Cardinal invariants of analytic quotients

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1. Classical cardinal invariants

2. Cardinal invariants of analytic quotients
   - Analytic ideals
   - Basic results for F_σ quotients
   - Consistency results for F_σ quotients

What are cardinal invariants of the continuum?

Cardinal invariants of the continuum are cardinals which typically lie between $\aleph_1$ and $\mathfrak{c} = 2^\omega$ and which describe the combinatorial structure of the real line.

Many of the classical cardinal invariants (introduced in the 70's and 80's) are defined in terms of $\mathcal{P}(\omega)/\mathrm{Fin}$.

The quotient $\mathcal{P}(\omega)/\mathrm{Fin}$

For $A, B \subseteq \omega$:

$$A \subseteq^* B \quad (A \text{ is almost contained in } B) \iff A \setminus B \text{ is finite}$$

$A =^* B$ iff $A \subseteq^* B$ and $B \subseteq^* A$ defines an equivalence relation. $\mathcal{P}(\omega)/\mathrm{Fin}$ is the collection of equivalence classes, ordered by

$$[A] \leq [B] \iff A \subseteq^* B$$

where $[A]$ is the class of A.

For simplicity we forget about equivalence classes and work with $([\omega]^\omega, \subseteq^*)$ while meaning $(\mathcal{P}(\omega)/\mathrm{Fin}, \leq)$.

Splitting and reaping: the classical case

Cardinal invariants: an example

For $A, B \in [\omega]^\omega$:

$$A \textit{ splits } B \Longleftrightarrow |A \cap B| = |B \setminus A| = \aleph_0$$

$\mathcal{F} \subseteq [\omega]^\omega$ is *splitting* if every member of $[\omega]^\omega$ is split by a member of $\mathcal{F}$.
$\mathcal{F} \subseteq [\omega]^\omega$ is *unsplit* (or *unreaped*) if no member of $[\omega]^\omega$ splits all members of $\mathcal{F}$, i.e.

$$\forall A \in [\omega]^\omega \; \exists B \in \mathcal{F} \; (|A \cap B| < \aleph_0 \text{ or } B \subseteq^* A)$$

$\mathfrak{s} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is splitting}\}$, the *splitting number*.
$\mathfrak{r} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is unsplit}\}$, the *reaping number*.

Splitting and reaping 2: the classical case

$\mathfrak{b} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is unbounded in } (\omega^\omega, \leq^*)\}$,
the *bounding number*.
$\mathfrak{d} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is cofinal in } (\omega^\omega, \leq^*)\}$, the *dominating number*.
All these cardinal invariants are uncountable (diagonal argument!).

Proposition

$\mathfrak{b} \leq \mathfrak{r}$ *and* $\mathfrak{s} \leq \mathfrak{d}$.

The order-relationship of $\mathfrak{b}$ and $\mathfrak{s}$ is not decidable.
The consistency of $\mathfrak{s} < \mathfrak{b}$ is easy.

Theorem (Shelah; Blass and Shelah)

$\mathfrak{b} < \mathfrak{s}$ *and* $\mathfrak{r} < \mathfrak{d}$ *are consistent.*

$\mathcal{P}(\omega)/\mathrm{Fin}$ as a forcing notion

We may also consider $(\mathcal{P}(\omega)/\mathrm{Fin}, \leq)$ as a forcing notion.
Maximal antichains correspond to *maximal almost disjoint families*:
$\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* if

$$|A \cap B| < \aleph_0$$

for distinct A and B from $\mathcal{A}$. $\mathcal{A}$ is *maximal almost disjoint* (*mad*) if it is almost disjoint and for all $C \in [\omega]^\omega$ there is $A \in \mathcal{A}$ with

$$|A \cap C| = \aleph_0$$

$\mathfrak{a} := \min\{|\mathcal{A}| : \mathcal{A} \text{ is mad}\}$, the *almost-disjointness number*.

Proposition

$\mathfrak{b} \leq \mathfrak{a}$.

$\mathcal{P}(\omega)/\mathrm{Fin}$ as a forcing notion 2

$\mathfrak{h} :=$ min$\{|\mathcal{D}| : \mathcal{D}$ is a family of open dense sets in $\mathcal{P}(\omega)/\mathrm{Fin}$ and $\bigcap \mathcal{D}$ is not dense$\}$, the *distributivity number*.

$\mathfrak{h}$ describes forcing-theoretic properties of $\mathcal{P}(\omega)/\mathrm{Fin}$:

Observation

$\mathfrak{h} =$ min$\{\kappa :$ $\mathcal{P}(\omega)/\mathrm{Fin}$ *adds a new function from* κ *to* $V\}$.

So $\mathcal{P}(\omega)/\mathrm{Fin}$ preserves all cardinals $\leq \mathfrak{h}$ and $> \mathfrak{c}$.
Everything in between is collapsed:

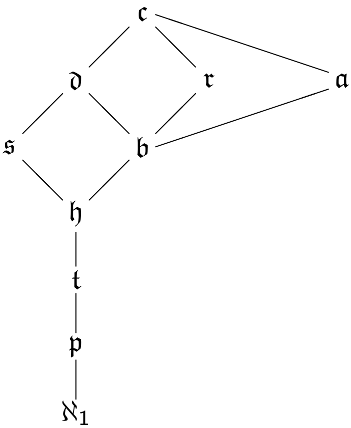
Theorem (Balcar, Pelant and Simon)

$\mathcal{P}(\omega)/\mathrm{Fin}$ *forces* $\mathfrak{c} = \mathfrak{h}^V$.

Proposition

$\aleph_1 \leq \mathfrak{h} \leq$ min$\{\mathfrak{b}, \mathfrak{s}\}$.

ZFC-inequalities: the classical case



1. Classical cardinal invariants

2. Cardinal invariants of analytic quotients
   - Analytic ideals
   - Basic results for F_σ quotients
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Quotients by ideals

Let $\mathcal{I}$ be an ideal on ω with $\mathrm{Fin} \subseteq \mathcal{I}$. Consider the quotient $\mathcal{P}(\omega)/\mathcal{I}$. Define cardinal invariants of this quotient in analogy to the classical cardinal invariants of $\mathcal{P}(\omega)/\mathrm{Fin}$.

We address the following questions:

- Can we prove similar inequalities for the new invariants as for their classical counterparts? Similar consistency results? (E.g., does $\mathfrak{s} \leq \mathfrak{d}$ generalize to $\mathfrak{s}(\mathcal{I}) \leq \mathfrak{d}$?)
- How do the new cardinals compare to the old ones? (E.g. what is the connection between $\mathfrak{s}$ and $\mathfrak{s}(\mathcal{I})$?)

Definable ideals

Which ideals should we consider?

1. $\mathcal{I}$ should be definable
   (so that it lives in any model of set theory)
2. Therefore consider only analytic ideals $\mathcal{I}$
3. Ideally, $\mathcal{I}$ should be F_σ
   (so that the quotient $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed)
4. We may also want to restrict to analytic P-ideals
   (because we have a nice structure theory for them)

Definable ideals 2

Via characteristic functions, identify $\mathcal{I} \subseteq \mathcal{P}(\omega)$ with a subset of 2^ω. Hence we may talk about $\mathcal{I}$ being Borel, analytic, F_σ, etc.

Observation

If $\mathcal{I}$ is F_σ, then $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed, i.e. $\mathfrak{t}(\mathcal{I}) \geq \aleph_1$.

$\mathfrak{t}(\mathcal{I}) := \min\{\kappa : \mathcal{P}(\omega)/\mathcal{I}$ not κ-closed$\}$, the *tower number* of $\mathcal{I}$.

Definition

$\mathcal{I}$ is a *P-ideal* if for all countable $\mathcal{A} \subseteq \mathcal{I}$ there is $B \in \mathcal{I}$ with $A \subseteq^* B$ for all $A \in \mathcal{A}$.

Structure theory for analytic ideals

Definition

$\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a *lower semicontinuous submeasure* if

- $\varphi(\emptyset) = 0, \varphi(\{n\}) < \infty$
- $\varphi(X) \leq \varphi(Y)$ for $X \subseteq Y$
- $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$
- $\varphi(X) = \lim_n \varphi(X \cap n)$

Structure theory for analytic ideals 2

$\mathrm{Exh}(\varphi) := \{X : \lim_n \varphi(X \setminus n) = 0\}$ is an $F_{\sigma\delta}$ P-ideal.
$\mathrm{Fin}(\varphi) := \{X : \varphi(X) < \infty\}$ is an F_σ-ideal.

Theorem (Mazur; Solecki)

- *$\mathcal{I}$ is an F_σ-ideal iff $\mathcal{I} = \mathrm{Fin}(\varphi)$ for some φ.*
- *$\mathcal{I}$ is an analytic P-ideal iff $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some φ.*
- *$\mathcal{I}$ is an F_σ P-ideal iff $\mathcal{I} = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ for some φ.*

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The quotient $\mathcal{P}(\omega)/\mathcal{I}$

Instead of working with $(\mathcal{P}(\omega)/\mathcal{I}, \leq)$, consider $(\mathcal{I}^+, \subseteq_\mathcal{I})$.
Here

$$A \subseteq_\mathcal{I} B \quad (A \text{ is contained in } B \text{ modulo } \mathcal{I}) \Longleftrightarrow A \setminus B \in \mathcal{I}$$

and

$$\mathcal{I}^+ := \mathcal{P}(\omega) \setminus \mathcal{I}$$

denotes the $\mathcal{I}$-*positive* sets.

Splitting and reaping: the F_σ case

Cardinal invariants: an example

For $A, B \in \mathcal{I}^+$:

$$A \textit{ splits } B \iff A \cap B, B \setminus A \in \mathcal{I}^+$$

$\mathcal{F} \subseteq \mathcal{I}^+$ is *$\mathcal{I}$-splitting* if every member of $\mathcal{I}^+$ is split by a member of $\mathcal{F}$.
$\mathcal{F} \subseteq \mathcal{I}^+$ is *$\mathcal{I}$-unsplit* (or *$\mathcal{I}$-unreaped*) if no member of $\mathcal{I}^+$ splits all members of $\mathcal{F}$, i.e.

$$\forall A \in \mathcal{I}^+ \; \exists B \in \mathcal{F} \; (A \cap B \in \mathcal{I} \text{ or } B \subseteq_{\mathcal{I}} A)$$

$\mathfrak{s}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \text{ is } \mathcal{I}\text{-splitting}\}$, the *$\mathcal{I}$-splitting number*.
$\mathfrak{r}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \text{ is } \mathcal{I}\text{-unsplit}\}$, the *$\mathcal{I}$-reaping number*.

Basic results for F_σ quotients 1

From now on assume all ideals are F_σ

Proposition

$\mathfrak{b} \leq \mathfrak{r}(\mathcal{I})$ *and* $\mathfrak{s}(\mathcal{I}) \leq \mathfrak{d}$.

Proposition

$\mathfrak{b} < \mathfrak{s}(\mathcal{I})$ *and* $\mathfrak{r}(\mathcal{I}) < \mathfrak{d}$ *are consistent.*

Proofs: Use $\mathcal{I} = \mathrm{Fin}(\varphi)$. $\square$

Proposition

Some $A \in \mathcal{I}^+$ *forces* $\mathfrak{c} = \mathfrak{h}(\mathcal{I})^V$.

Proposition

$\mathfrak{t}(\mathcal{I}) \leq \mathfrak{h}(\mathcal{I}) \leq \min\{\mathfrak{b}, \mathfrak{s}(\mathcal{I})\}$.

Basic results for F_σ quotients 2

Proposition (Farkas and Soukup)

$\mathfrak{b} \leq \mathfrak{a}(\mathcal{I})$ *for* F_σ *P-ideals.*

Proof: Use $\mathcal{I} = \mathrm{Exh}(\varphi)$. $\square$

Is this true for F_σ-ideals in general?
No!

Theorem

$\mathfrak{a}(\mathcal{ED}_{\mathrm{fin}}) < \mathfrak{b}$ *is consistent.*
(In fact, this holds in the Hechler model.)

Here, $\mathcal{ED}_{\mathrm{fin}}$ is one of the *eventually different* ideals introduced by Hernández and Hrušák.

The ideal $\mathcal{ED}_{\mathrm{fin}}$

Let

$$\Delta = \{(n, i) \in \omega \times \omega : i \leq n\}$$

be the triangle below the diagonal.

$\mathcal{ED}_{\mathrm{fin}}$ is the ideal on Δ generated by graphs of functions:
for $A \subseteq \Delta$:

$$A \in \mathcal{ED}_{\mathrm{fin}} \iff \exists m \, \forall n \, (|A_n| \leq m)$$

where $A_n = \{i : (n, i) \in A\}$ is the *vertical section* of A at n.
This is an F_σ-ideal.

Basic results for F_σ quotients 3

Observation

$\mathfrak{p} \leq \mathfrak{a}(\mathcal{I})$.

Proposition

$\mathfrak{a}(\mathcal{I})$ can be increased by a definable σ-centered forcing $\mathbb{P}$.

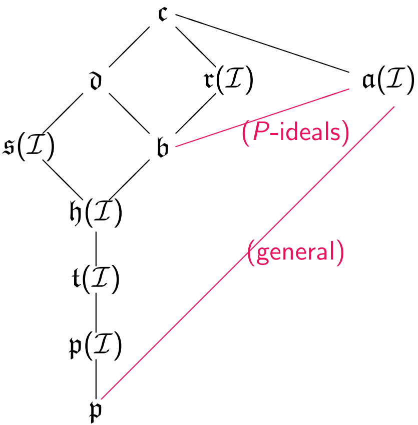
Corollary

$\mathfrak{a}(\mathcal{I}) > \mathfrak{d}$ is consistent.

Proof: Put $\mathbb{P}$ into Shelah's template framework. $\square$

... and many more.

ZFC-inequalities: the F_σ case



Summable ideals

Let $f : \omega \to \mathbb{R}^+$ with $\sum_n f(n) = \infty$.

For $A \subseteq \omega$:

$$A \in \mathcal{I}_f \iff \sum_{n \in A} f(n) < \infty$$

$\mathcal{I}_f$ is an F_σ P-ideal.

In fact $\mathcal{I}_f = \mathrm{Fin}(\mu_f) = \mathrm{Exh}(\mu_f)$ where

$$\mu_f(A) = \sum_{n \in A} f(n)$$

Splitting and reaping for summable ideals

Theorem

$\mathfrak{s}(\mathcal{I}) < \mathfrak{s}$ is consistent for any tall summable ideal $\mathcal{I}$.
Dually $\mathfrak{r} < \mathfrak{r}(\mathcal{I})$ is consistent.

Proof: $\mathcal{I} = \mathcal{I}_f$. Let $\varepsilon \gg \delta > 0$.
Say $g : \omega \to [\omega]^{<\omega}$ is an *ε-function* if

$$\mu_f(g(n)) \geq \varepsilon \text{ for all } n \text{ and } \limsup_n (\min g(n)) = \infty$$

$X \in [\omega]^\omega$ *δ-splits* g if

$$\exists^\infty n \left(\mu_f(g(n) \cap X) \geq \frac{\varepsilon}{2} - \delta \text{ and } \mu_f(g(n) \setminus X) \geq \frac{\varepsilon}{2} - \delta \right)$$

Splitting and reaping for summable ideals 2

Crucial Lemma

Let M, N be models of ZFC. Let $\mathcal{U}$ be an ultrafilter in M. Assume $X \in [\omega]^\omega \cap N$ satisfies

$$(\star_X^{M,N}): \quad \forall f, \varepsilon, \delta \, (X \ \delta\text{-splits } f)$$

Then there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in N such that $(\star_X^{M[G],N[G]})$ holds where G is $\mathbb{L}_\mathcal{V}$-generic over N (and thus $\mathbb{L}_\mathcal{U}$-generic over M).

Here $\mathbb{L}_\mathcal{U}$ denotes *Laver forcing* with $\mathcal{U}$,
i.e. forcing with Laver trees such that successor levels of splitnodes belong to $\mathcal{U}$.

Splitting and reaping for summable ideals 3

We continue the proof sketch of the Theorem.
Start with a model V for CH.
Add ω_1 many Cohen reals X_α and obtain the model W.
(The Cohen reals are intended as a witness for $\mathfrak{s}(\mathcal{I})$.)
Use the crucial lemma to build a matrix-like iterated forcing

$$(\mathbb{P}^\alpha_\gamma : \alpha \leq \omega_1, \gamma \leq \omega_2)$$

adding $\mathbb{L}_{\mathcal{U}_\gamma}$-generics ω_2 times with finite support and preserving $(\star^{V,W}_{X_\alpha})$ along the γ-iteration.
Obtain the models V' and W'.
The $\mathbb{L}_{\mathcal{U}_\gamma}$-generics witness $\mathfrak{s} = \aleph_2$.
$(\star^{V',W'}_{X_\alpha})$ shows the Cohen reals witness $\mathfrak{s}(\mathcal{I}) = \aleph_1$. $\square$

Distributivity for summable ideals

Conjecture

$\mathfrak{h}(\mathcal{I}) < \mathfrak{h}$ *is consistent for summable* $\mathcal{I}$.

Conjecture

$\mathfrak{h} < \mathfrak{h}(\mathcal{I})$ *is consistent for summable* $\mathcal{I}$.

Splitting, reaping, and distributivity for $\mathcal{ED}_{\mathrm{fin}}$

Theorem

$\mathfrak{s}(\mathcal{ED}_{\mathrm{fin}}) < \mathfrak{s}$ *and* $\mathfrak{r} < \mathfrak{r}(\mathcal{ED}_{\mathrm{fin}})$ *are consistent.*

Theorem

$\mathfrak{h}(\mathcal{ED}_{\mathrm{fin}}) < \mathfrak{h}$ *is consistent.*

Note that

- $\mathfrak{h}(\mathcal{ED}_{\mathrm{fin}}) \leq \mathfrak{h}$,
- $\mathfrak{s}(\mathcal{ED}_{\mathrm{fin}}) \leq \mathfrak{s}$, and
- $\mathfrak{r}(\mathcal{ED}_{\mathrm{fin}}) \geq \mathfrak{r}$

in *ZFC*.

Next RIMS meeting in Kyoto:

Combinatorial set theory and forcing theory

November 16 - 19, 2009

at Rakuyu Kaikan, Kyoto University, Japan

organized by Teruyuki Yorioka

http://www.ipc.shizuoka.ac.jp/∼styorio/rims09/