A reflecting principle compatible with the continuum large (1)

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ESI workshop on large cardinals and descriptive set theory

PFA implies $2^{\aleph_0} = \aleph_2$.
All known proofs of this implication use forcing notions that collapse ω_2 to ω_1.

Question: Does *FA*$(\{\mathbb{P} : \mathbb{P} \text{ proper}, |\mathbb{P}| = \aleph_1\})$ imply $2^{\aleph_0} = \aleph_2$?

I will isolate a certain subclass Γ of $\{\mathbb{P} : \mathbb{P} \text{ proper}, |\mathbb{P}| = \aleph_1\}$ and will sketch a proof that $FA(\Gamma) + 2^{\aleph_0} > \aleph_2$ is consistent.

$FA(\Gamma)$ will be strong enough to imply for example the negation of Justin Moore's $\mho$ and other strong forms of the negation of Club Guessing.

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Notation

If N is a set such that $N \cap \omega_1 \in \omega_1$, set $\delta_N = N \cap \omega_1$.

If $\mathcal{W}$ is a collection of countable sets and N is a set, $\mathcal{W}$ is N–stationary if for every ordinal $\gamma \in N$ and every function $Z : [\gamma]^{<\omega} \longrightarrow \gamma$, $Z \in N$ there is some $M \in \mathcal{W} \cap N$ such that $Z``[M]^{<\omega} \subseteq M$.

If $\mathbb{P}$ is a partial order, $\mathbb{P}$ is *nice* if

(a) conditions in $\mathbb{P}$ are functions with domain included in ω_1, and

(b) if $p, q \in \mathbb{P}$ are compatible, then the greatest lower bound r of p and q exists, $dom(r) = dom(p) \cup dom(q)$, and $r(\nu) = p(\nu) \cup q(\nu)$ for all $\nu \in dom(r)$ (where $f(\nu) = \emptyset$ if $\nu \notin dom(f)$).

Exercise: Every set–forcing for which $glb(p, q)$ exists whenever p and q are compatible conditions is isomorphic to a nice forcing.

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More notation

Given a nice partial order $(\mathbb{P}, \leq)$, a $\mathbb{P}$–condition p and a set M such that δ_M exists, we say that *M is good for p* iff, letting

$$X = \{s \in \mathbb{P} \cap M \,:\, s \leq p \restriction \delta_M,\ s \text{ compatible with } p\},$$

(i) $X \neq \emptyset$, and

(ii) for every $s \in X$ there is some $t \leq s$, $t \in M$, such that for all $t' \leq t$, if $t' \in M$, then $t' \in X$.

A class of posets

Let $\mathbb{P}$ be a nice poset. $\mathbb{P}$ is κ–suitable if there are a binary relation R and a club $C \subseteq \omega_1$ satisfying the following properties.

(1) If $p\, R\, (N, \mathcal{W})$, then the following conditions hold.

(1.1) N is a countable subset of $H(\kappa)$, $\mathcal{W}$ is an N–stationary subset of $[H(\kappa)]^{\aleph_0}$, and all members of $\mathcal{W} \cap N$ are good for p.

(1.2) If p' is a $\mathcal{P}$–condition extending p, then there is some $\mathcal{W}' \subseteq \mathcal{W}$ such that $p'\, R\, (N, \mathcal{W}')$.

(1.3) If $\mathcal{W}' \subseteq \mathcal{W}$ is N–stationary, then $p\, R\, (N, \mathcal{W}')$.

A class of posets

(2) For every $p \in \mathcal{P}$ and every finite set $\{(N_i, \mathcal{W}_i) : i < m\}$ such that

- (∘) each N_i is a countable subset of $H(\kappa)$ containing p, $\omega_1^{N_i} = \omega_1$, $\delta_{N_i} \in C$, $N_i \models ZFC^*$, and
- (∘) each $\mathcal{W}_i$ is N_i–stationary

there is a condition $q \in \mathcal{P}$ extending p and there are $\mathcal{W}_i' \subseteq \mathcal{W}_i$ ($i < m$) such that $q\, R\, (N_i, \mathcal{W}_i')$ for all $i < m$.

We will say that a nice partial order is *absolutely κ–suitable* if it is κ–suitable in every inner model W containing it and such that $\omega_1^W = \omega_1$.

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A class of posets

Let Γ_κ denote the class of all absolutely κ–suitable posets consisting of finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$.

Easy: For all $\kappa \geq \omega_2$, $\Gamma_\kappa \subseteq$ *Proper*.

FA(Γ_κ): For every $\mathbb{P} \in \Gamma_\kappa$ and every collection $\mathcal{D}$ of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

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One application of $FA(\Gamma_\kappa)$: Ω

Definition (Moore) $\mho$: There is a sequence $\langle g_\delta : \delta < \omega_1 \rangle$ such that each $g_\delta : \delta \longrightarrow \omega$ is continuous with respect to the order topology and such that for every club $C \subseteq \omega_1$ there is some $\delta \in C$ with $g_\delta``C = \omega$.

(○) Club Guessing implies $\mho$.

(○) $\mho$ preserved by ccc forcing, and in fact by ω–proper forcing.

(○) Each of *BPFA* and *MRP* implies $\Omega := \neg\mho$.

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Proposition: For every $\kappa \geq \omega_2$, $FA(\Gamma_\kappa)$ implies Ω.

Proof sketch:

Notation: Given X, a set of ordinals, and δ, an ordinal, set

- ($\circ$) $rank(X, \delta) = 0$ iff δ is not a limit point of X, and
- ($\circ$) $rank(X, \delta) > \eta$
  if and only if δ is a limit of ordinals ϵ such that $rank(X, \epsilon) \geq \eta$.

Given a sequence $\mathcal{G} = \langle g_\delta : \delta < \omega_1 \rangle$ of continuous colourings, let $\mathbb{P}_\mathcal{G}$ be the following poset:

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Conditions in $\mathbb{P}_{\mathcal{G}}$ are pairs $p = (f, \langle k_\xi : \xi \in D \rangle)$ satisfying the following properties:

(1) f is a finite function that can be extended to a normal function $F : \omega_1 \longrightarrow \omega_1$.

(2) For every $\xi \in dom(f)$, $rank(f(\xi), f(\xi)) \geq \xi$.

(3) $D \subseteq dom(f)$ and for every $\xi \in dom(f)$,

(3.1) $k_\xi < \omega$,
(3.2) $g_{f(\xi)}$"$range(f) \subseteq \omega \backslash \{k_\xi\}$, and
(3.3) $rank(\{\gamma < f(\xi) : g_{f(\xi)}(\gamma) \neq k_\xi\}, f(\xi)) = rank(f(\xi), f(\xi))$.

Given conditions $p_\epsilon = (f_\epsilon, (k^\epsilon_\xi : \xi \in D_\epsilon)) \in \mathbb{P}_{\mathcal{G}}$ for $\epsilon \in \{0, 1\}$, p_1 extends p_0 iff

(i) $f_0 \subseteq f_1$,

(ii) $D_0 \subseteq D_1$, and

(iii) $k^1_\xi = k^0_\xi$ for all $\xi \in D_0$.

Easy: If G is $\mathbb{P}_{\mathcal{G}}$–generic and $C = range(\bigcup\{f : (\exists\vec{k})(\langle f, \vec{k}\rangle \in G)\})$, then C is a club of ω_1^V and for every $\delta \in C$ there is $k_\delta \in \omega$ such that g_δ“$C \subseteq \omega\setminus\{k_\delta\}$.

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$\mathbb{P}_{\mathcal{G}} \in \Gamma_\kappa$ for every $\kappa \geq \omega_2$:

(•) We may easily translate $\mathbb{P}_{\mathcal{G}}$ into a nice forcing consisting of finite functions contained in $\omega_1 \times [\omega_1]^{<\omega}$.

(•) Given $p = (f, \langle k_\xi : \xi \in D \rangle) \in \mathbb{P}_{\mathcal{G}}$, $N \subseteq H(\kappa)$ countable such that $N \models ZFC^*$ and δ_N exists, and given $\mathcal{W}$ an N–stationary set, set

$$p\, R\, (N, \mathcal{W})$$

if and only if

(a) δ_N is a fixed point of f,

(b) $\delta_N \in D$, and

(c) for every $M \in \mathcal{W}$, $g_{\delta_N}(\delta_M) \neq k_{\delta_N}$.

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Easy to verify:

(1) in the definition of κ–suitable

[that is:

If $p\,R\,(N, \mathcal{W})$, then

(1.1) N is a countable subset of $H(\kappa)$, $\mathcal{W}$ is an N–stationary subset of $[H(\kappa)]^{\aleph_0}$, and all members of $\mathcal{W} \cap N$ are good for p,

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Let us check (2) in the definition of κ–suitable (with $C = \omega_1$)

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(2) For every $p \in \mathcal{P}$ and every finite set $\{(N_i, \mathcal{W}_i) : i < m\}$ such that

(a) each N_i is a countable subset of $H(\kappa)$ containing p, $\omega_1^{N_i} = \omega_1$, $\delta_{N_i} \in C$, $N_i \models ZFC^*$, and

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Let $p = (f, \langle k_\xi : \xi \in D \rangle) \in \mathbb{P}_{\mathcal{G}}$. Let $\{(N_i, \mathcal{W}_i) : i < m\}$ satisfy (a) and (b).

Let $(\delta_j)_{j<n}$ be the increasing enumeration of $\{\delta_{N_i} : i < m\}$.

Suppose $\{N_i : \delta_{N_i} = \delta_0\} = \{N_0, N_1, N_2\}$.
Let $\{k_0, \ldots k_{2^3-1}\}$ be 8 colours not touched by g_{δ_0}"$range(f)$.

There is $k^0 \in \{k_0, \ldots k_7\}$ such that, for all $i < 3$, $\mathcal{W}_i' = \{M \in \mathcal{W}_i : \delta_M \neq k^0\}$ is N_i–stationary.

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Now we continue with δ_1,
and get a colour k^1 we may avoid in the colouring g_{δ_1}. And so on.

In the end there is a condition $q = (f', \langle k'_\xi : \xi \in D' \rangle)$, $q \le p$, and N_i–stationary $\mathcal{W}'_i \subseteq \mathcal{W}_i$ $(i < m)$ such that

(a) f' has all δ_j $(j < n)$ as limit points and makes the promise k^j at each δ_j, and

(b) $q \, R \, (N_i, \mathcal{W}'_i)$ for all $i < m$.

Hence, $\mathbb{P}_\mathcal{G}$ is (isomorphic to) a forcing in Γ_κ.

An application of $FA(\{\mathbb{P}_\mathcal{G}\})$ gives now a witness of Ω for $\mathcal{G}$.
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$\square$

Another application of $FA(\Gamma_\kappa)$

Proposition: For every $\kappa \geq \omega_2$, $FA(\Gamma_\kappa)$ implies:

$\neg$*VWCG*: For every $\mathcal{C}$, if

(a) $|\mathcal{C}| = \aleph_1$ and
(b) for all $X \in \mathcal{C}$, $X \subseteq \omega_1$ and $ot(X) = \omega$,

then there is a club $C \subseteq \omega_1$ such that $|X \cap C| < \omega$ for all $X \in \mathcal{C}$.

$\neg$*VWCG* is equivalent to the following statement:

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The main theorem

Main Theorem (*CH*) Let κ be a cardinal such that $2^{<\kappa} = \kappa$, $\kappa^{\aleph_1} = \kappa$ and $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$. Then there is a partial order $\mathcal{P}$ such that

(1) $\mathcal{P}$ is proper,

(2) $\mathcal{P}$ has the $\aleph_2$–chain condition,

(3) $\mathcal{P}$ forces

- (•) *FA*(Γ_κ)
- (•) $2^{\aleph_0} = \kappa$

Proof sketch

Let $\Phi : \kappa \longrightarrow H(\kappa)$ be a bijection.

(Φ exists by $2^{<\kappa} = \kappa$.
Note: There is an ω_1–club of $\gamma < \kappa$ such that $\Phi``\gamma$ enumerates $[\gamma]^{\aleph_0}$.)

Proof sketch (continued)

Coherent systems of structures

$\{N_i : i < m\}$ is a *coherent systems of structures* if

$a1)$ $m < \omega$ and every N_i is a countable subset of $H(\kappa)$ such that $(N_i, \in, \Phi \cap N_i) \preccurlyeq (H(\kappa), \in, \Phi)$.

$a2)$ Given distinct i, i' in m, if $\delta_{N_i} = \delta_{N_{i'}}$, then there is an isomorphism

$$\Psi_{N_i, N_{i'}} : (N_i, \in, \Phi \cap N_i) \longrightarrow (N_{i'}, \in, \Phi \cap N_{i'})$$

Furthermore, $\Psi_{N_i, N_{i'}}$ is the identity on $\kappa \cap N_i \cap N_{i'}$.

Proof sketch (continued)

$a3$) For all i, j in m, if $\delta_{N_j} < \delta_{N_i}$, then there is some $i' < m$ such that $\delta_{N_{i'}} = \delta_{N_i}$ and $N_j \in N_{i'}$.

$a4$) For all i, i', j in m, if $N_j \in N_i$ and $\delta_{N_i} = \delta_{N_{i'}}$, then there is some $j' < m$ such that $N_{j'} = \Psi_{N_i, N_{i'}}(N_j)$.

Proof sketch (continued)

Our forcing will be the direct limit $\mathcal{P}_{\omega_2}$ of a sequence $\langle \mathcal{P}_\alpha : \alpha < \omega_2 \rangle$ of posets such that

(○) $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$ if $\alpha < \beta \leq \omega_2$, and

(○) a condition q in $\mathcal{P}_\alpha$ is an α–sequence p together with a certain system Δ_q of side conditions.

Unlike in a usual iteration, p will not consist of names, but of well–determined objects (finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$).

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Defining $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$

$\mathcal{P}_0$: Conditions are $p = \{(N_i, 0) : i < m\}$ where $\{N_i : i < m\}$ is a coherent system of structures.

$\leq_0$ is $\supseteq$.

Defining $\langle \mathcal{P}_\alpha : \alpha \le \omega_2 \rangle$ (continued)

Suppose $\mathcal{P}_\alpha$ defined and suppose conditions in $\mathcal{P}_\alpha$ are pairs (p, Δ_p) with p an α–sequence and $\Delta_p = \{(N, \beta_i) : i < m\}$.

Suppose $\mathcal{P}_\alpha$ has the $\aleph_2$–chain condition and $|\mathcal{P}_\alpha| = \kappa$.

By $\kappa^{\aleph_1} = \kappa$ we may fix an enumeration $\dot{\mathcal{Q}}^\alpha_i$ (for $i < \kappa$) of nice κ–suitable partial orders consisting of finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$ such that for every $\mathcal{P}_\alpha$–name $\dot{\mathcal{Q}}$ for such a poset there are κ–many $i < \kappa$ such that $\Vdash_{\mathcal{P}_\alpha} \dot{\mathcal{Q}} = \dot{\mathcal{Q}}^\alpha_i$.

We also fix $\mathcal{P}_\alpha$–names $\dot{R}^\alpha_i$ and $\dot{C}^\alpha_i$ (for $i < \kappa$) such that $\mathcal{P}_\alpha$ forces that $\dot{R}^\alpha_i$ and $\dot{C}^\alpha_i$ witness that $\dot{\mathcal{Q}}^\alpha_i$ is κ–suitable.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

Suppose $\mathcal{P}_\alpha$ defined and suppose conditions in $\mathcal{P}_\alpha$ are pairs (p, Δ_p) with p an α–sequence and $\Delta_p = \{(N, \beta_i) : i < m\}$.

Suppose $\mathcal{P}_\alpha$ has the $\aleph_2$–chain condition and $|\mathcal{P}_\alpha| = \kappa$.

By $\kappa^{\aleph_1} = \kappa$ we may fix an enumeration $\dot{\mathcal{Q}}_i^\alpha$ (for $i < \kappa$) of nice κ–suitable partial orders consisting of finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$ such that for every $\mathcal{P}_\alpha$–name $\dot{\mathcal{Q}}$ for such a poset there are κ–many $i < \kappa$ such that $\Vdash_{\mathcal{P}_\alpha} \dot{\mathcal{Q}} = \dot{\mathcal{Q}}_i^\alpha$.

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Defining $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

$\mathcal{P}_{\alpha+1}$: Conditions are

$$q = (p^\frown \langle f_i : i \in a \rangle, \{(N_i, \beta_i) : i < m\})$$

satisfying the following conditions.

*b*1) For all $i < m$, $\beta_i \leq (\alpha + 1) \cap sup(N_i \cap \omega_2)$.

*b*2) The restriction of q to α is a condition in $\mathcal{P}_\alpha$. This restriction is defined as the object $q|_\alpha := (p, \{(N_i, \beta_i^\alpha) : i < m\})$; where $\beta_i^\alpha = \beta_i$ if $\beta_i < \alpha + 1$, and $\beta_i^\alpha = \alpha$ if $\beta_i = \alpha + 1$. We denote $\{(N_i, \beta_i) : i < m\}$ by Δ_q.

*b*3) a is a finite subset of κ.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

$b4$) For each $i \in a$, f_i is a finite function included in $\omega_1 \times [\omega_1]^{<\omega}$ and $q|_\alpha$ forces (in $\mathcal{P}_\alpha$) that $f_i \in \dot{\mathcal{Q}}_i^\alpha$.

$b5$) For every N such that $(N, \alpha + 1) \in \Delta_q$ and $\alpha \in N$, $q|_\alpha$ forces that there is some $\mathcal{W}_N \subseteq \mathcal{W}^\alpha$ such that

$$f_i \dot{R}_i^\alpha (N, \mathcal{W}_N)$$

for all $i \in a \cap N$.

Here, $\mathcal{W}^\alpha$ denotes the collection of all M such that $(M, \alpha) \in \Delta_u$ for some $u \in \dot{G}_\alpha$.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

Given conditions

$$q_\epsilon = (p_\epsilon ^\frown \langle f_i^\epsilon : i \in a_\epsilon \rangle, \{(N_i^\epsilon, \beta_i^\epsilon) : i < m_\epsilon\})$$

(for $\epsilon \in \{0, 1\}$), we will say that $q_1 \leq_{\alpha+1} q_0$ if and only if the following holds.

$c\,1)$ $q_1|_\alpha \leq_\alpha q_0|_\alpha$

$c\,2)$ $a_0 \subseteq a_1$

$c\,3)$ For all $i \in a_0$, $q|_\alpha$ forces in $\mathcal{P}_\alpha$ that $f_i^1 \leq_{\dot{\mathcal{Q}}_i^\alpha} f_i^0$.

$c\,4)$ For all $i < m_0$ there exists $\widetilde{\beta}_i \geq \beta_i^0$ such that $(N_i^0, \widetilde{\beta}_i) \in \Delta_{q_1}$.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

Suppose $\alpha \leq \omega_2$ is a nonzero limit ordinal.

$\mathcal{P}_\alpha$ Conditions are $q = (p, \{(N_i, \beta_i) : i < m\})$ such that:

d 1) p is a sequence of length α.

d 2) For all $i < m$, $\beta_i \leq \alpha \cap sup(X_i \cap \omega_2)$. (Note that β_i is always less than ω_2, even when $\alpha = \omega_2$.)

d 3) For every $\varepsilon < \alpha$, the restriction $q|_\varepsilon := (p \restriction \varepsilon, \{(X_i, \beta_i^\varepsilon) : i < m\})$ is a condition in $\mathcal{P}_\varepsilon$; where $\beta_i^\varepsilon = \beta_i$ if $\beta_i \leq \varepsilon$, and $\beta_i^\varepsilon = \varepsilon$ if $\beta_i > \varepsilon$.

d 4) The set of $\zeta < \alpha$ such that $p(\zeta) \neq \emptyset$ is finite.

Defining $\langle \mathcal{P}_\alpha \,:\, \alpha \leq \omega_2 \rangle$ (continued)

Given conditions $q_1 = (p_1, \Delta_1)$ and $q_0 = (p_0, \Delta_0)$ in $\mathcal{P}_\alpha$, $q_1 \leq_\alpha q_0$ if and only if:

*e*1) For every $(X_i, \beta_i) \in \Delta_0$ there exists $\widetilde{\beta}_i \geq \beta_i$ such that $(X_i, \widetilde{\beta}_i) \in \Delta_1$.

*e*2) For every $\beta < \alpha$, $q_1|_\beta \leq_\beta q_0|_\beta$.

Notation: If $\alpha \leq \omega_2$ and $q = (p, \{(N_i, \beta_i) \,:\, i < m\} \in \mathcal{P}_\alpha$, we set $\mathcal{X}_q = \{N_i \,:\, i < m\}$.

Main facts about $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$

Lemma Let $\alpha \leq \beta \leq \omega_2$.

If $q = (p, \Delta_q) \in \mathcal{P}_\alpha$, $s = (r, \Delta_s) \in \mathcal{P}_\beta$ and $q \leq_\alpha s|_\alpha$, then $(p^\frown(r \restriction [\alpha, \beta)), \Delta_q \cup \Delta_s)$ is a condition in $\mathcal{P}_\beta$ extending s.

Therefore, $\mathcal{P}_\alpha$ can be seen as a complete suborder of $\mathcal{P}_\beta$.

Lemma For every $\alpha \leq \omega_2$, $\mathcal{P}_\alpha$ is $\aleph_2$–Knaster.

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Therefore, $\mathcal{P}_\alpha$ can be seen as a complete suborder of $\mathcal{P}_\beta$.

Lemma For every $\alpha \leq \omega_2$, $\mathcal{P}_\alpha$ is $\aleph_2$–Knaster.

Let $\langle \theta_\alpha : \alpha \leq \omega_2 \rangle$ be the following sequence of regular cardinals: $\theta_0 = (2^\kappa)^+$, $\theta_\gamma = (sup_{\alpha<\gamma}\theta_\alpha)^+$ if γ is a nonzero limit ordinal, and $\theta_{\alpha+1} = (2^{\theta_\alpha})^+$.

Also, for each $\alpha \leq \omega_2$ let $\mathcal{M}_\alpha$ be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing $\langle \theta_\beta : \beta < \alpha \rangle$, Φ and $\mathcal{P}_\alpha$.

All $\mathcal{P}_\alpha$ are proper:

Let $\langle \theta_\alpha : \alpha \leq \omega_2 \rangle$ be the following sequence of regular cardinals: $\theta_0 = (2^\kappa)^+$, $\theta_\gamma = (sup_{\alpha<\gamma}\theta_\alpha)^+$ if γ is a nonzero limit ordinal, and $\theta_{\alpha+1} = (2^{\theta_\alpha})^+$.

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All $\mathcal{P}_\alpha$ are proper:

Lemma Suppose $\alpha \leq \omega_2$ and $N^* \in \mathcal{M}_\alpha$. Then,

$(1)_\alpha$ for every $q \in N^* \cap \mathcal{P}_\alpha$ there is $q' \leq_\alpha q$ such that $(N^* \cap H(\kappa), \alpha \cap sup(N^* \cap \omega_2)) \in \Delta_{q'}$, and

$(2)_\alpha$ for every $q \in \mathcal{P}_\alpha$, if there is some N such that $(N, \alpha \cap sup(N \cap \omega_2)) \in \Delta_q$ and such that either

(a) $N^* \cap H(\kappa) = N$ or

(b) $N^* \cap H(\kappa) = \Phi``(\gamma \cap N)$ for some $\gamma \in N \cap \kappa$ such that $\Phi \restriction \gamma$ enumerates $[\gamma]^{\aleph_0}$,

then q is $(N^*, \mathcal{P}_\alpha)$–generic.

The proof is by induction on α.

Proof sketch of $(2)_\alpha$ in the case $\alpha = \sigma + 1$:

Let E be an open and dense subset of $\mathcal{P}_\alpha$ in N^*.
It suffices to show that every q satisfying the hypothesis of $(2)_\alpha$ is compatible with some condition in $E \cap N^*$.
By density of E we may assume, without loss of generality, that $q \in E$.
We may also assume that $a^q \neq \emptyset$.

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We may also assume that $a^q \neq \emptyset$.

Claim

For every $i \in \kappa \setminus N^*$ there are ordinals $\alpha_i < \beta_i$ such that

(a) $\alpha_i \in N^*$ *and* $\beta_i \in (\kappa \cap N^*) \cup \{\kappa\}$,

(b) $\alpha_i < i < \beta_i$, *and*

(c) $[\alpha_i, \beta_i) \cap N' \cap N^* = \emptyset$ *whenever* $N' \in \mathcal{X}_q \setminus N^*$ *is such that* $\delta_{N'} < \delta_{N^*}$.

[This is proved using the fact that all $\Psi_{\overline{N},N}$ fix $\kappa \cap \overline{N} \cap N$ and are continuous (for $\overline{N} \in \mathcal{X}_q$ with $\delta_{\overline{N}} = \delta_N$), meaning that $\Psi_{\overline{N},N}(\xi) = sup(\Psi_{\overline{N},N}``\xi)$ whenever $\xi \in \overline{N}$ is an ordinal of countable cofinality.]

Suppose $a^q \setminus N^* = \{i_0, \dots i_{n-1}\}$, and for each $k < n$ let $\alpha_k < \beta_k$ be ordinals realizing the above claim for i_k.

Let us work in $V^{\mathcal{P}_\sigma \restriction (q|_\sigma)}$. By condition $b5$) in the definition of $\mathcal{P}_{\sigma+1}$ we know that there is a stationary $\mathcal{W}_N \subseteq \mathcal{W}^\sigma$ such that $f_i^q \dot{R}_i^\sigma (N, \mathcal{W}_N)$ for all $i \in a^q \cap N$.

By an inductive construction (using (1) in the definition of κ–suitable) we may find an N–stationary $\mathcal{W} \subseteq \mathcal{W}_N$ such that $f_i^q \dot{R}_i^\sigma (N, \mathcal{W})$ for all $i \in a^q \cap N$ and such that each $M \in \mathcal{W}$ is good for f_j^q for every $j \in a^q \cap M$.

Since $N^* \cap H(\kappa)$ is an $\in$–initial segment of N and since $[N^* \cap H(\kappa)]^{\aleph_0} \cap N \subseteq N^*$, every N–stationary subset of $[H(\kappa)]^{\aleph_0}$ is also N^*–stationary.

Hence, we may find M^* and M in N^* such that

(a) $M^* \in \mathcal{M}_\sigma$ and M^* contains $\mathcal{P}_{\sigma+1}$, E, $a^q \cap N^*$, $f_i^q \restriction \delta_{N^*}$ for every $i \in a^q \cap N^*$, α_k for every $k < n$, and β_k for every $k < n$ with $\beta_k < \kappa$.

(b) $(M, \sigma) \in \Delta_u$ for some $u \in \dot{G}_\sigma$,

(c) $M^* \cap H(\kappa) = \Phi``(\gamma \cap M)$ for some ordinal $\gamma \in M$ such that $\Phi \restriction \gamma$ enumerates $[\gamma]^{\aleph_0}$, and

(d) M is good for f_i^q for every $i \in a^q \cap N^*$.

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(b) $(M, \sigma) \in \Delta_u$ for some $u \in \dot{G}_\sigma$,

(c) $M^* \cap H(\kappa) = \Phi``(\gamma \cap M)$ for some ordinal $\gamma \in M$ such that $\Phi \restriction \gamma$ enumerates $[\gamma]^{\aleph_0}$, and

(d) M is good for f_i^q for every $i \in a^q \cap N^*$.

From (d), together with $\delta_M = \delta_{M^*}$, we have that M^* is good for f_i^q for every $i \in a^q \cap N^*$. For every such i let f_i be a $\dot{\mathcal{Q}}_i^\sigma$–condition in M^* extending $f_i^q \restriction \delta_{M^*} = f_i^q \restriction \delta_N$ and such that every $\dot{\mathcal{Q}}_i^\sigma$–condition in M^* extending f_i is compatible with f_i^q.

By extending q below σ we may assume that $(M, \sigma) \in \Delta_q$ and that q_σ decides f_i for every $i \in a^q$.

The result of replacing f_i^q with $glb(f_i, f_i^q)$ in q for every $i \in a^q \cap N^*$ is a $\mathcal{P}_{\sigma+1}$–condition.

Hence, by further extending q if necessary we may assume that every $\dot{\mathcal{Q}}_i^\sigma$–condition in M^* extending $f_i^q \restriction \delta_{M^*}$ is compatible with f_i^q.

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Let now G be a $\mathcal{P}_\sigma$–generic filter over the ground model with $q|_\sigma \in G$.
By correctness of $M^*[G]$ within $H(\theta_\sigma)[G]$ we know that in $M^*[G]$ there is a condition q° satisfying the following conditions.

(a) $q^\circ \in E$ and $q^\circ|_\sigma \in G$.

(b) $a^{q^\circ} = (a^q \cap N^*) \cup \{i_0^\circ, \ldots i_{n-1}^\circ\}$ with $\alpha_k < i_k^\circ < \beta_k$ for all $k < n$.

(c) For all $i \in a^q \cap N^*$, $f_i^{q^\circ}$ extends $f_i^q \restriction \delta_{M^*}$ in $\dot{\mathcal{Q}}_i^\sigma$.

(the existence of such a q° is witnessed, in $V[G]$, by q itself).

By induction hypothesis, $q|_\sigma$ is $(M^*, \mathcal{P}_\sigma)$–generic. Hence, $M^*[G] \cap V = M^*$. It follows that q° is in M^*.

By extending q below σ we may assume that q decides q° and also that it extends $q^\circ|_\sigma$. The proof in this case will be finished if we show that q and q° are compatible.

It is not difficult to find f_i^* (for $i \in a^q \cup \{i_0^\circ, \dots i_{n_1}^*\}$) extending f_i^q and/or $f_{i_k^\circ}^{q^\circ}$ (for $k < n$) for which, in $V^{\mathcal{P}_\sigma \restriction (q|_\sigma)}$, we can verify condition $b5$) with respect to all N' such that $(N', \sigma + 1) \in \Delta_q \cup \Delta_{q^\circ}$ and $\sigma \in N'$.

If $\delta_{N'} \geq \delta_N$, we use condition (2) (and (1)) in the definition of κ–suitable.

If $\delta_{N'} < \delta_N$ and $N' \in M^*$ (that is, $(N', \sigma + 1) \in \Delta_{q^\circ}$), we use condition (1) in the definition of κ–suitable.

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The only potentially problematic case is when $\delta_{N'} < \delta_N$ and $N' \in \mathcal{X}_q \backslash M^*$. But we are safe also in this case since then $(a^q \cup \{i_0^\circ, \dots i_{n_1}^*\}) \cap N' = a^q \cap N'$. We apply again (1) in the definition of κ–suitable.

Finally we extend q below σ once more to a condition q' deciding f_i^*. Now we amalgamate q' and q° and get a legal $\mathcal{P}_\alpha$–condition (note that in extending q below σ we are not adding new pairs $(N', \sigma + 1)$ to Δ).

This finishes the (very sketchy) proof of the lemma in this case.
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This finishes the (very sketchy) proof of the lemma in this case.
$\square$

Given ordinals $\alpha < \omega_2$ and $i < \kappa$, we let $\dot{G}_i^\alpha$ be a $\mathcal{P}_{\alpha+1}$ for the collection of all f_i^q, where $q \in \dot{G}_{\alpha+1}$, $\alpha \in Psupp(q)$, and $i \in a^q$.

Lemma

For every $\alpha < \omega_2$ and every $i < \kappa$, $\mathcal{P}_{\alpha+1}$ forces that $\dot{G}_i^\alpha$ is a $V^{\mathcal{P}_\alpha}$–generic filter over $\dot{\mathcal{Q}}_i^\alpha$.

From the above lemmas it is easy to see by standard arguments that $\mathcal{P}_{\omega_2}$ forces $FA(\Gamma_\kappa)$ and $2^{\aleph_0} = \kappa$. □

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From the above lemmas it is easy to see by standard arguments that $\mathcal{P}_{\omega_2}$ forces $FA(\Gamma_\kappa)$ and $2^{\aleph_0} = \kappa$. $\square$

An enhanced version of the Main Theorem

Given a class Γ of partial orders and a cardinal λ, $FA(\Gamma)_{<\lambda}$ means:

For every $\mathbb{P} \in \Gamma$ and collection $\mathcal{D}$ of size less than λ consisting of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

An enhanced version of the Main Theorem

Theorem (*CH*) Let κ be a cardinal such that $2^{<\kappa} = \kappa$, $\kappa^{\aleph_1} = \kappa$ and $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$. Then there is a partial order $\mathcal{P}$ such that

(1) $\mathcal{P}$ is proper,

(2) $\mathcal{P}$ has the $\aleph_2$–chain condition,

(3) $\mathcal{P}$ forces

- $FA(\Gamma_\kappa)_{<cf(\kappa)}$
- $2^{\aleph_0} = \kappa$

Another strong failure of Club Guessing

Definition (Moore): *Measuring*: For every sequence $(C_\delta : \delta < \omega_1)$ such that each C_δ is a closed subset of δ there is a club $D \subseteq \omega_1$ such that for every limit point $\delta \in D$ of D,

(a) either a tail of $D \cap \delta$ is contained in C_δ,

(b) or a tail of $D \cap \delta$ is disjoint from C_δ.

(○) *Measuring* follows from *BPFA* and also from *MRP*.

(○) *Measuring* implies the negation of Weak Club Guessing and implies Ω.

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- (•) Measuring
- (•) $2^{\aleph_0} = \kappa$