Nonseparable UHF algebras
(or: Graphs, groups, and noncommutative tori)

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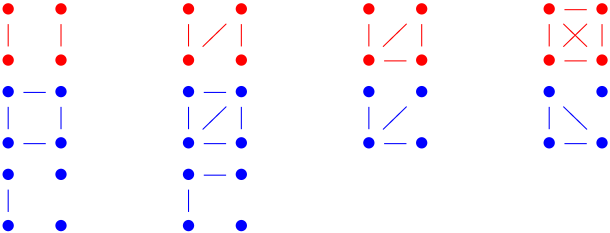


Hilbert space

H:	a complex Hilbert space
$(\mathcal{B}(H), +, \cdot, ^*, \|\cdot\|)$:	the algebra of bounded linear operators on H

Definition

A (concrete) *C*-algebra* is a norm-closed subalgebra of $\mathcal{B}(H)$.

Theorem (Gelfand–Naimark–Segal)

A Banach algebra with involution A is isomorphic to a concrete C-algebra if and only if*

$$\|aa^*\| = \|a\|^2$$

for all $a \in A$.

The simplest C*-algebras

$$\mathcal{B}(H)$$

$$M_n(\mathbb{C}), \text{ for } n \in \mathbb{N}.$$

Full matrix algebras

(Unital) embeddings

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C})$$

via

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix}$$

or, in short

$$a \mapsto a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Fact

All embeddings between C-algebras are norm-preserving.*

CAR (Fermion) algebra: 'the E_0 of C*-algebras'

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C}) \hookrightarrow M_{16}(\mathbb{C}) \hookrightarrow \dots$$

$$M_{2^\infty}(\mathbb{C}) = \varinjlim M_{2^n}(\mathbb{C}) = \bigotimes_{n \in \mathbb{N}} M_2(\mathbb{C}).$$

(where $\varinjlim$ means 'completion of the direct limit.')

Uniformly Hyperfinite algebras, Approximately Matricial algebas and Locally Matricial algebras

Definition

1. A is UHF if A is a tensor product of full matrix algebras.
2. A is AM is A is a direct limit of full matrix algebras.
3. A is LM if $\forall \varepsilon > 0$ and for every finite $F \subseteq A$ there is a full matrix algebra $M \subseteq A$ such that $F \subseteq_{\varepsilon} M$.

The question

Theorem (J. Glimm)

If A is separable and unital then

$$UHF \Leftrightarrow AM \Leftrightarrow LM$$

Question (J. Dixmier)

If A is unital, does

$$UHF \Leftrightarrow AM \Leftrightarrow LM?$$

We have a complete answer to the problem, but I will concentrate on AM vs. UHF in this talk.

Characterizing $M_2(\mathbb{C})$

$$u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$u^2 = 1 \qquad v^2 = 1$$

$$uv = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad vu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Lemma

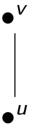
A is isomorphic to $M_2(\mathbb{C})$ if and only if $A = C^*(\{u, v\})$, with $u^2 = v^2 = 1$ and $uv = -vu$.

Proof.

($\Leftarrow$) A is a linear span of u, v, uv, and 1.

The only noncommutative C*-algebra that is 4-dimensional as a vector space is $M_2(\mathbb{C})$. $\square$

Graphs and noncommutative tori



$$M_2(\mathbb{C})$$

means

$$uv = -vu$$

(and $u^2 = v^2 = 1$).

More graphs and noncommutative tori

$\bullet^{v}$

$\mathbb{C}^4$

$\bullet^{u}$

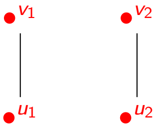
means

$$uv = vu$$

(and $u^2 = v^2 = 1$).

Example # 2

Which C*-algebra is coded by the following graph?



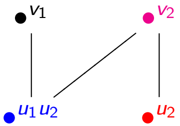
$M_4(\mathbb{C})$

means

$$M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$$

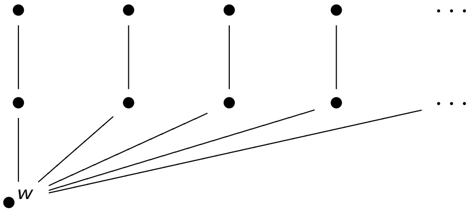
Example # 3

Which C*-algebra is coded by the following graph?



$M_4(\mathbb{C})$

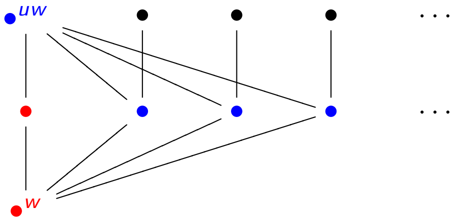
Examples $\#4$—$\omega + 1$



$$M_4(\mathbb{C}) M_8(\mathbb{C}) M_{16}(\mathbb{C}) M_{2^\infty}(\mathbb{C}) - theCARalgebra$$

Let's denote this algebra by $B(\kappa)$, where $\kappa = |G|$.

Analysis of $B(\aleph_0)$



So we have proved

Lemma

$B(\aleph_0)$ is isomorphic to $M_{2^\infty}(\mathbb{C})$.
For every infinite κ, $B(\kappa)$ is AM.

Relative commutant

Definition

If A is a subalgebra of B, let

$$Z_B(A) = \{b \in B : ab = ba \text{ for all } a \in A\}.$$

Let $Z(A) = Z_A(A)$.

Fact

1. *if A is LM (or AM, or UHF) then $Z(A) = \mathbb{C}I$.*
2. $Z_{A \otimes B}(A) \supseteq B$.

Complemented subalgebras

A subalgebra A of B is *complemented in* B if

$$C^*(A, Z_B(A)) = B.$$

Note that A is complemented in $A \otimes C$.

Lemma

If A is UHF then club many of its separable subalgebras are complemented.

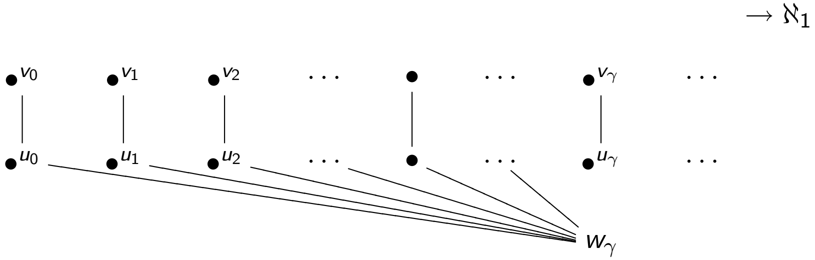
Theorem (Farah–Katsura)

$B(\kappa)$ is AM but not UHF if κ is uncountable.

Proof.

Club many of its separable subalgebras are not complemented. □

Modifying $M_{2^{\aleph_1}}(\mathbb{C})$ further



$M_{2^{\aleph_1}}(\mathbb{C})$
For $S \subseteq \aleph_1$ let $B(S)$ be given by the graph with vertices

$$\{u_\gamma, v_\gamma : \gamma < \aleph_1\} \cup \{w_\gamma : \gamma \in S\}.$$

Many AM algebras

Theorem (Farah–Katsura)

If $B(S) \cong B(T)$ then $S \Delta T \in \mathsf{NS}_{\omega_1}$.

Corollary

There are $2^{\aleph_1}$ nonisomorphic AM algebras of character density $\aleph_1$ for any uncountable regular κ.

UHF algebras can be classified

$$\bigotimes_{\kappa_2} M_2(\mathbb{C}) \otimes \bigotimes_{\kappa_3} M_3(\mathbb{C}) \otimes \bigotimes_{\kappa_5} M_5(\mathbb{C}) \otimes \bigotimes_{\kappa_7} M_7(\mathbb{C}) \otimes \dots$$

so there are only $2^{\aleph_0}$ in character density $\aleph_{\omega_1}$.

Summary and more

Theorem (Farah–Katsura)

AM $\not\Rightarrow$ UHF in any uncountable character density.
LM $\Leftrightarrow$ AM in character density $\leq \aleph_1$.
LM $\not\Rightarrow$ AM in character density $\geq \aleph_2$.

Representation density

Question (M. Takesaki, 2008)

What about C-algebras faithfully represented on a separable Hilbert space?*
Does LM $\Rightarrow$ AM in this case?

A nonseparable UHF algebra cannot be faithfully represented on a separable Hilbert space.

Irreducible representations

An isomorphic embedding

$$A \xrightarrow{\pi} \mathcal{B}(H)$$

π is an *irreducible representation* (*irrep*) if H has no nontrivial closed subspace invariant for $\pi[A]$.

Homogeneity of the pure state space

Theorem (Kishimoto–Ozawa–Sakai, 2003)

Assume A is simple and separable. Then its space of irreps is homogeneous: For all irreps π_1, π_2 there exist automorphisms α, β such that

$$\begin{array}{ccc} A & \xrightarrow{\pi_1} & \mathcal{B}(H_1) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\pi_2} & \mathcal{B}(H_1) \end{array}$$

commutes.

There is a nonseparable simple algebra with nonhomogeneous space of irreps.

Nuclearity

The class of nuclear C*-algebras is the most studied class of C*-algebras. All LM algebras are nuclear.

Question (Kishimoto–Ozawa–Sakai, 2003)

Assume A is simple and nuclear. Is its space of irreps homogeneous?

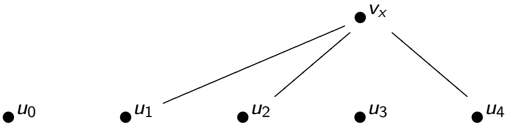
A positive answer would imply that $\diamond_\kappa$ implies there is a counterexample to Naimark's problem.
(The case $\kappa = \aleph_1$ is a theorem of Akemann–Weaver.)

More graphs

For $\mathbb{A} \subseteq 2^\kappa$ define a bipartite graph $G = G(\kappa, \mathbb{A})$ and a the corresponding C*-algebra $B(\kappa, \mathbb{A})$.

$$V(G) = \kappa \cup \mathbb{A}.$$

For $i \in \kappa$ and $x \in \mathbb{A}$ let u_i and v_x be adjacent if $i \in x$.



$$x = \{1, 2, 4\}.$$

Lemma

If $\mathbb{A}$ is dense in 2^κ and independent, then $B(\kappa, \mathbb{A})$ is AM and it has a faithful irreducible representation on $\ell_2(\kappa)$.

An answer to a question that Takesaki did not ask

Theorem (Farah–Katsura)

AM $\not\Rightarrow$ UHF for separably represented C-algebras.*

Proof.

Take $M(\mathbb{N}, \mathbb{A})$ for an uncountable dense independent family $\mathbb{A} \subseteq 2^{\mathbb{N}}$. It is nonseparable, AM, and has a faithful irreducible representation on $\ell_2(\mathbb{N})$. So it cannot be UHF. □

Proposition (Farah–Katsura)

CH implies that for separably respresented algebras LM implies AM.

The dual of $\mathbb{A}$

For $\mathbb{A}$ define the dual $\hat{\mathbb{A}} = \{y_i : i \in \kappa\} \subseteq 2^{\mathbb{A}}$ by

$$x \in y_i \Leftrightarrow i \in x$$

Lemma

$B(\kappa, \mathbb{A}) \cong B(\mathbb{A}, \hat{\mathbb{A}})$.

Lemma

1. $\hat{\hat{\mathbb{A}}} = \mathbb{A}$.
2. *$\mathbb{A}$ is dense iff $\hat{\mathbb{A}}$ is independent.*
3. *$\mathbb{A}$ is independent iff $\hat{\mathbb{A}}$ is dense.*

A nuclear C*-algebra with a nonhomogeneous irrep space

Theorem (Farah)

There is an AM (therefore simple nuclear) C-algebra that has nonhomogeneous space of irreps.*

Proof.

Take a dense, independent $\mathbb{A} \subseteq 2^{\mathbb{N}}$ of cardinality $2^{\aleph_0}$. Then $B(\mathbb{N}, \mathbb{A}) \cong B(2^{\aleph_0}, \hat{\mathbb{A}})$ has irreps on $\ell_2(\mathbb{N})$ and on $\ell_2(2^{\aleph_0})$. □

Recall that the UHF algebras can be classified

$$\bigotimes_{\kappa_2} M_2(\mathbb{C}) \otimes \bigotimes_{\kappa_3} M_3(\mathbb{C}) \otimes \bigotimes_{\kappa_5} M_5(\mathbb{C}) \otimes \bigotimes_{\kappa_7} M_7(\mathbb{C}) \otimes \dots$$

An embarrassing open problem.

Question

Does $\bigotimes_{\aleph_1} M_2(\mathbb{C})$ *embed into* $\bigotimes_{\aleph_0} M_2(\mathbb{C}) \otimes \bigotimes_{\aleph_1} M_3(\mathbb{C})$ *for all* κ*?*

(The answer is 'no' for smaller cardinals.)