Polarized partitions on the second level of the projective hierarchy

Yurii Khomskii

Joint work with Jörg Brendle (Kobe University, Japan)

Amsterdam set theory workshop, 2 June 2010
Question: Let \( m_0, \ldots, m_{k-1} \) be fixed integers \( \geq 2 \). Is it true that for every partition \( \pi : \omega^k \rightarrow 2 \) there is a sequence \( H_0, \ldots, H_{k-1} \) of finite subsets of \( \omega \) with \( |H_i| = m_i \), which is homogeneous for \( \pi \), i.e., \( \pi \) is constant on \( \prod_{i<k} H_i \)?

Notation

\[
\begin{pmatrix}
\omega \\
\vdots \\
\omega
\end{pmatrix} \rightarrow \begin{pmatrix}
m_0 \\
\vdots \\
m_{k-1}
\end{pmatrix}
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Answer: Yes
Finite polarized partitions 1

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Answer: Yes, because $\pi$ induces a partition of $\omega^k$, which we identify with $[\omega]^k$. 

- Notation

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\begin{pmatrix}
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\vdots \\
\omega
\end{pmatrix} \to 
\begin{pmatrix}
m_0 \\
\vdots \\
m_{k-1}
\end{pmatrix}
$$
Finite polarized partitions 1

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**Answer:** Yes, because $\pi$ induces a partition of $\omega^{\uparrow k}$, which we identify with $[\omega]^k$. By the finite Ramsey theorem there is an infinite $H$ s.t. $\pi$ is constant on $[H]^k$. 
Finite polarized partitions 1

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- $H_0 :=$ first $m_0$ members of $H$, 

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\vdots \\
m_{k-1} \\
\end{pmatrix}
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- $H_0 :=$ first $m_0$ members of $H$,
- $H_1 :=$ next $m_1$ members of $H$, etc.
Question: Let \( m_0, \ldots, m_{k-1} \) be fixed integers \( \geq 2 \). Is it true that for every partition \( \pi : \omega^k \to 2 \) there is a sequence \( H_0, \ldots, H_{k-1} \) of finite subsets of \( \omega \) with \( |H_i| = m_i \), which is homogeneous for \( \pi \), i.e., \( \pi \) is constant on \( \prod_{i<k} H_i \)?

Answer: Yes, because \( \pi \) induces a partition of \( \omega^{\uparrow k} \), which we identify with \( [\omega]^k \). By the finite Ramsey theorem there is an infinite \( H \) s.t. \( \pi \) is constant on \( [H]^k \). Now let

- \( H_0 := \) first \( m_0 \) members of \( H \),
- \( H_1 := \) next \( m_1 \) members of \( H \), etc.

Then \( x \in \prod_i H_i \rightarrow \text{ran}(x) \subseteq H \), hence it is easy to see that \( \langle H_i \rangle_{i \in \omega} \) is homogeneous.
Question: Let $m_0, \ldots, m_{k-1}$ and $n_0, \ldots, n_{k-1}$ be fixed integers $\geq 2$. Is it true that for every partition $\pi : \prod_{i<k} n_i \to 2$ there is a sequence $H_0, \ldots, H_{k-1}$ with $|H_i| = m_i$ and $H_i \subseteq n_i$, which is homogeneous for $\pi$, i.e., $\pi$ is constant on $\prod_{i<k} H_i$?

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  n_0 \\
  \vdots \\
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\end{pmatrix} \to \begin{pmatrix}
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Question: Let $m_0, \ldots, m_{k-1}$ and $n_0, \ldots, n_{k-1}$ be fixed integers $\geq 2$. Is it true that for every partition $\pi : \prod_{i<k} n_i \to 2$ there is a sequence $H_0, \ldots, H_{k-1}$ with $|H_i| = m_i$ and $H_i \subseteq n_i$, which is homogeneous for $\pi$, i.e., $\pi$ is constant on $\prod_{i<k} H_i$?

Notation

$$
\left(
\begin{array}{c}
n_0 \\
\vdots \\
n_{k-1}
\end{array}
\right) \rightarrow 
\left(
\begin{array}{c}
m_0 \\
\vdots \\
m_{k-1}
\end{array}
\right)
$$

Answer:

- Cannot use the finite Ramsey theorem.
Question: Let $m_0, \ldots, m_{k-1}$ and $n_0, \ldots, n_{k-1}$ be fixed integers $\geq 2$. Is it true that for every partition $\pi : \prod_{i<k} n_i \to 2$ there is a sequence $H_0, \ldots, H_{k-1}$ with $|H_i| = m_i$ and $H_i \subseteq n_i$, which is homogeneous for $\pi$, i.e., $\pi$ is constant on $\prod_{i<k} H_i$?

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  \vdots \\
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\to
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\]

Answer:

- Cannot use the finite Ramsey theorem.
- False if the $n_i$ are too small, e.g.: $\left( \begin{array}{c} 2 \\ 2 \end{array} \right) \not\to \left( \begin{array}{c} 2 \\ 2 \end{array} \right)$. 

Yuri Khomskii (University of Amsterdam)
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**Answer:**
- Cannot use the finite Ramsey theorem.
- False if the \( n_i \) are too small, e.g.: \( \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \not\to \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \).
- However, by induction we can compute the \( n_i \)'s from the \( m_i \)'s so that the partition holds.
Finite polarized partitions 2

\[(?) \rightarrow (m_0)\]
Finite polarized partitions 2

\[(2m_0) \rightarrow (m_0)\]
Finite polarized partitions 2

\[
\begin{pmatrix}
2m_0 \\
\? \\
\end{pmatrix} \rightarrow \begin{pmatrix}
m_0 \\
m_1 \\
\end{pmatrix}
\]

If \((n_0 \cdot \cdot n_k - 1) \rightarrow (m_0 \cdot \cdot m_k - 1)\) holds and \(m_k\) is given, then by defining \(n_k := 2 \cdot m_k \cdot \prod_{i < k} (n_i m_i)\) the partition \((n_0 \cdot \cdot n_k) \rightarrow (m_0 \cdot \cdot m_k)\) holds as well.
Finite polarized partitions 2

\[ \left( \begin{array}{c} 2m_0 \\ 2m_1 \end{array} \right) \left( \begin{array}{c} 2m_0 \\ m_0 \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \end{array} \right) \]
\[
\left( \begin{array}{c}
2m_0 \\
2m_1 \binom{2m_0}{m_0}
\end{array} \right) \rightarrow \left( \begin{array}{c}
m_0 \\
m_1
\end{array} \right)
\]

e tc...
Finite polarized partitions 2

\[
\begin{pmatrix}
2m_0 \\
2m_1 \left( \begin{smallmatrix} 2m_0 \\ m_0 \end{smallmatrix} \right)
\end{pmatrix} \rightarrow \begin{pmatrix}
m_0 \\
m_1
\end{pmatrix}
\]

etc. . .

If

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the partition

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holds as well.
Now let’s extend this to infinite dimensions.
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Answer: By AC, cannot be true for all partitions. If $\pi$ is analytic, then the partition holds by Silver’s theorem (all analytic sets are Ramsey).
Infinite polarized partitions 1

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This question came under attention only recently, in works of DiPrisco, Llopis, Todorčević and Zapletal.
Analytic partitions

- DiPrisco & Todorčević, 2004: $\left( \begin{array}{c} n_0 \\ n_1 \\ \vdots \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \\ \vdots \end{array} \right)$ holds for analytic partitions. Bounds $\langle n_i \rangle_{i \in \omega}$ are computed from $\langle m_i \rangle_{i \in \omega}$ in terms of recursive, but not primitive-recursive algorithm.
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Analytic partitions

- DiPrisco & Todorčević, 2004: \((\binom{n_0}{n_1}) \rightarrow (\binom{m_0}{m_1})\) holds for analytic partitions. Bounds \(\langle n_i \rangle_{i \in \omega}\) are computed from \(\langle m_i \rangle_{i \in \omega}\) in terms of recursive, but not primitive-recursive algorithm.


Our goal is to see what happens at the next level of the projective hierarchy: \(\Delta^1_2\) and \(\Sigma^1_2\).
The second level

Questions about regularity on the second level of the projective hierarchy are typically independent of ZFC.

E.g. if $V = L$, then there is a $\Delta^1_2$ set which is non-Lebesgue measurable, there is a $\Delta^1_2$ set which doesn't have the Baire property, and there is a $\Delta^1_2$ set which doesn't have Ramsey property.

On the other hand, if $\forall a (\aleph_L[a] = \aleph_0)$ then all $\Sigma^1_2$ sets are Lebesgue-measurable, have the Baire property and the Ramsey property.

In fact, regularity of $\Sigma^1_2$ and $\Delta^1_2$ sets indicates the level of transcendence over $L$. 
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In fact, regularity of $\Sigma^1_2$ and $\Delta^1_2$ sets indicates the level of transcendent over $L$.
Examples

- Judah & Shelah, 1989: the following are equivalent:
  1. all $\Delta^1_2$ sets are Lebesgue-measurable,
  2. for all $a \in \omega^\omega$ there is a random real over $L[a]$.

- Brendle & Löwe, 1999: the following are equivalent:
  1. all $\Delta^1_2$ sets are Sacks-measurable (Marczewski-measurable)
  2. all $\Sigma^1_2$ sets are Sacks-measurable,
  3. for all $a \in \omega^\omega$ there is a real not in $L[a]$.

Same for Miller-measurable (super-perfect trees) & unbounded reals, and Laver-measurable & dominating reals.
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Same for Miller-measurable (super-perfect trees) & unbounded reals, and Laver-measurable & dominating reals.
Examples 2

- Ikegami, 2008: for a wide class of forcings $\mathbb{P}$, one can canonically define $\mathbb{P}$-measurability and a notion of $\mathbb{P}$-transcendence, such that the following are equivalent:
  1. all $\Delta^1_2$ sets are $\mathbb{P}$-measurable,
  2. for all $a \in \omega^\omega$ there is a $\mathbb{P}$-transcendent real over $L[a]$. 

Advantage: we can control regularity on the second level by iterated forcing constructions over $L[a]$. 
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1. all $\Delta^1_2$ sets are $P$-measurable,
2. for all $a \in \omega^\omega$ there is a $P$-transcendent real over $L[a]$.

**Advantage:** we can control regularity on the second level by iterated forcing constructions over $L$. 
Comparing regularities

Traditionally one has investigated the strength of various regularity properties on the second level by comparing them with each other:
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- Bartoszyński, 1984; Raisonnier & Stern, 1985: if all $\Sigma^1_2$ sets are Lebesgue measurable then all $\Sigma^1_2$ sets have the Baire property (but not vice versa.)
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- Bartoszyński, 1984; Raisonnier & Stern, 1985: if all $\Sigma^1_2$ sets are Lebesgue measurable then all $\Sigma^1_2$ sets have the Baire property (but not vice versa.)
- Judah & Shelah, 1989: The following implications hold, and none other.

$\Sigma^1_2$(Ramsey) $\rightarrow$ $\Sigma^1_2$(Baire) $\rightarrow$ $\Sigma^1_2(K_\sigma$-regularity)
Comparing regularities

\[
\forall a \ (\mathcal{N}_1^{L[a]} = \mathcal{N}_0) \iff \Sigma^1_2(E) = \Sigma^4_2(D)
\]

\[
\Sigma^1_2(R) = \Delta^1_2(R) \quad \Sigma^1_2(C) = \Delta^1_2(D) \quad \Delta^1_2(E) \quad \Delta^1_2(B)
\]

\[
\Sigma^1_2(L) = \Delta^1_2(L) \quad \Delta^1_2(C) \quad \Sigma^1_2(V) \quad \Delta^1_2(V)
\]

\[
\forall a \ (\omega^\omega \cap L[a] \neq \omega^\omega) \iff \Sigma^1_2(S) = \Delta^1_2(S)
\]

Diagram: Brendle & Löwe, *Eventually different functions and inaccessible cardinals*
Back to the polarized partitions.

What can we say about the statements \((\omega \omega \rightarrow m_0 m_1)\) holds for \(\Sigma_{1/2} / \Delta_{1/2}\) partitions" and \((n_0 n_1 \rightarrow m_0 m_1)\) holds for \(\Sigma_{1/2} / \Delta_{1/2}\) partitions?"
Back to the polarized partitions. What can we say about the statements

- \[ \left( \begin{array}{c} \omega \\ \omega \\ . \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \\ . \end{array} \right) \] holds for \( \Sigma^1_2/\Delta^1_2 \) partitions” and

- \[ \left( \begin{array}{c} n_0 \\ n_1 \\ . \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \\ . \end{array} \right) \] holds for \( \Sigma^1_2/\Delta^1_2 \) partitions”? \]
Results

Easy to see:

1. If \( \binom{n_0}{n_1} \to \binom{m_0}{m_1} \) holds for \( \Sigma^1_2/\Delta^1_2 \) partitions, then \( \omega \omega \to \omega \omega \) holds on the same level.

2. If all \( \Sigma^1_2/\Delta^1_2 \) sets are Ramsey then \( \omega \omega \to \omega \omega \) holds for \( \Sigma^1_2 \) partitions.

Theorem (Brendle)

If \( \omega \omega \to \omega \omega \) holds for \( \Delta^1_2 \) partitions, then for all \( a \in \omega \omega \) there is an eventually different real over \( L[a] \).
Results

Easy to see:

1. If \( \left( \begin{array}{c} n_0 \\ n_1 \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \end{array} \right) \) holds for \( \Sigma_2^1 / \Delta_2^1 \) partitions, then
\( \left( \begin{array}{c} 3 \\ 3 \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \end{array} \right) \) holds on the same level.

2. If all \( \Sigma_2^1 / \Delta_2^1 \) sets are Ramsey then \( \left( \begin{array}{c} 3 \\ 3 \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \end{array} \right) \) holds for \( \Sigma_2^1 \) partitions.
Results

Easy to see:

1. If \( \binom{n_0}{n_1} \rightarrow \binom{m_0}{m_1} \) holds for \( \Sigma^1_2/\Delta^1_2 \) partitions, then \( \binom{3}{3} \rightarrow \binom{m_0}{m_1} \) holds on the same level.

2. If all \( \Sigma^1_2/\Delta^1_2 \) sets are Ramsey then \( \binom{3}{3} \rightarrow \binom{m_0}{m_1} \) holds for \( \Sigma^1_2 \) partitions.
Results

Easy to see:

1. If \( \left( \begin{array}{l} n_0 \\ n_1 \end{array} \right) \rightarrow \left( \begin{array}{l} m_0 \\ m_1 \end{array} \right) \) holds for \( \Sigma^1_2/\Delta^1_2 \) partitions, then \( \left( \begin{array}{l} 3 \\ 3 \\ \vdots \end{array} \right) \rightarrow \left( \begin{array}{l} m_0 \\ m_1 \end{array} \right) \) holds on the same level.

2. If all \( \Sigma^1_2/\Delta^1_2 \) sets are Ramsey then \( \left( \begin{array}{l} 3 \\ 3 \\ \vdots \end{array} \right) \rightarrow \left( \begin{array}{l} m_0 \\ m_1 \end{array} \right) \) holds for \( \Sigma^1_2 \) partitions.

Theorem (Brendle)

If \( \left( \begin{array}{l} 3 \\ 3 \\ \vdots \end{array} \right) \rightarrow \left( \begin{array}{l} m_0 \\ m_1 \end{array} \right) \) holds for \( \Delta^1_2 \) partitions, then for all \( a \in \omega^\omega \) there is an eventually different real over \( L[a] \).
Results

Easy to see:

1. If \( \binom{n_0}{n_1} \rightarrow \binom{m_0}{m_1} \) holds for \( \Sigma^1_2/\Delta^1_2 \) partitions, then
   \( \binom{3}{3} \rightarrow \binom{m_0}{m_1} \) holds on the same level.

2. If all \( \Sigma^1_2/\Delta^1_2 \) sets are Ramsey then
   \( \binom{3}{3} \rightarrow \binom{m_0}{m_1} \) holds for \( \Sigma^1_2 \) partitions.

Theorem (Brendle)

If \( \binom{3}{3} \rightarrow \binom{m_0}{m_1} \) holds for \( \Delta^1_2 \) partitions, then for all \( a \in \omega^\omega \) there is an eventually different real over \( L[a] \).
$\Delta^1_2$ level: diagram of implications

$\Delta^1_2(\text{Ramsey}) \implies \Delta^1_2(\bar{\omega} \rightarrow \bar{m}) \implies \Delta^1_2(n \rightarrow m) \implies \forall a \exists \text{ ev. diff.}/L[a]$
$\Delta^1_2$ level: diagram of implications

$\Delta^1_2(\text{Ramsey})$

$\Delta^1_2(\exists \text{ dominating}/ \mathbb{L}[a])$

$\forall a \exists \text{ dominating}/ \mathbb{L}[a]$

$\Delta^1_2(\omega \rightarrow \bar{m})$

$\Delta^1_2(\bar{n} \rightarrow \bar{m})$

$\forall a \exists \text{ ev. diff.}/ \mathbb{L}[a]$
\[ \Delta^1_2 \text{ level: diagram of implications} \]

\[ \Delta^1_2(\text{Ramsey}) \]
\[ \Delta^1_2(\text{Laver}) \quad \forall a \ni \text{dominating/}L[a] \]
\[ \Delta^1_2(\text{Miller}) \quad \forall a \ni \text{unbounded/}L[a] \]

\[ \Delta^1_2(\bar{\omega} \to m) \]
\[ \forall a \ni \text{ev. diff./}L[a] \]
Question: which implications cannot be reversed?
$\Delta^1_2$ level: diagram of implications

Question: which implications cannot be reversed?
Mathias model

**Theorem (Brendle-Kh)**

In the $\omega_1$-iteration of Mathias forcing starting from $L$, all $\Delta^1_2$ sets are Ramsey but there is a $\Delta^1_2$ partition which violates $\begin{pmatrix} n_0 \\ n_1 \end{pmatrix} \rightarrow \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}$ (regardless of the values $n_i$, as long as they are computable from $m_i$).
\[ \Delta^1_2 \text{ level: diagram of implications} \]

- \[ \Delta^1_2(\text{Ramsey}) \]
  - \[ \Delta^1_2(\text{Laver}) \]
    - \[ \forall a \exists \text{dominating}/L[a] \]
  - \[ \Delta^1_2(\text{Miller}) \]
    - \[ \forall a \exists \text{unbounded}/L[a] \]
  - \[ \Delta^1_2(\bar{\omega} \rightarrow \bar{m}) \]
    - \[ \forall a \exists \text{ev. diff.}/L[a] \]
  - \[ \Delta^1_2(\bar{n} \rightarrow \bar{m}) \]

*False*

*True*
Mathias model

**Theorem (Brendle-Kh)**

In the $\omega_1$-iteration of Mathias forcing starting from $L$, all $\Delta^1_2$ sets are Ramsey but there is a $\Delta^1_2$ partition which violates $\left( \begin{array}{c} n_0 \\ n_1 \\ \cdot \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \\ \cdot \end{array} \right)$ (regardless of the values $n_i$, as long as they are computable from $m_i$).

In the proof, we use the fact that Mathias forcing satisfies the *Laver property*. 
$\Delta^1_2$ level: diagram of implications

- $\Delta^1_2$(Ramsey)
  - $\forall \alpha \exists$ dominating/$\mathbf{L}[\alpha]$

- $\Delta^1_2$(Laver)
  - $\forall \alpha \exists$ dominating/$\mathbf{L}[\alpha]$

- $\Delta^1_2$(Miller)
  - $\forall \alpha \exists$ unbounded/$\mathbf{L}[\alpha]$

- $\Delta^1_2(\vec{\omega} \to \vec{m})$
  - $\forall \alpha \exists$ ev. diff./$\mathbf{L}[\alpha]$

- $\Delta^1_2(\vec{n} \to \vec{m})$
$\Delta^1_2$ level: diagram of implications

- $\Delta^1_2 (\text{Ramsey})$
- $\Delta^1_2 (\text{Laver})$
  \[ \forall a \exists \text{dominating}/L[a] \]
- $\Delta^1_2 (\text{Miller})$
  \[ \forall a \exists \text{unbounded}/L[a] \]
- $\Delta^1_2 (\bar{\omega} \to \bar{m})$
  \[ \forall a \exists \text{ev. diff.}/L[a] \]
- $\Delta^1_2 (\bar{n} \to \bar{m})$
Theorem (Brendle-Kh)

There is a model in which \( \left( \begin{array}{c} n_0 \\ n_1 \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \end{array} \right) \) holds for \( \Delta^1_2 \) partitions, but not all \( \Delta^1_2 \) sets are Miller-measurable (i.e., there are no unbounded reals).
Diagram of implications

$\Delta_2^1$(Ramsey)

$\Delta_2^1$(Laver)
$\forall a \exists$ dominating/$L[a]$

$\Delta_2^1$(Miller)
$\forall a \exists$ unbounded/$L[a]$

$\Delta_2^1(\bar{\omega} \rightarrow \bar{m})$

$\Delta_2^1(\bar{n} \rightarrow \bar{m})$

False

True

$\forall a \exists$ ev. diff./$L[a]$
Theorem (Brendle-Kh)

There is a model in which \( \binom{n_0}{n_1} \rightarrow \binom{m_0}{m_1} \) holds for \( \Delta^1_2 \) partitions, but not all \( \Delta^1_2 \) sets are Miller-measurable (i.e., there are no unbounded reals).

The proof uses a creature forcing \( P_{KSZ} \) due to [Kellner-Shelah, 2009] and [Shelah-Zapletal, 2010].
Theorem (Brendle-Kh)

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\binom{n_0}{n_1} \rightarrow \binom{m_0}{m_1}
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Forcing conditions look like uniform finitely branching trees with a lower bound on the branching size. However, ordering is not simply inclusion.

\( P_{KSZ} \) adds a generic real \( x_G := \bigcup \{ \text{stem}(p) \mid p \in G \} \)
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The proof uses a creature forcing \( \mathbb{P}_{KSZ} \) due to [Kellner-Shelah, 2009] and [Shelah-Zapletal, 2010].

Forcing conditions look like uniform finitely branching trees with a lower bound on the branching size. However, ordering is not simply inclusion.

\( \mathbb{P}_{KSZ} \) adds a generic real \( x_G := \bigcup \{ \text{stem}(p) \mid p \in G \} \), but the generic filter is not determined from the generic real in the usual fashion and \( \mathbb{P}_{KSZ} \) is not in general representable as \( \mathcal{B}(\omega^\omega)/I \) for a \( \sigma \)-ideal \( I \).
Interesting fact: computations of bounds $\langle n_i \rangle_{i \in \omega}$ follows from purely forcing-theoretic considerations. Assuming all $m_i = 2$, we get:

\[
n_i := 2 \left( (2 \prod_{j < i} n_j)^i \right)
\]
$\Delta^1_2$ level: diagram of implications

$\Delta^1_2 (\text{Ramsey})$

$\Delta^1_2 (Laver)$
$\forall a \exists$ dominating/$L[a]$

$\Delta^1_2 (\text{Miller})$
$\forall a \exists$ unbounded/$L[a]$

$\Delta^1_2 (\bar{m} \rightarrow \bar{m})$

$\forall a \exists$ ev. diff./$L[a]$

$\Delta^1_2 (\vec{n} \rightarrow \vec{m})$
$\Delta^1_2$ level: diagram of implications

$\Delta^1_2(\text{Ramsey})$ \quad $\Delta^1_2(\bar{m} \to \bar{m})$

$\Delta^1_2(\text{Laver}) \forall a \exists \text{dominating}/L[a]$

$\Delta^1_2(\text{Miller}) \forall a \exists \text{unbounded}/L[a]$

$\forall a \exists \text{ev. diff.}/L[a]$
Open question

Still open: Is the implication “$\Delta^1_2(\vec{\omega} \rightarrow \vec{m}) \rightarrow \forall a \exists$ eventually different real over $L[a]$” irreversible?
Open question

Still open: Is the implication \( \Delta^1_2(\bar{\omega} \rightarrow \bar{m}) \implies \forall a \exists \text{ eventually different real over } L[a] \) irreversible?

Conjecture

\( \Delta^1_2(\bar{\omega} \rightarrow \bar{m}) \) fails in the Random model.
Can we extend the result about $\mathbb{P}_{\text{KSZ}}$ to $\Sigma^1_2$?
The $\Sigma^1_2$ level

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Not a priori, since $\mathbb{P}_{KSZ}$ only adds one generic real.
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In [DiPrisco & Todorčević, 2003] a forcing is introduced which adds a 

*generic product* $H = \langle H_i \rangle_{i \in \omega}$ satisfying the following “*clopification property*”:

For every Borel set $B$ in the ground model, $B \cap \prod_i H_i$ is relatively clopen in the induced topology of $\prod_i H_i$. 

Theorem (Brendle-Kh)

An $\omega_1$-iteration of any (proper) forcing notion satisfying the clopification property yields a model where $(n_0 \cdot \cdot \cdot n) \to (m_0 \cdot \cdot \cdot m)$ holds for $\Sigma^1_2$ partitions.
The $\Sigma^1_2$ level

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The $\Sigma^1_2$ level

Problem with using the DiPrisco-Todorčević forcing: difficult to see whether it is $\omega^\omega$-bounding or not.
The $\Sigma^1_2$ level

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So instead, we combine elements of the DiPrisco-Todorčević forcing with $\mathcal{P}_{KSZ}$,
The $\Sigma^1_2$ level

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**Corollary**

There is a model in which

$$\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$$

holds for $\Sigma^1_2$ partitions but $\Sigma^1_2(Miller)$ fails.
Computation of bounds

Some final words on the computation of $\langle n_i \rangle_{i \in \omega}$ from $\langle m_i \rangle_{i \in \omega}$:

1. Finite partitions: $n_i := 2m_i \cdot \prod_{j < i} (n_j m_j)$.
2. For Borel and analytic sets: $n_i := 2(2 \prod_{j < i} n_j)^i$.
3. For $\Delta^1_2$ sets: same as above.
4. For $\Sigma^1_2$ sets: currently much higher: non-primitive recursive.
Some final words on the computation of $\langle n_i \rangle_{i \in \omega}$ from $\langle m_i \rangle_{i \in \omega}$:

1. Finite partitions: $n_i := 2 \cdot m_i \cdot \prod_{j < i} \binom{n_j}{m_j}$. 

Yuri Khomskii (University of Amsterdam)
Some final words on the computation of \( \langle n_i \rangle_{i \in \omega} \) from \( \langle m_i \rangle_{i \in \omega} \):

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Some final words on the computation of $\langle n_i \rangle_{i \in \omega}$ from $\langle m_i \rangle_{i \in \omega}$:

1. Finite partitions: $n_i := 2 \cdot m_i \cdot \prod_{j<i} \binom{n_j}{m_j}$.

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3. For $\Delta^1_2$ sets: same as above.
Some final words on the computation of $\langle n_i \rangle_{i \in \omega}$ from $\langle m_i \rangle_{i \in \omega}$:

1. **Finite partitions:** $n_i := 2 \cdot m_i \cdot \prod_{j < i} \left( \frac{n_j}{m_j} \right)$.

2. **For Borel and analytic sets:** $n_i := 2 \left( \left( 2 \prod_{j < i} n_j \right)^i \right)$.

3. **For $\Delta^1_2$ sets:** same as above.

4. **For $\Sigma^1_2$ sets:** currently much higher: non-primitive recursive.
Thank you!

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