Complexity of isomorphism between Banach spaces and inevitable list of Gowers

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June 25th, 2009

Isomorphic classification of Banach spaces[1]

The isomorphic classification of finite dimensional Banach spaces or of Hilbert spaces is easy (thanks to Hilbert bases).

Spaces and subspaces will always be separable Banach spaces of infinite dimension.

As we shall see, the case of infinite dimensional spaces is extremely complicated.

However, there exist two directions for a “loose” classification of Banach spaces.

[1] The author received the support of CAPES, processo AEX 0713/09-0

Isomorphic classification of Banach spaces

However, there exist two directions for a “loose” classification of Banach spaces.

(1) the measure of the complexity of a space X, seen as the complexity of isomorphism between its subspaces,

(2) a classification by the presence in X of elementary subspaces (*Gowers' project*), either from a

  (a) *“theoretical”* point of view, by structural dichotomy theorems, or from an

  (b) *“empirical”* point of view, by the study or construction of examples.

Isomorphic classification of Banach spaces

The two directions are related:

- A Banach space will be at least as complex as each of the elementary subspaces it contains.
- Inversely, some spaces will be labelled elementary or simple when isomorphism (or embedding, or biembedding) between their subspaces is simple.

Gowers' results

At the beginning of the 1990's, Gowers revolutionated the theory of Banach spaces.

Theorem (Gowers: Banach's hyperplane problem, 1932)

There exists a Banach space G_u which is not isomorphic to its proper subspaces.

In that sense, G_u is said to be exotic.

Theorem (Gowers - Maurey: the unconditional basic sequence problem, 1940's)

There exists a Banach space GM which does not contain any unconditional basic sequence.

Gowers' results

Theorem (Gowers - Maurey: the indecomposable space problem, 1970's)

Furthermore, GM is HI: none of its subspaces is decomposable (isomorphic to a direct sum of subspaces).

Theorem (Gowers, Komorowski - Tomczak-Jagermann, Banach's homogeneous space problem, 1932)

Any homogeneous Banach space (isomorphic to all its subspaces) is isomorphic to the Hilbert space.

Tsirelson's space T (1974) is an ancester of the spaces GM and G_u. It was the first known example of a space which did not contain a copy of c_0 or ℓ_p, $1 \leq p < +\infty$.

Gowers' results

In his paper "An infinite Ramsey theorem and some Banach space dichotomies", Gowers proved that these spaces form an inevitable list of Banach spaces.

Theorem (Gowers, 1995)

Every Banach space contains a subspace:

- *either of the type of GM,*
- *or of the type of G_u,*
- *or of the type of T,*
- *or of the type of c_0 and ℓ_p.*

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- *either is HI, as GM,*
- *or of the type of G_u,*
- *or of the type of T,*
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Theorem (Gowers, 1995)

Every Banach space contains a subspace which:

- *either is HI, as GM,*
- *or has an unconditional basis such that disjointly supported subspaces never are isomorphic, as G_u,*
- *or of the type of T,*
- *or of the type of c_0 or ℓ_p.*

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Every Banach space contains a subspace which:

- *either is HI, as GM,*
- *or has an unconditional basis such that disjointly supported subspaces never are isomorphic, as G_u,*
- *or has an unconditional basis, is quasi-minimal, without a minimal subspace, as T,*
- *or of the type of c_0 or ℓ_p.*

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- *either is HI, as GM,*
- *or has an unconditional basis such that disjointly supported subspaces never are isomorphic, as G_u,*
- *or has an unconditional basis, is quasi-minimal, without a minimal subspace, as T,*
- *or has an unconditional basis and is minimal, as c_0 or ℓ_p.*

Gowers' results

The inevitability of this list of of Banach spaces is defined by Gowers as follows:

a) if a space X belongs to a class, then all its subspaces belong to this same class (or all its block subspaces in the case of a property related to a Schauder basis of X),
b) every space has a subspace in one of the classes,
c) the classes are obviously disjoint,
d) belonging to a class gives a lot of information on the operators defined on the space and its subspaces.

Some questions of Gowers

In that article, Gowers formulates three questions:

Question (Gowers' 1st question)

How is it possible to refine Gowers' inevitable list?

In particular, the more regular class of the list should be the class of spaces isomorphic to c_0 or ℓ_p, and not the class of minimal spaces, which contains other spaces (such as the dual T^* of T or Schlumprecht's space S).

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In that article, Gowers formulates three questions:

Question (Gowers' 1st question)

How is it possible to refine Gowers' inevitable list?

Question (Gowers' 2nd question)

How is it possible to relate Gowers' list to Banach's hyperplane problem?

One should know, for each class of the list, whether spaces in that class may or may not be isomorphic to their proper subspaces.

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In that article, Gowers formulates three questions:

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How is it possible to refine Gowers' inevitable list?

Question (Gowers' 2nd question)

How is it possible to relate Gowers' list to Banach's hyperplane problem?

Question (Gowers' 3rd question)

For every space X, let $P(X)$ be the set of subspaces of X equiped with the relation of isomorphic embeddability, modulo biembeddability. What is the possible structure of $P(X)$?

More precisely, Gowers asks for which partially ordered sets P one may find an X such that not only $P(X) \simeq P$, but also every subspace Y of X contains a subspace Z such that $P(Z) \simeq P$.

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For every space X, let $P(X)$ be the set of subspaces of X equiped with the relation of isomorphic embeddability, modulo biembeddability. What is the possible structure of $P(X)$?

Let us note that when $|P(X)| = 1$, X embeds into all of its subspaces; X is then said to be minimal.

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For every space X, let $P(X)$ be the set of subspaces of X equiped with the relation of isomorphic embeddability, modulo biembeddability. What is the possible structure of $P(X)$?

We now give partial answers to these three questions.

1st question: Gowers' two dichotomies

Gowers' inevitable list of classes is based on two dichotomies.

Theorem (Gowers' 1st dichotomy, 1993)

Every Banach space contains a HI subspace or a subspace with an unconditional basis.

Theorem (Gowers' 2nd dichotomy, 1995)

Every Banach space contains a quasi minimal *subspace (any two subspaces have isomorphic subspaces) or a subspace with a basis such that disjointly supported subspaces never are isomorphic.*

1st question: Gowers' two dichotomies

Recall that if X is a space with a Schauder basis (e_n), then

- the support $\mathrm{supp}\, x$ of $x = \sum_i a_i e_i \in X$ is the set of i such that $a_i \neq 0$. The range of x is the interval of integers $[\min \mathrm{supp}\, x, \max \mathrm{supp}\, x]$.
- A block basis of X is a sequence (y_n) of successive vectors (or blocks) of X, that is such that for every n, $\max \mathrm{supp}\, x_n < \min \mathrm{supp}\, x_{n+1}$.
- A block subspace Y is a subspace of X generated by a block basis (y_n) of X; in this case one writes $Y = [y_n]$ and observes that (y_n) is a Schauder basis of Y.

1st question: Gowers' two dichotomies

Since every subspace of a given Banach space contains a perturbation of a block subspace, it is possible, for dichotomy theorems, to restrict oneself to block subspaces.

The set of block subspaces has a good topological structure, and this allows Gowers to prove a Ramsey type theorem for block bases and deduce from it his two dichotomies.

Unless specified otherwise, subspaces will be block subspaces.

Gowers' 1st question: three new dichotomies

There exist three new dichotomies related to different forms of minimality.

Theorem (3rd dichotomy, Ferenczi - Rosendal, 2007)

Every Banach space contains a minimal subspace or a tight subspace.

Theorem (4th dichotomy, Ferenczi - Rosendal, 2007)

Every Banach space contains a sequentially minimal subspace or a subspace which is tight by range.

Theorem (5th dichotomy, Ferenczi - Rosendal, 2007)

Every Banach space contains a locally minimal subspace or a subspace which is tight with constants.

1st question: diagram of dichotomies

The relations between the different dichotomies may be visualized in the following diagram:

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Disjointly sup. non $\simeq$	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$*$ 4th dichotomy $*$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal
$\Uparrow$		$\Downarrow$
Tight with const.	$*$ 5th dichotomy $*$	Loc. minimal

Gowers' 1st question: the 3rd dichotomy

Definition (Rosenthal)

A Banach space X is minimal if every subspace of X contains an isomorphic copy of X.

Examples: c_0, ℓ_p, $1 \leq p < +\infty$, but also T^* or S.

Gowers' 1st question: the 3rd dichotomy

Unconditional basis	$* \text{ 1st dichotomy } *$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Disjointly sup. non $\simeq$	$* \text{ 2nd dichotomy } *$	Quasi minimal
		$\Uparrow$
		Minimal

A space X is minimal if every subspace of X contains an isomorphic copy of X. Examples: c_0, ℓ_p, but also T^* or S.

Gowers' 1st question: the 3rd dichotomy

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Disjointly sup. non $\simeq$	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
No minimal subspace	$*$? $*$	Minimal

A space X is minimal if every subspace of X contains an isomorphic copy of X. Examples: c_0, ℓ_p, but also T^* or S.

Gowers' 1st question: the 3rd dichotomy

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Disjointly sup. non $\simeq$	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal

A space X is minimal if every subspace of X contains an isomorphic copy of X. Examples: c_0, ℓ_p, but also T^* or S.

Gowers' 1st question: the 3rd dichotomy

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Disjointly sup. non $\simeq$	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal

Definition

A space X with a Schauder basis (e_n) is tight if for every subspace Y of X there exists a successive sequence $I_0 < I_1 < I_2 < \cdots$ of subsets of $\mathbb{N}$ such that for every infinite subset A of $\mathbb{N}$, $Y \not\hookrightarrow [e_n, n \notin \cup_{i \in A} I_i]$.

Definition

A space X with a Schauder basis (e_n) is tight if for every subspace Y of X there exists a successive sequence $I_0 < I_1 < I_2 < \cdots$ of subsets of $\mathbb{N}$ such that for every infinite subset A of $\mathbb{N}$, $Y \not\hookrightarrow [e_n, n \notin \cup_{i \in A} I_i]$.

Examples: Tsirelson's space T, Gowers' space G_u.

In the case of G_u, one may choose $I_k = \mathrm{supp}\, y_k$ if $Y = [y_k]$; therefore we shall say that G_u is tight by support.

A tight space admits few embeddings of any space Y, and in particular may not contain a minimal subspace.

Gowers' 1st question: the 3rd dichotomy

The proof of the 3rd dichotomy combines:

- results on infinite asymptotic games (Rosendal, 2006), evolving from ideas of Odell - Schlumprecht (2002),
- the notion of generalized asymptotic game (Ferenczi, 2006),
- techniques of Pełczar (2003) then Ferenczi (2006) for relational games.

Gowers' 1st question: generalized games

I	n_0		n_1		n_2		$\dots$	$\dots$
II		$x_0 \geq n_0$		$x_1 \geq n_1$		$x_2 \geq n_2$	$\dots$	$\dots$

Infinite asymptotic game

I	n_0		n_1		$\dots$
II		$m_0 \geq n_0, x_0,$ supp $x_0 \subset [n_0, m_0]$		$m_1 \geq n_1, x_1,$ supp $x_1 \subset [n_0, m_0] \cup [n_1, m_1]$	

Generalized asymptotic game

Gowers' 1st question: generalized games

I	Y_0		Y_1		Y_2		$\dots$	$\dots$
II		$x_0 \in Y_0$		$x_1 \in Y_1$		$x_2 \in Y_2$	$\dots$	$\dots$

Infinite Gowers game

I	Y_0		Y_1		$\dots$	
II		$F_0 \subset Y_0$, $x_0 \in F_0$		$F_1 \subset Y_1$, $x_1 \in F_0 + F_1$		$F_2 \subset Y_2$, $x_2 \in F_0 + F_1 + F_2$

Generalized Gowers game

Gowers' 1st question: generalized games

I		$n_0 \leq x_0 \subseteq L$ m_0		$n_1 \leq x_1 \subseteq L$ m_1	$\dots$
II	n_0		$m_0 \leq y_0 \subseteq M$ n_1		$m_1 \leq y_1 \subseteq M$ n_2

Asymptotic relational game $P_{L,M}$

I		$n_0 \leq E_0 \subseteq L$ $x_0 \in E_0, m_0$		$n_1 \leq E_1 \subseteq L$ $x_1 \in E_0 + E_1, m_1$	
II	n_0		$m_0 \leq F_0 \subseteq M$ $y_0 \in F_0, n_1$		$\dots$

Generalized relational game $G_{L,M}$

Gowers' 1st question: generalized games

While Gowers obtains winning strategies for II to play inside an analytic set in his game, we obtain winning strategies for II to play inside a closed set in the generalized game.
In other words, to work in a more general setting, we must restrict our attention to more simple sets.

Observe that in T^*, there is no winning strategy in Gowers' game for II to produce a block sequence spanning a subspace isomorphic to T^*, but there is a strategy in the generalized Gowers' game to produce a (non block) sequence with such a property.

Gowers' 1st question: the 4th dichotomy

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal

Gowers' 1st question: the 4th dichotomy

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomy $*$	Quasi minimal
		Subseq. minimal
		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal

A space $X = [x_n]$ is subsequentially minimal if every subspace of X contains a copy of a *subsequence* of (x_n).

Examples: all minimal spaces with a Schauder basis, T.

Gowers' 1st question: the 4th dichotomy

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomy $*$	Quasi minimal
		$\Uparrow$
		Seq. minimal
		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal

A space $X = [x_n]$ is subsequentially minimal if every subspace of X contains a copy of *subsequence* of (x_n).

One may also define a hereditary form of this property, called sequential minimality.

Gowers' 1st question: the 4th dichotomy

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$*$ 4th dichotomy $*$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal

A space $X = [x_n]$ is tight by range if it is tight and if for every $Y = [y_k]$, one may choose $I_k = \mathrm{ran}\ y_k$.
Examples: G_u, G.

It may be observed that such a space is *never* isomorphic to its proper subspaces.

Gowers' 1st question: the 5th dichotomy

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$*$ 4th dichotomy $*$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal
$\Uparrow$		
Tight with const.		

A space is tight with constants if it is tight and if for every Y, one may choose I_k such that $Y \not\hookrightarrow_k [e_n, n \notin I_k]$.
Examples: T, $T^{(p)}$, and more generally, every strongly asymptotically ℓ_p space which does not contain a copy of ℓ_p, $1 \leq p < +\infty$.

Gowers' 1st question: the 5th dichotomy

Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$*$ 4th dichotomy $*$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal
$\Uparrow$		$\Downarrow$
Tight with const.	$*$ 5th dichotomy $*$	Loc. minimal

A space X is locally minimal if for some $K \geq 1$, every subspace of X contains a K-isomorphic copy of every *finite dimensional* subspace of X.

Examples: all minimal spaces, GM^*, G_u^*,...

Gowers' 1st question: the 5th dichotomy

Unconditional basis	∗ 1st dichotomy ∗	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	∗ 2nd dichotomy ∗	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	∗ 4th dichotomy ∗	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	∗ 3rd dichotomy ∗	Minimal
$\Uparrow$		$\Downarrow$
Tight with const.	∗ 5th dichotomy ∗	Loc. minimal

The 4th and 5th dichotomies are a consequence of certain aspects of Gowers' Ramsey theorem (in the version of Bagaria and López-Abad) and of ideas of coding due to López-Abad.

Gowers' 1st question: the 6 dichotomies

St. as. $\ell_p, 1 \leq p < +\infty$	$*$ Tcaciuc $*$	Unif. inhomogeneous
$\Downarrow$		$\Uparrow$
unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$*$ 4th dichotomy $*$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal
$\Uparrow$		$\Downarrow$
Tight with const.	$*$ 5th dichotomy $*$	Loc. minimal

Finally there exists a 6th dichotomy, due to A. Tcaciuc, opposing strongly asymptotically ℓ_p spaces to uniformly inhomogeneous spaces, such as S, G_u, or all HI spaces.

Gowers' 1st question: the 6 dichotomies

St. as. $\ell_p, 1 \leq p < +\infty$	$*$ Tcaciuc $*$	Unif. inhomogeneous
$\Downarrow$		$\Uparrow$
unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$*$ 4th dichotomy $*$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal
$\Uparrow$		$\Downarrow$
Tight with const.	$*$ 5th dichotomy $*$	Loc. minimal

By combining the 6 dichotomies, $2^6 = 64$ classes of Banach spaces should be obtained, but because of the different relations between the properties, there are only 19 classes.

Gowers' 1st question: the 6 dichotomies

St. as. $\ell_p, 1 \leq p < +\infty$	$*$ Tcaciuc $*$	Unif. inhomogeneous
$\Downarrow$		$\Uparrow$
unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomy $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$*$ 4th dichotomy $*$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomy $*$	Minimal
$\Uparrow$		$\Downarrow$
Tight with const.	$*$ 5th dichotomy $*$	Loc. minimal

More precisely, 6 classes are obtained by using the first 4 dichotomies, and 19 subclasses by also using the last two.

Gowers' 1st question: a list of 19 classes

Theorem (Ferenczi - Rosendal, 2007)

Every infinite dimensional Banach space contains a subspace of one of the following 19 types:

Type	*Properties*	*Examples*
(1a)-(1b)	*HI, tight by range*	G^* *(1b)*, G
(2a)-(2b)	*HI, tight, sequentially minimal*	*?*
(3a)-(3d)	*tight by support*	G_u^* *(3b)*, X_u *(3c)*, X_u^* *(3d)*, G_u
(4a)-(4d)	*unc. basis, quasi min., tight by range*	*?*
(5a)-(5d)	*unc. basis, tight, seq. minimal*	T, $T^{(p)}$
(6a)-(6c)	*unconditional basis, minimal*	S, S^* *(6a)*, T^* *(6b)*, c_0, ℓ_p *(6c)*

Theorem (Ferenczi - Rosendal, 2007)

Every infinite dimensional Banach space contains a subspace of one of the following 19 types:

Type	Properties	Examples
(1a)	HI, tight by range and with const.	?
(1b)	HI, tight by range, loc. min.	G^*
(2)	HI, tight, sequentially minimal	?
(3)	tight by support	G_u, G_u^*, X_u, X_u^*
(4)	unc. basis, quasi min., tight by range	?
(5)	unc. basis, tight, seq. minimal	T, $T^{(p)}$
(6)	unconditional basis, minimal	c_0, ℓ_p, T^*, S, S^*

Gowers' 1st question: a list of 19 classes

Theorem (Ferenczi - Rosendal, 2007)

Every infinite dimensional Banach space contains a subspace of one of the following 19 types:

Type	Properties	Examples
(1)	HI, tight by range	G, G^*
(2)	HI, tight, sequentially minimal	?
(3)	tight by support	G_u, G_u^*, X_u, X_u^*
(4)	unc. basis, quasi min., tight by range	?
(5)	unc. basis, tight, seq. minimal	T, $T^{(p)}$
(6)	unconditional basis, minimal	c_0,ℓ_p,T^*, S,S^*

Gowers' 1st question: a list of 19 classes

Theorem (Ferenczi - Rosendal, 2007)

Every infinite dimensional Banach space contains a subspace of one of the following 19 types:

Type	Properties	Examples
(1)	HI, tight by range	G, G^*
(2)	HI, tight, sequentially minimal	?
(3)	tight by support and	
(3a)	with constants, unif. inhomogeneous	?
(3b)	locally minimal, unif. inhomogeneous	G_u^*
(3c)	strongly as. ℓ_p, $1 \leq p < \infty$	X_u
(3d)	strongly as. ℓ_∞	X_u^*
(4)	unc. basis, quasi min., tight by range	?
(5)	unc. basis, tight, seq minimal	T, $T^{(p)}$
(6)	unconditional basis, minimal	c_0, ℓ_p, T^*, S, S^*

Gowers' 1st question: a list of 19 classes

Theorem (Ferenczi - Rosendal, 2007)

Every infinite dimensional Banach space contains a subspace of one of the following 19 types:

Type	Properties	Examples
(1)	HI, tight by range	G, G^*
(2)	HI, tight, sequentially minimal	?
(3)	tight by support	G_u, G_u^*, X_u, X_u^*
(4)	unc. basis, quasi min., tight by range	?
(5)	unc. basis, tight, seq. minimal	T, $T^{(p)}$
(6)	unconditional basis, minimal	c_0,ℓ_p,T^*, S,S^*

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Every infinite dimensional Banach space contains a subspace of one of the following 19 types:

Type	Properties	Examples
(1)	HI, tight by range	G, G^*
(2)	HI, tight, sequentially minimal	?
(3)	tight by support	G_u, G_u^*, X_u, X_u^*
(4)	unc. basis, quasi min., tight by range	?
(5)	unc. basis, seq. minimal, tight, and	
(5a)	tight with constants, unif. inhomog.,	?
(5b)	loc. minimal, unif. inhomog.	?
(5c)	tight with cons., st. as. ℓ_p, $1 \leq p < \infty$,	T, $T^{(p)}$
(5d)	st. as. ℓ_∞	?
(6)	unconditional basis, minimal	c_0, ℓ_p, T^*, S, S^*

Gowers' 1st question: a list of 19 classes

Theorem (Ferenczi - Rosendal, 2007)

Every infinite dimensional Banach space contains a subspace of one of the following 19 types:

Type	Properties	Examples
(1)	HI, tight by range	G, G^*
(2)	HI, tight, sequentially minimal	?
(3)	tight by support	G_u, G_u^*, X_u, X_u^*
(4)	unc. basis, quasi min., tight by range	?
(5)	unc. basis, tight, seq. minimal	T, $T^{(p)}$
(6)	unconditional basis, minimal, and	
(6a)	unif. inhomogeneous	S,S^*
(6b)	reflexive, st. as. ℓ_∞	T^*
(6c)	isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$	c_0, ℓ_p

Gowers' 2nd question: exotic spaces

Let us reformulate the second question of Gowers:

Question (Gowers' 2nd question)

How is it possible to relate the six dichotomies and Banach's hyperplane problem?

HI spaces (Gowers-Maurey 1993), spaces of type (3) (Gowers 1991), but also those of type (4) do not have subspaces which embed in further subspaces.

Minimal spaces (type (6)), the space T are saturated with subspaces which embed in further subspaces.

Gowers' 2nd question: exotic spaces

Theorem (Ferenczi - Rosendal, 2007)

Every infinite dimensional Banach space contains a subspace of one of the following 19 types:

Type	Properties	Examples
	all subs. non-$\simeq$ to further subs.	
(1)	HI, tight by range	G, G^*
(2)	HI, tight, sequentially minimal	?
(3)	tight by support	G_u, G_u^*, X_u, X_u^*
(4)	unc. basis, quasi min., tight by range	?
	?	
(5)	unc. basis, tight, seq. minimal	T, $T^{(p)}$, ?
	all subs. $\simeq$ to further subs.	
(6)	unconditional basis, minimal	c_0, ℓ_p, T^*, S, S^*

Gowers' 2nd question: exotic spaces

Theorem (Ferenczi-Rosendal, 2007)

Every Banach space contains either a negative answer to Banach's hyperplane problem, either a subspace with a subsequential form of minimality.

Gowers' 2nd question: complex structures

There exists a totally different direction of research to compare a space and its hyperplanes.

Definition

Let X be a real space. Then X admits a complex structure if there exists an operateur J on X such that $J^2 = -Id$.

Theorem (Ferenczi - Galego, 2008)

There exist real even Banach spaces, in the sense that they admit a complex structure, but their hyperplanes do not.

Examples: some HI spaces, some $C(K)$ due to P. Koszmider, some spaces with an unconditional basis.

Note that even Banach spaces are structurally different from their hyperplanes.

Gowers' 3rd question

Question (Gowers' 3rd question)

For every space X, let $P(X)$ be the set of subspaces of X equiped with the relation of isomorphic embeddability, modulo biembeddability. What is the possible structure of $P(X)$?

Theorem (Ferenczi - Rosendal, 2007)

Let X be a Banach space. Then

- *either X contains a minimal subspace Y - and therefore $(Sev(Y), \hookrightarrow)$ is trivial,*
- *or X does not contain a minimal subspace - and then the relation of inclusion up to finite sets on $\mathcal{P}(\mathbb{N})$ embeds into $(Sev(X), \hookrightarrow)$*

Gowers' 3rd question

Question (Gowers' 3rd question)

For every space X, let $P(X)$ be the set of subspaces of X equiped with the relation of isomorphic embeddability, modulo biembeddability. For which partially ordered sets P may one find an X such that $P(X) \simeq P$ and every subspace Y of X contains a subspace Z such that $P(Z) \simeq P$?

The previous theorem implies that such P's

- either are of cardinality 1, when X is minimal,
- or are extremely complex, in particular
  (a) every partial order of order at most $\aleph_1$ embeds into P,
  (b) every closed partial order on a Polish space embeds into P.

Complexity of isomorphism: Godefroy's question

Theorem (Gowers, Komorowski - Tomczak-Jaegermann, 1990's)

Any homogeneous Banach space is isomorphic to the Hilbert space.

Question (Godefroy, 1999)

How many mutually non isomorphic subspaces must a non-Hilbertian space contain?

The question of Godefroy takes its interest in the context of the theory of complexity of analytic equivalence relations on Polish spaces.

Complexity of isomorphism

One may see separable Banach spaces as closed subspaces of $U = C([0,1])$ with Effros Borel structure.

The relation of isomorphism between separable Banach spaces (as well as isometry, biembeddability,...) is then analytic.

Complexity of isomorphism

Theorem (Ferenczi - Louveau - Rosendal, 2006)

The complexity of isomorphism between separable Banach spaces is E_{max}, the $\leq_B$-maximum relation among analytic relations on Polish spaces.

The same result holds for

- *isomorphic biembeddability, complemented isomorphic biembeddability, Lipschitz isomorphism of separable Banach spaces,*
- *permutative equivalence of normalized Schauder bases,*
- *uniform homemorphism of complete metric spaces,*
- *isomorphism of Polish groups,*
- *...*

Definition

For any separable Banach space X, we define the complexity of X as the complexity of the relation of isomorphism between the subspaces of X.

Examples: the complexity of ℓ_2 is trivial, and the complexity of the universal unconditional space P of Pełczýnski is E_{max}.

We may therefore reformulate the question of Godefroy in the following manner:

Question

Let X be a space which is not isomorphic to the Hilbert space. What is the complexity of X?

Complexity of separable Banach spaces

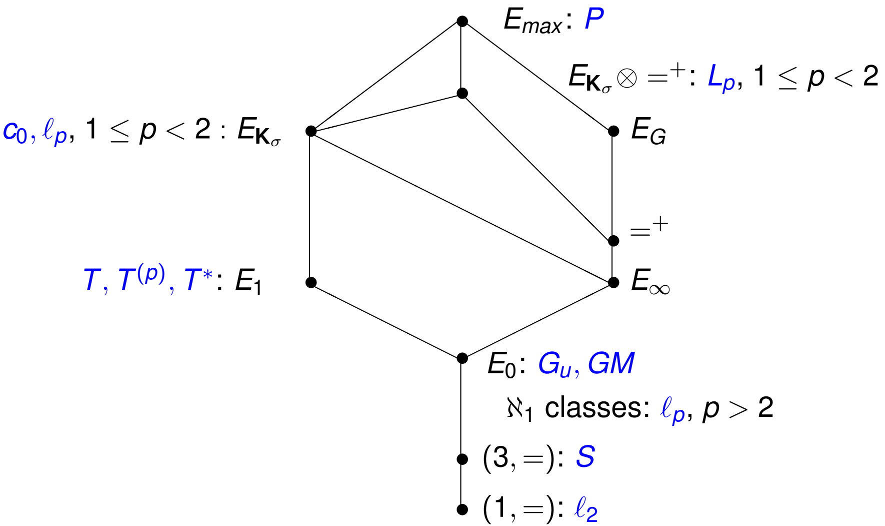
Question

Let X be a space which is not isomorphic to the Hilbert space. What is the complexity of X?

Lower bounds of complexity were obtained for a certain number of spaces:

- E_0 for G_u (Bossard, 2002) and GM (Rosendal, 2004),
- E_1 for T and T^* (Rosendal, 2003),
- E_{K_σ} for c_0 and ℓ_p, $1 \leq p < 2$ (Ferenczi - Galego, 2004),
- $E_{K_\sigma} \otimes =^+$ for L_p, $1 \leq p < 2$ (Ferenczi - Galego, 2004).

A few lower bounds of complexity



Complexity of separable Banach spaces

Theorem (Ferenczi, 2006)

Every separable Banach space either contains a minimal subspace, or is of complexity at least E_0.

Conjecture

Every separable Banach space which is not isomorphic to the Hilbert space is of complexity at least E_0.

Complexity and list of Gowers

Finally we may establish a relation between the two directions of classification by determining lower bounds of complexity for each type and subtype of the list of Gowers.

Type	isomorphism	biembeddability
(1)(2) HI, tight	E_0	E_0
(3)(4)(5) unc. basis, tight	E_0	E_0
(6a) minimal,	2	t
unif. inhomogeneous,		r
(6b) minimal,	E_0	i
reflexive, st. as. ℓ_∞		v
(6c) - $\ell_p, 1 \leq p < 2$ or c_0	E_{K_σ}	i
- $\ell_p, p > 2$	$\aleph_1$ classes	a
- ℓ_2	trivial	l

Lower bounds of complexity
for each class of the list of Gowers

Conclusion

Several notions of regular or classical space may be distinguished:

(1) any space with sufficiently many homogeneities (isomorphism with its hyperplanes, with its square,...).
(2) any space equipped with a sufficiently good structure (unconditional basis, UFDD...),
(3) any space of low complexity in the sense of $\leq_B$,
(4) any sequence space with a norm which is simple to define,

In this context,

- the spaces c_0 and ℓ_p satisfy (1)(2)(4), and in some sense, (3),

Conclusion

Several notions of regular or classical space may be distinguished:

(1) any space with sufficiently many homogeneities (isomorphism with its hyperplanes, with its square,...).
(2) any space equipped with a sufficiently good structure (unconditional basis, UFDD...),
(3) any space of low complexity in the sense of $\leq_B$,
(4) any sequence space with a norm which is simple to define,

In this context,

- the space T satisfies (1),(2), and also (4) in comparison with GM, G_u,

Conclusion

Several notions of regular or classical space may be distinguished:

(1) any space with sufficiently many homogeneities (isomorphism with its hyperplanes, with its square,...).

(2) any space equipped with a sufficiently good structure (unconditional basis, UFDD...),

(3) any space of low complexity in the sense of $\leq_B$,

(4) any sequence space with a norm which is simple to define,

- but spaces with an unconditional basis en general do not satisfy (1) (3) nor (4). For (1) however, several problems in the theory of Banach spaces which have solutions in the general case, remain open in the unconditional case: for exemple, the problem of uniqueness of complex structures (Bourgain, 1986) or the Schroeder-Bernstein problem (Gowers, 1996).

Conclusion

Several notions of regular or classical space may be distinguished:

(1) any space with sufficiently many homogeneities (isomorphism with its hyperplanes, with its square,...).
(2) any space equipped with a sufficiently good structure (unconditional basis, UFDD...),
(3) any space of low complexity in the sense of $\leq_B$,
(4) any sequence space with a norm which is simple to define,

Some results were presented here in the direction of the relations between (1) (2) and (3). But what we wish to mean exactly by (1), (3), and especially (4), and the exact relations between (1)-(4), remain to understand.

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