathbf{V} \models 2^{\aleph_1} = \aleph_2$); more generally, $\mathbf{V} \models$ GCH implies $\Vdash_{\mathbb{P}_\alpha}$ GCH. So condition (1) holds for α.

(2) By condition (1) (for α), $\mathbb{P}_\alpha$ preserves all cardinals and cofinalities, and $\Vdash_{\mathbb{P}_\alpha} 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2$; by Lemma 3.35,

$$\Vdash_{\mathbb{P}_\alpha} \text{“there are } {\aleph_2}^\mathbf{V} \text{ many unbounded p-filters”},$$

so there is a family of $\mathbb{P}_\alpha$-names as required in (2).

(3) Since (β, η) is selected by our bookkeeping device ι in such a way that $\beta \leq \alpha$, the name $\dot{\mathcal{F}}_{\beta,\eta}$ has already been defined and can be viewed as a $\mathbb{P}_\alpha$-name for a filter (see the discussion above); so $\dot{\mathbb{Q}}_\alpha = \mathbb{P}(\dot{\mathcal{F}}_{\beta,\eta})^\omega$ is the $\mathbb{P}_\alpha$-name for a forcing notion.

(4) Because of condition (2) for β (which is $\leq \alpha$),

$$\Vdash_{\mathbb{P}_\beta} \text{“}\dot{\mathcal{F}}_{\beta,\eta} \text{ is an unbounded p-filter”}. \tag{3.37}$$

We have to show that $\dot{\mathcal{F}}_{\beta,\eta}$ remains an unbounded p-filter (if viewed at stage α), i.e., $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathcal{F}}_{\beta,\eta}$ is an unbounded p-filter" (see again the discussion above). The forcing $\mathbb{P}_\alpha$ is isomorphic to $\mathbb{P}_\beta \star (\mathbb{P}_\alpha : \mathbb{P}_\beta)$, where $\mathbb{P}_\alpha : \mathbb{P}_\beta$ is a $\mathbb{P}_\beta$-name for the forcing carrying us from stage β to stage α (the so-called quotient forcing); by [6, Theorem 4.6 on page 329], $\mathbb{P}_\alpha : \mathbb{P}_\beta$ is in turn isomorphic to a $\mathbb{P}_\beta$-name for a countable support iteration with the (appropriately translated) iterands $\dot{\mathbb{Q}}_\gamma$, $\beta \leq \gamma < \alpha$, which are proper and ω^ω-bounding by condition (4) (for $\gamma < \alpha$). By Theorem 3.32 and Theorem 3.33, $\Vdash_{\mathbb{P}_\beta}$ "$\mathbb{P}_\alpha : \mathbb{P}_\beta$ is a proper and ω^ω-bounding forcing", so (together with (3.37)) Lemma 3.21 proves

$$\Vdash_{\mathbb{P}_\alpha} \text{"}\dot{\mathcal{F}}_{\beta,\eta} \text{ is an unbounded p-filter".}$$

Consequently (for $\dot{\mathbb{Q}}_\alpha$ was defined to be $\mathbb{P}(\dot{\mathcal{F}}_{\beta,\eta})^\omega$)

$$\Vdash_{\mathbb{P}_\alpha} \text{"}\dot{\mathbb{Q}}_\alpha \text{ is proper and } \omega^\omega\text{-bounding"}$$

holds by Lemma 3.26, so condition (4) is true for α.

(5) $\Vdash_{\mathbb{P}_\alpha} 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2$ (by conditions (1) for α); so condition (5) follows from Lemma 3.35.

Now consider the countable support limit

$$\mathbb{P}_{\omega_2} := \lim\nolimits_{\text{count}} \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \ \alpha < \omega_2 \rangle.$$

$\mathbb{P}_{\omega_2}$ is a proper and ω^ω-bounding forcing notion satisfying the $\aleph_2$-c.c., hence it preserves all cardinals and cofinalities; the proof is the same as above (cf. condition (1) and note that Theorem 3.36 is still true for $\alpha = \omega_2$). Similarly, $\Vdash_{\mathbb{P}_{\omega_2}} 2^{\aleph_1} = \aleph_2$ still holds ($\mathbb{P}_{\omega_2}$ is the direct limit of the $\mathbb{P}_\alpha$, $\alpha < \omega_2$, so $|\mathbb{P}_{\omega_2}| \leq 2^{\aleph_1}$, and due to the $\aleph_2$-c.c., we can again use Lemma 3.39).

However, Theorem 3.38 cannot be applied, since it holds only for $\alpha < \omega_2$. In fact, $\Vdash_{\mathbb{P}_{\omega_2}} 2^{\aleph_0} = \aleph_2$; this is because new reals are added at every successor ordinal $\alpha \in \omega_2$ (the forcing $\dot{\mathbb{Q}}_\alpha = \mathbb{P}(\dot{\mathcal{F}}_{\beta,\eta})^\omega$ adds a new real each time, cf. (3.1)); so $\Vdash_{\mathbb{P}_{\omega_2}} 2^{\aleph_0} \geq \aleph_2$, but $2^{\aleph_0} > \aleph_2$ is impossible due to $\Vdash_{\mathbb{P}_{\omega_2}} 2^{\aleph_1} = \aleph_2$. If $\mathbf{V} \models$ GCH, then the continuum function in the final model is as follows:

$$\Vdash_{\mathbb{P}_{\omega_2}} \ 2^{\aleph_0} = \aleph_2 \quad \wedge \quad \forall \alpha \geq 1 : \ 2^{\aleph_\alpha} = \aleph_{\alpha+1}. \tag{3.38}$$

Now fix some generic filter $G \subseteq \mathbb{P}_{\omega_2}$ over $\mathbf{V}$, yielding a model $\mathbf{V}[G]$ of ZFC. We claim that

$$\mathbf{V}[G] \models \text{ "There are no p-points."}$$

For each $\alpha \leq \omega_2$, let $G_\alpha = G \cap \mathbb{P}_\alpha$, which is $\mathbb{P}_\alpha$-generic over $\mathbf{V}$; moreover, $\mathbf{V}[G_\beta] \subseteq \mathbf{V}[G_\alpha] \subseteq \mathbf{V}[G]$ for each $\beta \leq \alpha \leq \omega_2$ (cf. Fact 3.31).

Assume (towards a contradiction) that there is a p-point $\mathcal{F}$ in the final model $\mathbf{V}[G]$. By Lemma 3.6, $\mathcal{F}$ is an unbounded p-filter.

For each $\alpha \leq \omega_2$, define the filter

$$\mathcal{F}_\alpha := \mathcal{F} \cap \mathbf{V}[G_\alpha].$$

Note that each $\mathcal{F}_\alpha$ is a subset of $\mathbf{V}[G_\alpha]$, but not necessarily an element of $\mathbf{V}[G_\alpha]$.

We say the filter $\mathcal{F}_\alpha$ is an *unbounded p-filter with respect to* β (for $\beta \leq \alpha$) if (compare with Definition 1.9 and Definition 3.4)

- each countable collection of sets from $\mathcal{F}_\beta$ has a pseudo-intersection within $\mathcal{F}_\alpha$, and
- for each $g \in \omega^\omega \cap \mathbf{V}[G_\beta]$ there is an $X \in \mathcal{F}_\alpha$ such that $f_X(k) \geq g(k)$ for infinitely many $k \in \omega$.

Note that Lemma 3.34 applies to each direct limit of our countable support iteration (since uncountable cofinalities remain uncountable due to properness); so if $\gamma \leq \omega_2$ has uncountable cofinality, each real within $\mathbf{V}[G_\gamma]$ already appears at some earlier stage, i.e., there is an $\varepsilon < \gamma$ such that the real lies in $\mathbf{V}[G_\varepsilon]$. We will repeatedly use this fact in the following two lemmas:

Lemma 3.41. *For each $\beta < \omega_2$, there is a $J(\beta)$, $\beta < J(\beta) < \omega_2$, such that $\mathcal{F}_{J(\beta)}$ is an unbounded p-filter with respect to β.*

Proof. Each countable collection of sets from $\mathcal{F}_\beta$ has a pseudo-intersection $X \in \mathcal{F}_{\omega_2} = \mathcal{F}$ (since $\mathcal{F}_\beta \subseteq \mathcal{F}$ and $\mathcal{F}$ is a p-filter in $\mathbf{V}[G]$), so – by Lemma 3.34 (and $\mathrm{cf}(\omega_2) > \omega$) – there is an $\varepsilon < \omega_2$ with $X \in \mathbf{V}[G_\varepsilon]$, i.e., $X \in \mathcal{F}_\varepsilon$. Since there are only $|\mathcal{F}_\beta|^{\aleph_0} = {\aleph_1}^{\aleph_0} = \aleph_1$ many countable collections of sets from $\mathcal{F}_\beta$ (note that CH holds within $\mathbf{V}[G_\beta]$), we can define $\alpha_{\text{p-filter}} < \omega_2$ to be the supremum of the respective ε's.

Similarly, we go through all $g \in \omega^\omega \cap \mathbf{V}[G_\beta]$ (which are again just $\aleph_1$ many), obtaining an $\alpha_{\text{unbounded}} < \omega_2$.

Then $J(\beta) := \max(\alpha_{\text{p-filter}}, \alpha_{\text{unbounded}}) < \omega_2$ satisfies the required properties (since $\mathcal{F}_\varepsilon \subseteq \mathcal{F}_{J(\beta)}$ for each $\varepsilon < J(\beta)$); w.l.o.g., we can choose $J(\beta)$ such that $J(\beta) > \beta$. □

By iterating Lemma 3.41 ω_1-many times, we get

Lemma 3.42. *There is an $\alpha < \omega_2$ such that $\mathcal{F}_\alpha$ is an unbounded p-filter with respect to α.*

Proof. Using $J(\cdot)$, we obtain a continuous ω_1-sequence: starting with $\beta_0 := 0$, we define $\beta_{\xi+1} := J(\beta_\xi)$, and take the supremum at each limit ordinal ξ. Let $\alpha := \sup_{\xi<\omega_1} \beta_\xi < \omega_2$. We claim that $\mathcal{F}_\alpha$ is an unbounded p-filter with respect to α. Note that $\mathrm{cf}(\alpha) = \omega_1$.

Let $\{Y_n : n \in \omega\} \subseteq \mathcal{F}_\alpha$ be a countable collection of sets from $\mathcal{F}_\alpha$. For each $n \in \omega$, $Y_n \in \mathcal{F}_\alpha \subseteq \mathbf{V}[G_\alpha]$, so – by Lemma 3.34 (and $\mathrm{cf}(\alpha) > \omega$) – there is an $\varepsilon < \alpha$ such that $Y_n \in \mathbf{V}[G_\varepsilon]$. Consequently (again due to $\mathrm{cf}(\alpha) > \omega$), we can find a $\xi < \omega_1$ such that $\{Y_n : n \in \omega\} \subseteq \mathbf{V}[G_{\beta_\xi}]$. Since $\beta_{\xi+1} = J(\beta_\xi)$, $\mathcal{F}_{\beta_{\xi+1}}$ is a "p-filter with respect to β_ξ", so the collection $\{Y_n : n \in \omega\}$ has a pseudo-intersection within $\mathcal{F}_{\beta_{\xi+1}} \subseteq \mathcal{F}_\alpha$.

The part concerning unboundedness is similar. $\square$

So this filter $\mathcal{F}_\alpha$ can be viewed as an "external" unbounded p-filter of $\mathbf{V}[G_\alpha]$, i.e., it is a subset of $\mathbf{V}[G_\alpha]$ and behaves like an unbounded p-filter "in $\mathbf{V}[G_\alpha]$", apart from the fact that it does not belong to $\mathbf{V}[G_\alpha]$ as an element.

Remark. Note that all the limit processes above can be viewed as "intersecting certain club sets".

Quite similar to Lemma 3.34, the following holds: whenever $M \in \mathbf{V}[G_\gamma]$ is an object in $\mathbf{V}[G_\gamma]$ of size at most $\aleph_1$ and $\Vdash_{\mathbb{P}_\gamma} \mathrm{cf}(\gamma) > \omega_1$, then there is an $\varepsilon < \gamma$ such that $M \in \mathbf{V}[G_\varepsilon]$.

Since our filter $\mathcal{F}_\alpha \in \mathbf{V}[G_{\omega_2}]$ is indeed of size $\aleph_1$ (note that $\mathbf{V}[G_\alpha] \models \mathrm{CH}$ due to $\alpha < \omega_2$), there is an $\varepsilon < \omega_2$ such that the filter (generated by) $\mathcal{F}_\alpha$ lies (as an object) within $\mathbf{V}[G_\varepsilon]$, i.e., $\mathcal{F}_\alpha \in \mathbf{V}[G_\varepsilon]$.

$\mathcal{F}_\alpha$ is an unbounded p-filter with respect to α; like in the proof of condition (4) on page 81, $\Vdash_{\mathbb{P}_\alpha}$ "$\mathbb{P}_\varepsilon : \mathbb{P}_\alpha$ is a proper and ω^ω-bounding forcing", so Lemma 3.21 shows that $\mathcal{F}_\alpha$ is an unbounded p-filter in $\mathbf{V}[G_\varepsilon]$. (Note that $\mathcal{F}_\alpha$ is not an element of $\mathbf{V}[G_\alpha]$, in contrast to the assumption in Lemma 3.21, but this makes no difference.)

We summarize the situation: under the assumption that there is a p-point $\mathcal{F}$ in the final model $\mathbf{V}[G]$, we have found a filter $\mathcal{F}_\alpha \subseteq \mathcal{F}$ and an ordinal $\varepsilon < \omega_2$ such that $\mathcal{F}_\alpha \in \mathbf{V}[G_\varepsilon]$ is an unbounded p-filter. According to the construction of our forcing iteration, for each unbounded p-filter which appears at some intermediate stage, the respective forcing $\mathbb{P}(\cdot)^\omega$ is applied somewhere in the course of the iteration; therefore, there has to be a ζ with $\varepsilon \leq \zeta < \omega_2$ such that the ζ's iterand $\mathbb{Q}_\zeta$ actually is $\mathbb{P}(\mathcal{F}_\alpha)^\omega$ (where $\mathcal{F}_\alpha \in \mathbf{V}[G_\zeta]$ is an unbounded p-filter in $\mathbf{V}[G_\zeta]$).

Now we can derive a contradiction from Lemma 3.27 (our "single step killing lemma"): like above, $\Vdash_{\mathbb{P}_{\zeta+1}}$ "$\mathbb{P}_{\omega_2} : \mathbb{P}_{\zeta+1}$ is ω^ω-bounding"; so there is no p-point in the final model $\mathbf{V}[G]$ which extends $\mathcal{F}_\alpha$, since $\mathbb{P}(\mathcal{F}_\alpha)^\omega$ "has killed $\mathcal{F}_\alpha$". But $\mathcal{F} \in \mathbf{V}[G]$ was assumed to be a p-point and $\mathcal{F} \supseteq \mathcal{F}_\alpha$, a contradiction. □

Remark. Note that it is even possible to find a p-point $\mathcal{F}_\alpha$ at some intermediate stage. (Just replace the condition for unboundedness by the ultrafilter property in the above arguments.) Nevertheless, the forcing $\mathbb{P}(\cdot)^\omega$ will be applied only for an unbounded p-filter, since the filter $\mathcal{F}_\alpha$ will potentially lose its ultrafilter property when "passing to a later stage".

3.9 Open questions concerning $2^{\aleph_0} \geq \aleph_3$

Let's sum up what we have found out about the existence of p-points. If $2^{\aleph_0} = \aleph_1$, then there exists a p-point (see Theorem 1.13). If $2^{\aleph_0} = \aleph_2$, then either case is possible: if MA holds (or at least $\mathfrak{d} = 2^{\aleph_0}$) then there exists a p-point (see Corollary 1.21 and Theorem 1.24), but in Shelah's model there are no p-points (and $2^{\aleph_0} = \aleph_2$, see Theorem 3.40 and (3.38)).

Remark. Consequently, $\mathfrak{d} = \aleph_1$ in Shelah's model (since $\aleph_1 \leq \mathfrak{d} \leq \aleph_2 = 2^{\aleph_0}$ and $\mathfrak{d} = 2^{\aleph_0}$ would imply the existence of p-points). Of course, this can also be seen directly: the forcing $\mathbb{P}_{\omega_2}$ is ω^ω-bounding, so $(\omega^\omega)^{\mathbf{V}}$ (which has size $\aleph_1$ due to CH in $\mathbf{V}$) is a dominating family in the final model $\mathbf{V}[G]$.

There are also models of ZFC with p-points and larger continuum (cf. Corollary 1.21). The question arises if there are models with larger continuum in which there are no p-points. Interestingly, this seems to be unknown:

Open question. *Is there a model of ZFC without a p-point and $2^{\aleph_0} \geq \aleph_3$?*

We conclude the chapter with the following remark: a *q-point* is an ultrafilter $\mathcal{F}$ such that for every partition of ω into finite sets $s_0, s_1, s_2, \ldots$, there is a set $X \in \mathcal{F}$ with $|X \cap s_k| \leq 1$ for each $k \in \omega$. Note that each Ramsey ultrafilter is both a p-point and a q-point (cf. Definition 1.22); in fact, an ultrafilter is a Ramsey ultrafilter if and only if it is a p-point and a q-point, as can be easily shown. The following question is open as well:

Open question. *Is there a model of ZFC with no p-point and no q-point?*

This question is assumed to be much more difficult than the first one; in fact, a positive answer would solve the first question: it can be shown that $\mathfrak{d} = \aleph_1$ implies the existence of a q-point; due to "$\mathfrak{d} = 2^{\aleph_0}$ implies the existence of a p-point", in each model with no p-point and no q-point $2^{\aleph_0} \geq \aleph_3$ will hold. In particular, there is a q-point in Shelah's model.

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Index