COFINAL TYPES OF *definable* DIRECTED ORDERS

Tamás Mátrai

UNIVERSITY OF TORONTO

Vienna

June 23, 2009

DIRECTED ORDERS

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$$p, q \in P \implies p \vee q \in P \text{ (least upper bound)}$$

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$\nearrow$

Hyperspace: compact subsets of 2^ω

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S. Todorčević

i.e. cofinal types of directed orders on ω_1

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i.e. $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic, ideal

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1. Definability problem

2. Primality problem

S. Solecki, S. Todorčević

I.E. THEORY OF DIRECTED BASIC ORDERS

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Theorem. $(P, \leq), (Q, \leq)$ basic,

$P \leq_T Q \Longrightarrow \exists f : P \to Q$ Souslin measurable Tukey reduction

$\mathcal{I}_{\mathrm{MAX}}$

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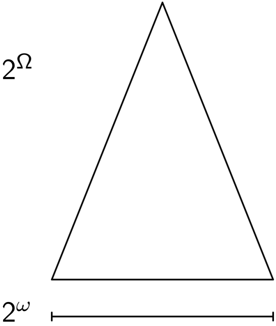

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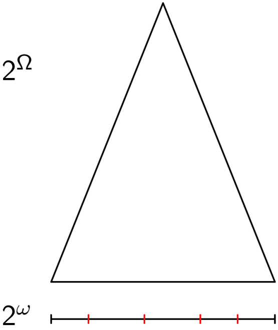

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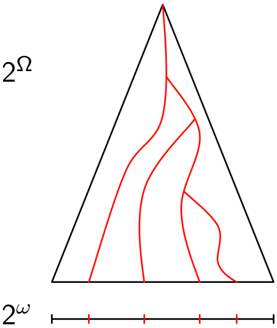

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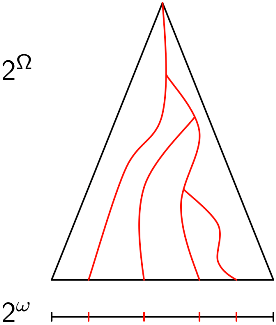

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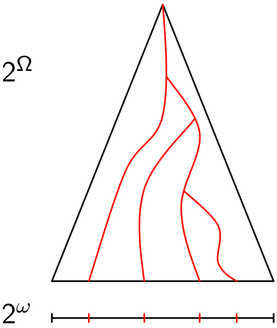

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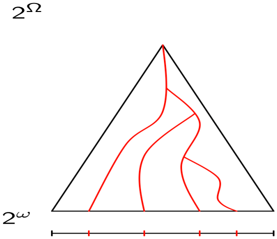

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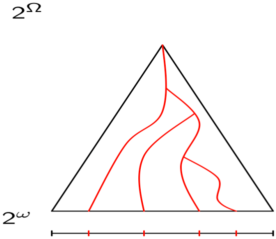

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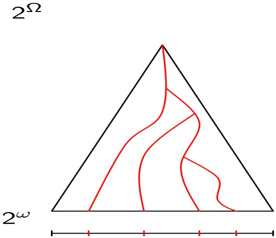

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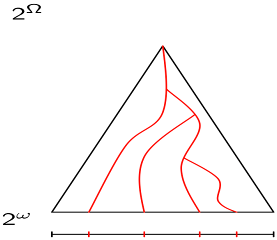

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$\mathcal{A} \in \mathcal{P}(\Omega)$

$\mathcal{I}_{\max}$ is NOT basic (Solecki-Todorčević theory does not apply)

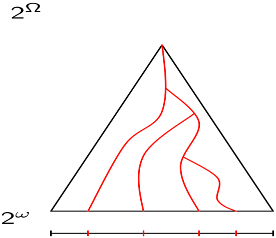
Particular problems: $\mathcal{J}, \mathcal{K} \subseteq \mathcal{P}(\omega)$ analytic ideals

$\mathcal{I}_{\text{MAX}}$

I.E. HOW TO COPY IDEALS OF COMPACT SETS INTO $\mathcal{P}(\omega)$

$[\mathfrak{c}]^{<\omega}$ maximal cofinal type of analytic ideals

$[\mathfrak{c}]^{<\omega} \rightsquigarrow [2^\omega]^{<\omega} \subseteq \mathcal{K}(2^\omega)$ is F_σ $\rightsquigarrow$ $\mathcal{I}_{\max} \subseteq \mathcal{P}(\Omega)$ is F_σ



$$\Omega = 2^{<\omega}$$

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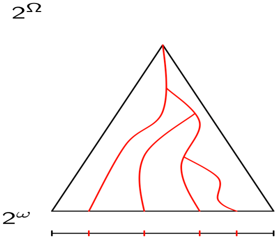

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2. $\mathcal{I}_{\max} \leq_T \mathcal{J} \oplus \mathcal{K} \overset{?}{\Longrightarrow} \mathcal{I}_{\max} \leq_T \mathcal{J}$ or $\mathcal{I}_{\max} \leq_T \mathcal{K}$

S. TODORČEVIĆ

I.E. PRIMALITY OF $\mathcal{I}_{\text{MAX}}$ FOR SOUSLIN MEASURABLE REDUCTIONS

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S. TODORČEVIĆ

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$$\mathcal{I}_{\max} \leq_{ST} \mathcal{J} \oplus \mathcal{K}$$

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S. Todorčević

i.e. primality of $\mathcal{I}_{\text{max}}$ for Souslin measurable reductions

$\mathcal{J}, \mathcal{K} \subseteq \mathcal{P}(\omega)$ analytic ideals,

Tukey reducibility witnessed by Souslin measurable function

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Corollary. CH $\Longrightarrow \exists \mathcal{J} \subseteq \mathcal{P}(\omega)$ analytic ideal:

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INFINITE DIMENSIONAL PERFECT SET THMS

I.E. CONSISTENT POSITIVE ANSWER TO DEFINABILITY PROBLEM

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$$[\mathfrak{c}]^{<\omega} = \mathcal{I}_{\max} \leq_T \mathcal{J} \Leftrightarrow \exists \mathcal{H} \subseteq \mathcal{J} \text{ strongly unbounded: } |\mathcal{H}| = \mathfrak{c}$$

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- $\exists \mathcal{H} \subseteq \mathcal{J}$ strongly unbdd, $|\mathcal{H}| = \mathfrak{c}$

  $\Rightarrow \exists P \subseteq \mathcal{J}$ perfect strongly unbdd

INFINITE DIMENSIONAL PERFECT SET THMS

I.E. CONSISTENT POSITIVE ANSWER TO DEFINABILITY PROBLEM

$\mathcal{J} \subseteq \mathcal{P}(\omega)$ analytic ideal $\qquad \looparrowdownleft \forall \mathcal{C} \in [\mathcal{H}]^{\omega} \ \bigcup \mathcal{C} \notin \mathcal{J}$

$$[\mathfrak{c}]^{<\omega} = \mathcal{I}_{\max} \leq_T \mathcal{J} \Leftrightarrow \exists \mathcal{H} \subseteq \mathcal{J} \text{ strongly unbounded: } |\mathcal{H}| = \mathfrak{c}$$

$$\mathbb{J} = \left\{ (A_n)_{n<\omega} \in \mathcal{P}(\omega)^{\omega} \colon A_n \in \mathcal{J} \ (n < \omega), \bigcup_{n<\omega} A_n \notin \mathcal{J} \right\} \text{ is } \sigma(\Sigma^1_1)$$

$IS_{\omega}(\mathcal{H})$: injective $\omega \to \mathcal{H}$ functions

$$\mathcal{H} \subseteq \mathcal{J} \text{ strongly unbdd} \Leftrightarrow IS_{\omega}(\mathcal{H}) \subseteq \mathbb{J} \Leftrightarrow \mathcal{H} \text{ is } \mathbb{J}\text{-homogeneous}$$

Theorem. Under suitable assumptions,

- $\exists \mathcal{H} \subseteq \mathcal{J} \colon |\mathcal{H}| = \mathfrak{c}, IS_{\omega}(\mathcal{H}) \subseteq \mathbb{J} \Rightarrow \exists P \subseteq \mathcal{J}$ perfect: $IS_{\omega}(P) \subseteq \mathbb{J}$
- $\exists \mathcal{H} \subseteq \mathcal{J}$ strongly unbdd, $|\mathcal{H}| = \mathfrak{c}$
  $\Rightarrow \exists P \subseteq \mathcal{J}$ perfect strongly unbdd
- $\mathcal{I}_{\max} \leq_T \mathcal{J} \Rightarrow \exists f \colon \mathcal{I}_{\max} \to \mathcal{J}$ continuous Tukey map

PRODUCT GAMES

I.E. HOW TO GET INFINITE DIMENSIONAL PERFECT SET THMS

PRODUCT GAMES

I.E. HOW TO GET INFINITE DIMENSIONAL PERFECT SET THMS

Example: Banach-Mazur game

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Playground: X, payoff set: $A \subseteq X$

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$$\begin{array}{lcccccccc}
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 & & & & & & & & \\
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- $U(n), V(n) \subseteq X$ $(n < \omega)$ non-empty open sets

Product games

i.e. how to get infinite dimensional perfect set thms

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Theorem.

† I has a winning strategy $\Leftrightarrow \exists U$ non-empty open: $A \cap U$ is meager

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- I has a winning strategy $\Leftrightarrow \exists U$ non-empty open: $A \cap U$ is meager

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PRODUCT GAMES

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n^{th} move: $U_0(n-1) \times \cdots \times U_{n-1}(n-1) \times X^{\omega \setminus n}$,

$$V_0(n-1) \times \cdots \times V_{n-1}(n-1) \times X^{\omega \setminus n}$$

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meager $\rightsquigarrow$ $\mathcal{Z} = \left\{ Z \subseteq X^\omega : \exists M_n \subseteq X^n \text{ meager} \left(Z \subseteq \bigcup_{0<n<\omega} M_n \times X^{\omega \setminus n} \right) \right\}$

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open set $\rightsquigarrow$ open "tower" (not even box open)

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- I has a winning strategy $\Leftrightarrow \exists \mathcal{U}$ non-empty open tower: $A \cap [\mathcal{U}] \in \mathcal{Z}$
  $\rightsquigarrow R \subseteq X$ everywhere non-meager $\Rightarrow IS_\omega(R) \not\subseteq A$
- II has a winning strategy $\Leftrightarrow X^\omega \setminus A \in \mathcal{Z}$
  $\rightsquigarrow \exists P \subseteq X$ perfect: $IS_\omega(P) \subseteq A$

To summarize. . .

i.e. consistent positive answer to Definability problem

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$\mathcal{J} \subseteq \mathcal{P}(\omega)$ analytic ideal

TO SUMMARIZE...

I.E. CONSISTENT POSITIVE ANSWER TO DEFINABILITY PROBLEM

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$$\mathbb{J} = \left\{ (A_n)_{n<\omega} \in \mathcal{P}(\omega)^\omega \colon A_n \in \mathcal{J} \ (n < \omega), \bigcup_{n<\omega} A_n \notin \mathcal{J} \right\} \text{ is } \sigma(\Sigma^1_1)$$

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Theorem. Under suitable assumptions,

- $\exists \mathcal{H} \subseteq \mathcal{J} : |\mathcal{H}| = \mathfrak{c}, IS_{\omega}(\mathcal{H}) \subseteq \mathbb{J} \Rightarrow \exists P \subseteq \mathcal{J}$ perfect : $IS_{\omega}(P) \subseteq \mathbb{J}$
- $\exists \mathcal{H} \subseteq \mathcal{J}$ strongly unbdd, $|\mathcal{H}| = \mathfrak{c}$

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Recall:

$\dashv$ I has a winning strategy $\rightsquigarrow \mathcal{H} \subseteq X$ everywhere non-meager $\Rightarrow IS_\omega(\mathcal{H}) \nsubseteq \mathbb{J}$

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Corollary. Under the same assumptions,

$$\mathcal{I}_{\max} \leq_T \mathcal{J} \oplus \mathcal{K} \Longrightarrow \mathcal{I}_{\max} \leq_T \mathcal{J} \text{ or } \mathcal{I}_{\max} \leq_T \mathcal{K}$$

Tukey picture update

i.e. many cofinal types of definable directed orders

TUKEY PICTURE UPDATE

I.E. MANY COFINAL TYPES OF DEFINABLE DIRECTED ORDERS

$$\begin{tikzcd}
 & & (\mathcal{P}(\omega), \subseteq^\star) \arrow[d, hook] & \\
\textit{measure leaf} \arrow[r] & \mathcal{Z}_0 & \in \{\text{analytic } P\text{-ideals}\} \leq_T & \mathcal{I}_{1/n} \\
1 <_T \omega <_T \omega^\omega \arrow[ur, phantom, "<_T" sloped] \arrow[dr, phantom, "<_T" sloped] & & & \\
\textit{category leaf} \arrow[r] & \mathcal{N} \cap \mathcal{K}(2^\omega) \arrow[uu, phantom, "<_T" sloped] \arrow[rr, phantom, "\leq_T"] & & \mathcal{M} \cap \mathcal{K}(2^\omega) \arrow[uu, phantom, "<_T" sloped]
\end{tikzcd}$$

TUKEY PICTURE UPDATE

I.E. MANY COFINAL TYPES OF DEFINABLE DIRECTED ORDERS

$$\begin{tikzcd}
\boxed{\{H \subseteq \omega\colon \lim_{n<\omega} |H \cap n|/n = 0\}} \arrow[drr] & & & (\mathcal{P}(\omega), \subseteq^\star) \arrow[d, hook] & \\
\textit{measure leaf} \arrow[r] & & \mathcal{Z}_0 & \in \{\text{analytic } P\text{-ideals}\} \leq_T & \mathcal{I}_{1/n} \\
& 1 <_T \omega <_T \omega^\omega \arrow[ur, phantom, sloped, "<_T"] \arrow[dr, phantom, sloped, "<_T"] & & & \\
\textit{category leaf} \arrow[rr] & & \mathcal{N} \cap \mathcal{K}(2^\omega) \arrow[uu, phantom, sloped, "<_T"] & \leq_T & \mathcal{M} \cap \mathcal{K}(2^\omega) \arrow[uu, phantom, sloped, "<_T"]
\end{tikzcd}$$

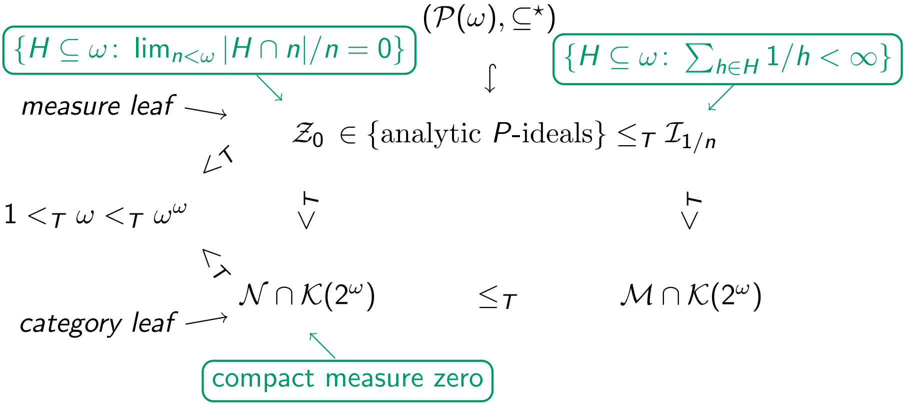
TUKEY PICTURE UPDATE

I.E. MANY COFINAL TYPES OF DEFINABLE DIRECTED ORDERS

$$\begin{tikzcd}
\boxed{\{H \subseteq \omega \colon \lim_{n<\omega} |H \cap n|/n = 0\}} \arrow[dr] & & (\mathcal{P}(\omega), \subseteq^\star) \arrow[d, hook] & \boxed{\{H \subseteq \omega \colon \sum_{h \in H} 1/h < \infty\}} \arrow[d] \\
\textit{measure leaf} \arrow[r] & \mathcal{Z}_0 \arrow[r, phantom, "\in"] \arrow[dd, phantom, sloped, "<_T"] & \{\text{analytic } P\text{-ideals}\} \arrow[r, phantom, "\leq_T"] & \mathcal{I}_{1/n} \arrow[dd, phantom, sloped, "<_T"] \\
1 <_T \omega <_T \omega^\omega \arrow[ur, phantom, sloped, "<_T"] \arrow[dr, phantom, sloped, "<_T"] & & & \\
\textit{category leaf} \arrow[r] & \mathcal{N} \cap \mathcal{K}(2^\omega) \arrow[rr, phantom, "\leq_T"] & & \mathcal{M} \cap \mathcal{K}(2^\omega)
\end{tikzcd}$$

TUKEY PICTURE UPDATE

I.E. MANY COFINAL TYPES OF DEFINABLE DIRECTED ORDERS



$(\mathcal{P}(\omega), \subseteq^\star)$

$\{H \subseteq \omega \colon \lim_{n<\omega} |H \cap n|/n = 0\}$

$\{H \subseteq \omega \colon \sum_{h \in H} 1/h < \infty\}$

measure leaf

$\mathcal{Z}_0 \in \{\text{analytic } P\text{-ideals}\} \le_T \mathcal{I}_{1/n}$

$<_T$

$1 <_T \omega <_T \omega^\omega$

$\le_T$

$\le_T$

$<_T$

$\mathcal{N} \cap \mathcal{K}(2^\omega) \le_T \mathcal{M} \cap \mathcal{K}(2^\omega)$

category leaf

compact measure zero

TUKEY PICTURE UPDATE

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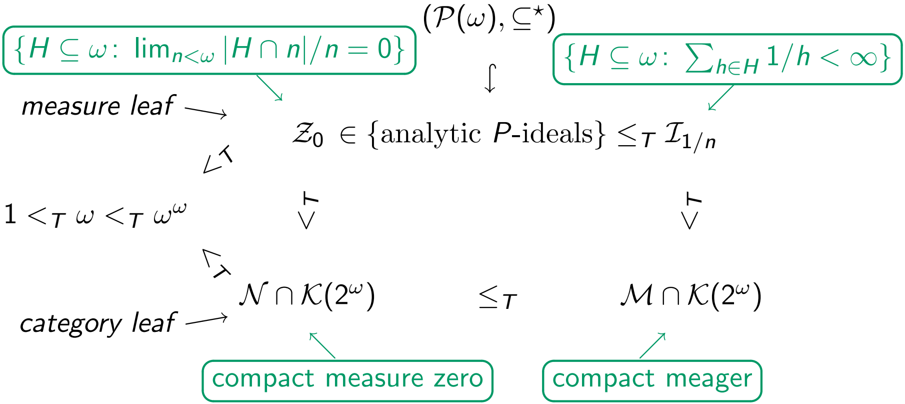

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TUKEY PICTURE UPDATE

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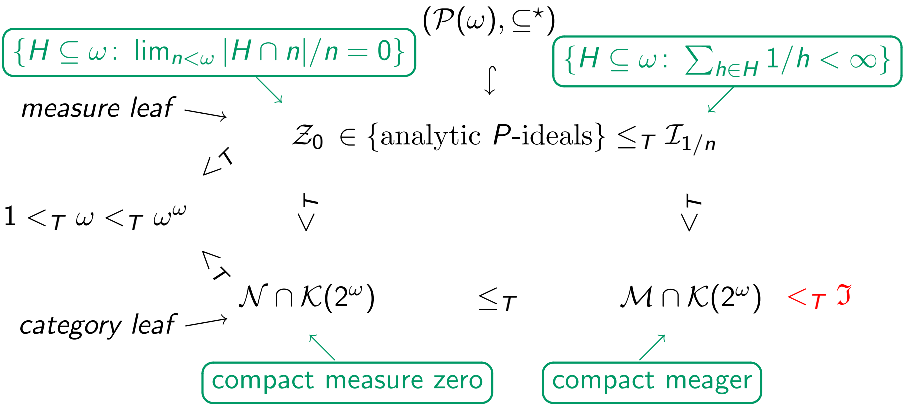

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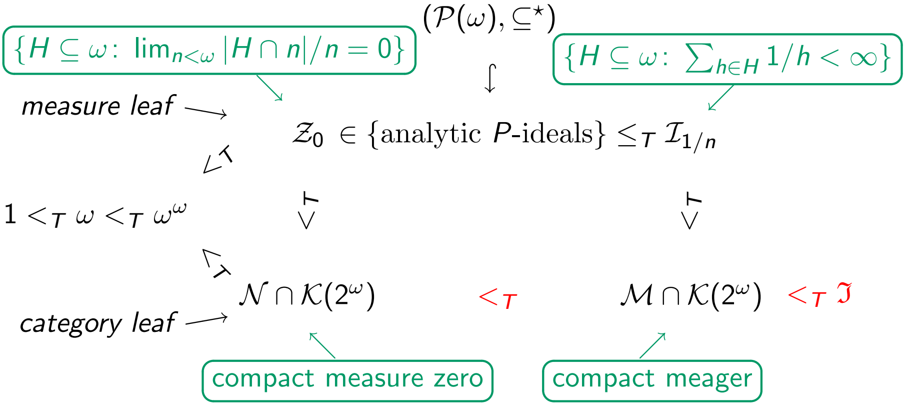
Tukey picture update

i.e. many cofinal types of definable directed orders



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$1 <_T \omega <_T \omega^\omega$

$<_T$ $<_T$

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category leaf

compact measure zero

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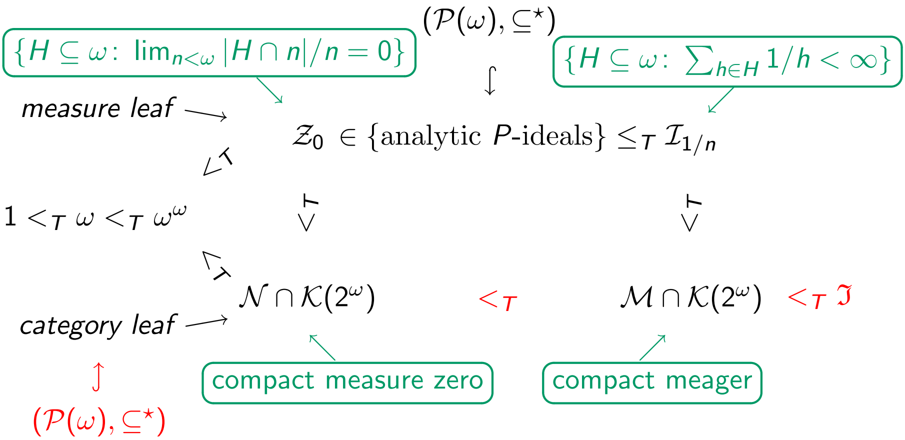

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Tree calibration

TREE CALIBRATION

T : tree

$\mathcal{S}$: set of subtrees of T

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$(P, \leq)$ is $(T, \mathcal{S})$-calibrated:

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$(P, \leq)$ is $(T, \mathcal{S})$-calibrated:

$\exists D\colon T \to 2^P$ with $D(\emptyset) = 2^P$ and $D(t) \subseteq \bigcup\{D(t')\colon t' \in \mathrm{succ}_T(t)\}$

$\forall S \in \mathcal{S}$

$\forall d\colon S \to P$ with $d(s) \in D(s)$ $(s \in S)$

$$\{d(s)\colon s \in S\} \subseteq P \text{ is bounded}$$

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Theorem.

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Theorem.

1. $Q \leq_T P$, P is $(T, \mathcal{S})$-calibrated $\Rightarrow$ Q is $(T, \mathcal{S})$-calibrated

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Theorem.

1. $Q \leq_T P$, P is $(T, \mathcal{S})$-calibrated $\Rightarrow$ Q is $(T, \mathcal{S})$-calibrated
2. $Q \not\leq_T P \Rightarrow \exists (T, \mathcal{S})\colon P$ is $(T, \mathcal{S})$-calibrated,
   Q is not $(T, \mathcal{S})$-calibrated

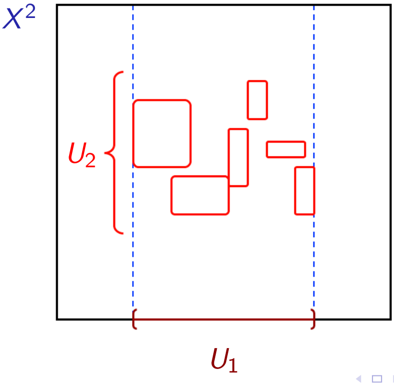
Back to the picture

OPEN TOWERS

X topological space

$\mathcal{U} = (U_n)_{0<n<\omega}$ open tower:

- $U_n \subseteq X^n$ open $(0 < n < \omega)$
- $U_n \Delta \mathrm{Pr}_{X^n}(U_{n+1})$ is nowhere dense $(0 < n < \omega)$



OPEN TOWERS

X topological space

$\mathcal{U} = (U_n)_{0<n<\omega}$ open tower: $[\mathcal{U}] = \bigcap_{0<n<\omega} U_n \times X^{\omega\setminus n}$

- $U_n \subseteq X^n$ open $(0 < n < \omega)$
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