Symbiosis and Upwards Reflection

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A little preview

Theorem (Löwenheim-Skolem)

If $\mathcal{A} \models \phi$ *then there is a countable* $\mathcal{B} \models \phi$*.*

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If $\mathcal{A} \models \phi$ then there is a countable $\mathcal{B} \models \phi$.

Proof.

Let $\mathcal{H}_\theta$ be sufficiently large, containing $\mathcal{A}$, and $\mathcal{H}_\theta \models (\mathcal{A} \models \phi)$. Let $M \prec \mathcal{H}_\theta$ be a countable elementary submodel with $\mathcal{A} \in M$. Let $\pi : M \cong \bar{M}$ be the transitive collapse and $\mathcal{B} = \pi(\mathcal{A})$. Since $\bar{M}$ is countable and transitive, $\mathcal{B}$ is countable. By elementarity $M \models (\mathcal{A} \models \phi)$, so $\bar{M} \models (\mathcal{B} \models \phi)$. But "$\mathcal{B} \models \phi$" is Δ_1, so by absoluteness $\mathcal{B} \models \phi$. □

A little preview

Is this (only) a joke?

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Notice that we only used that "$\mathcal{A} \models \phi$" is Δ_1. In fact Σ_1 would have been sufficient.

Theorem

Let $\mathcal{L}$ be any logic extending FOL, such that "$\mathcal{A} \models_{\mathcal{L}} \phi$" is Σ_1. Then the (downward) Löwenheim-Skolem Theorem holds for $\mathcal{L}$.

Remark: For most interesting extensions $\mathcal{L}$ of FOL, the satisfaction relation is not Σ_1. But if our set theory satisfies a **stronger reflection principle** then the same argument can work.

Model Theory vs. Set Theory

Logicians have two ways to describe a class of structures:

- **defining** in set theory: $\{\mathcal{A} \mid \Phi(\mathcal{A})\}$
- **axiomatizing** by logic: $\{\mathcal{A} \mid \mathcal{A} \models \phi\}$

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Example 1

Describe the class of all structures with 3 or more elements.

- In set theory: $\{\mathcal{A} \mid \Phi(\mathcal{A})\}$, where $\Phi(x)$ is "$|x| \geq 3$"
- In logic: $\{\mathcal{A} \mid \mathcal{A} \models \phi\}$ where ϕ is

$$\exists x_1 x_2 x_3 (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3)$$

Note: Φ can be Δ_0

Model Theory vs. Set Theory

Example 2

Describe the class of infinite structures.

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Describe the class of infinite structures.

- In set theory: $\{\mathcal{A} \mid \Phi(\mathcal{A})\}$, where $\Phi(x)$ is "$|\omega| \leq A$".
- Impossible in $\mathcal{L}_{\omega\omega}$. But using $\mathcal{L}_{\omega_1\omega_1}$, $\mathcal{A}$ is infinite iff $\mathcal{A} \models \phi$, where

$$\phi \equiv \exists x_0, x_1, \cdots \bigwedge_{i \neq j} x_i \neq x_j$$

- Alternatively, we can add a **generalized quantifier** Q_∞ saying "there are infinitely many". Then $\mathcal{A}$ is infinite iff $\mathcal{A} \models Q_\infty x(x = x)$

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Describe the class of structures $(\mathcal{A}, P)$ such that

$$|\{x \in A \mid P(x)\}| = |\{x \in A \mid \neg P(x)\}|$$

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- In set theory: $\{\mathcal{A} \mid \Phi(\mathcal{A})\}$, where $\Phi(x)$ is as above.
- In $\mathcal{L}_{\omega\omega}$ impossible. In $\mathcal{L}_{\omega_1\omega_1}$ or $\mathcal{L}_{\omega\omega}(Q_\infty)$ also impossible. But we can add the so-called **Härtig quantifier** I defined by

$$\mathcal{A} \models \mathsf{I}xy\ \phi(x)\psi(y) \ :\Leftrightarrow\ |\{a \in A : \mathcal{A} \models \phi[a]\}| = |\{b \in A : \mathcal{A} \models \psi[b]\}|$$

Then this model class is axiomatizable by $\phi \equiv$ "$\mathsf{I}xyP(x)\neg P(x)$" in the logic $\mathcal{L}_{\omega\omega}(\mathsf{I})$.

Note: Φ can be Δ_2 but not Δ_1 (cardinalities are not absolute).

Set Theory vs. Logic: who is stronger?

Question

Who is stronger: set theory $\{\mathcal{A} \mid \Phi(\mathcal{A})\}$ or logic $\{\mathcal{A} \mid \mathcal{A} \models \phi\}$?

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1. Since the satisfaction relation for $\mathcal{L}_{\omega\omega}$ is Δ_1, any $\mathcal{L}_{\omega\omega}$ model class $\mathrm{Mod}(\phi)$ is Δ_1.

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3. One can show that every Δ_1 class (if closed under isomorphisms), is axiomatizable by the logic $\Delta(\mathcal{L}_{\omega\omega}(\mathsf{I}))$.

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4. But not vice versa, e.g., $|\{x \in A \mid P(x)\}| = |\{x \in A \mid \neg P(x)\}|$ is $\Delta(\mathcal{L}_{\omega\omega}(\mathsf{I}))$-axiomatizable but not Δ_1.

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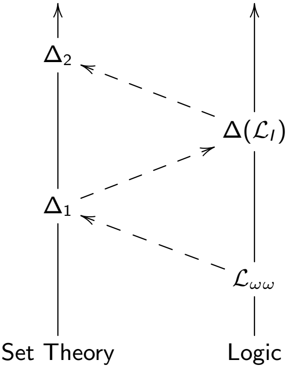
4. But not vice versa, e.g., $|\{x \in A \mid P(x)\}| = |\{x \in A \mid \neg P(x)\}|$ is $\Delta(\mathcal{L}_{\omega\omega}(\mathsf{I}))$-axiomatizable but not Δ_1.
5. It is Δ_2, but that's again too strong ...

Set theoretic vs. logical strength



Symbiosis

In his PhD Dissertation (1977), Väänänen introduced the concept **Symbiosis**, aiming to find an **exact ballance of power** between set-theoretic and model-theoretic strength.

It turns out that the interesting cases take place **between** Δ_1 **and** Δ_2

If R is a set-theoretic predicate, focus on $\Delta_1(R)$-classes, for a fixed Σ_1 or Π_1 predicate R.

$\Delta_1(R)$-classes

Definition

Let R be a fixed set-theoretic predicate. Then a formula ϕ is $\Sigma_1(R)$ if it is Σ_1 in the extended language of set theory with the R-predicate. The same holds for $\Pi_1(R)$ and $\Delta_1(R)$.

Example:

1. $Cd(x) \leftrightarrow x$ is a cardinal.
2. $Rg(x) \leftrightarrow x$ is a regular cardinal'.
3. $PwSt(x, y) \leftrightarrow y = \mathcal{P}(x)$.

For instance "x is uncountable" can be expressed in a $\Sigma_1(Cd)$ way:

$$\exists\alpha\exists f(Cd(\alpha) \wedge \alpha \neq \omega \wedge f : \alpha \hookrightarrow x)$$

If R is Π_1 or Σ_1 then $\Delta_1(R) \subseteq \Delta_2$.

The complexity of $\models_{\mathcal{L}}$

Using this notion, we can compute the set-theoretic power of $\models_{\mathcal{L}}$ more accurately.

Lemma

$\models_{\mathcal{L}_{\omega\omega}(\mathsf{I})}$ *is* $\Delta_1(Cd)$

Proof.

Call a model M of set theory *Cd*-**correct** if $M \models Cd(\alpha)$ iff $Cd(\alpha)$. Then "$\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(\mathsf{I})} \phi$" is absolute between models of set theory which are *Cd*-correct. Thus

$\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(\mathsf{I})} \phi$ iff

$$\exists M(M \text{ trans.} \wedge M \models \mathrm{ZFC}^* \wedge \mathcal{A} \in M \wedge \forall\alpha(M \models Cd(\alpha) \leftrightarrow Cd(\alpha)) \wedge M \models (\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(\mathsf{I})} \phi))$$

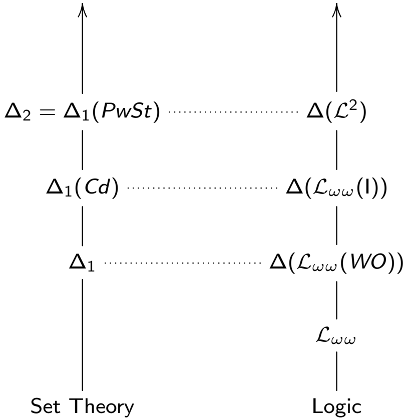
iff

$$\forall M(M \text{ trans.} \wedge M \models \mathrm{ZFC}^* \wedge \mathcal{A} \in M \wedge \forall\alpha(M \models Cd(\alpha) \leftrightarrow Cd(\alpha)) \rightarrow M \models (\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(\mathsf{I})} \phi))$$

This gives a $\Sigma_1(Cd)$ and a $\Pi_1(Cd)$ definition. □

Symbiosis

By **Symbiosis**, we want to capture the idea that $\mathcal{L}$ has **the same** expressive power as $\Delta_1(R)$, for some Π_1 predicate R.



Applications of Symbiosis

Applications:

1. Large Cardinal strength of principles of $\mathcal{L}$ (such as Löwenheim-Skolem and Compactness)
2. Relating properties of $\mathcal{L}$ to set-theoretic reflection principles for $\Sigma_1(R)$- and $\Delta_1(R)$- classes
3. Large Cardinal strength of reflection principles
4. Probably more . . .

Symbiosis

Definition (Väänänen)

$\mathcal{L}$ and R are **symbiotic** if

1. $\models_{\mathcal{L}}$ is $\Delta_1(R)$,
2. ... ?

What should ... say?

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$\mathcal{L}$ and R are **symbiotic** if

1. $\models_{\mathcal{L}}$ is $\Delta_1(R)$, equiv: for every $\mathcal{L}$-sentence ϕ, $\mathrm{Mod}(\phi)$ is $\Delta_1(R)$.
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What if the class is not closed under isomorphisms?

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Second attempt: "every $\Delta_1(R)$-class of τ-structures closed under isomorphisms is of the form $\mathrm{Mod}(\phi)$".

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Symbiosis only works for strong logics of a special form: $\Delta(\mathcal{L})$

Δ-operation

Definition

Let $\tau \subseteq \tau'$ be **many-sorted** vocabularies. If $\mathcal{A}$ is a τ'-structure, then the τ-**reduct** $\mathcal{A}\restriction\tau$ is defined by ignoring all symbols not in τ' and **restricting the domain to the sorts in** τ.

Definition

A class $\mathcal{K}$ of τ-structures is $\Sigma(\mathcal{L})$-**axiomatizable** if $\mathcal{K} = \{\mathcal{A}\restriction\tau \ : \mathcal{A} \models_{\mathcal{L}} \phi\}$ for some ϕ in an extended language τ'. A class $\mathcal{K}$ is $\Delta(\mathcal{L})$-**axiomatizable** if both $\mathcal{K}$ and its complement are $\Sigma(\mathcal{L})$-axiomatizable.

Δ-operation

The Δ-operation has many applications in abstract model theory.

1. It is convenient to regard $\Delta(\mathcal{L})$ itself as an **abstract logic**.
2. $\Delta(\mathcal{L}_{\omega\omega}) = \mathcal{L}_{\omega\omega}$
3. If $\mathcal{L}$ satisfies **Craig interpolation** then $\Delta(\mathcal{L}) = \mathcal{L}$.
4. The Δ-operation preserves many properties of the logic $\mathcal{L}$, in particular downward Löwenheim-Skolem theorems.

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Theorem (Bagaria & Väänänen)

1. $\mathcal{L}_{\omega\omega}(\mathrm{I})$ *and* Cd *are symbiotic.*
2. $\mathcal{L}^2$ *and* $PwSt$ *are symbiotic.*
3. ... *and many others.*

The class $\mathbb{Q}_R$

Definition (Väänänen)

$\mathcal{L}$ and R are **symbiotic** if

1. $\models_{\mathcal{L}}$ is $\Delta_1(R)$, and
2. Every $\Delta_1(R)$-class closed under isomorphisms is $\Delta(\mathcal{L})$-axiomatizable.

Instead of 2, we can consider a special case which is easier to both prove and apply.

Definition

For a predicate R, let $\mathbb{Q}_R$ be the class of **all** R**-correct** ZFC***-models** closed under isomorphisms, i.e.,

$$\mathbb{Q}_R = \{(N,E) \mid (N,E) \cong (M,\in) \text{ for some } R\text{-correct model } (M,\in) \models \text{ZFC}^*\}$$

Equivalent formulation of condition ②

Lemma

The following conditions (of Symbiosis) are equivalent:

- ② *Every $\Delta_1(R)$-class closed under isomorphisms is $\Delta(\mathcal{L})$-axiomatizable.*
- ②* *$\mathbb{Q}_R$ is $\Delta(\mathcal{L})$-axiomatizable.*

Proof.

②* $\Rightarrow$ ②:

$(N, E) \in \mathbb{Q}_R$ iff

$\exists M\, ((M, \in) \cong (N, E) \land (M, \in) \models \mathrm{ZFC}^* \land \forall x \in M\, ((M \models R(x)) \leftrightarrow R(x)))$

iff

E wellfounded & extensional $\land \forall M$
$((M, \in) \cong (N, E) \land M$ transitive $\rightarrow (M, \in) \models \mathrm{ZFC}^* \land \forall x \in M\, ((M \models R(x)) \leftrightarrow R(x)))$

Therefore $\mathbb{Q}_R$ is $\Delta_1(R)$ and we are done. □

Equivalent formulation of condition ②

Proof.

② $\Rightarrow$ ②*:
Let $\mathcal{K}$ be a class of τ-structures and consider first the $\Sigma_1(R)$ formula Φ defining the class, i.e., $\mathcal{A} \in \mathcal{K} \Leftrightarrow \Phi(\mathcal{A})$.

For simplicity, assume τ has only one unary predicate symbol P.

- Consider τ as being of sort s_1.
- Extend the language with a new sort s_0, with a binary relation E and a constant c.
- New function symbol F, from s_1 to s_0.

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- Consider τ as being of sort s_1.
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Let $\mathcal{K}^*$ be the class of all models $\mathcal{N} = (N, A, E, c, P, F)$ in the extended language, such that

1. (N, E) is an R-correct ZFC*-model
2. $(N, E) \models \Phi(c)$ (expressed in E)
3. $\mathcal{N} \models F$ is an isomorphic between "c written using E" and (A, P).

□

Equivalent formulation of condition 2

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Now ① essentially says "$(N, E) \in \mathbb{Q}_R$". By assumption $(2)^*$, this statement is $\Delta(\mathcal{L})$-axiomatizable, in particular $\Sigma(\mathcal{L})$-axiomatizable.

② and ③ are in FOL.

Therefore the class $\mathcal{K}^*$ is $\Sigma_1(\mathcal{L})$-axiomatizable.

So we will be done if we can prove that $\mathcal{K} = \{\mathcal{N}\restriction\tau \mid \mathcal{N} \in \mathcal{K}^*\}$. □

Equivalent formulation of condition ②

Proof.

Claim: $\mathcal{K} = \{\mathcal{N}\restriction\tau \mid \mathcal{N} \in \mathcal{K}^*\}$.

First suppose $(A, P) \in \mathcal{K}$. Let V_α be sufficiently large so that $V_\alpha \models \mathrm{ZFC}^*$ and R is absolute for V_α (if R is Π_1, use Π_1-reflection). Then

$$(V_\alpha, A, \in, (A, P), P, id_A)$$

is an element of $\mathcal{K}^*$.

Equivalent formulation of condition 2

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Conversely, suppose $(N, A, E, c, P, F) \in \mathcal{K}^*$. Let $\pi : (N, E) \cong (M, \in)$ and let $\mathcal{B} = \pi(c)$. Then $M \models \Phi(\mathcal{B})$. But since Φ **is** $\Sigma_1(R)$ **and** M **is** R**-correct**, by absoluteness $\Phi(\mathcal{B})$ is true. Therefore, $\mathcal{B} \in \mathcal{K}$. But by condition (3), $\mathcal{B}$ is isomorphic to (A, P). Since $\mathcal{K}$ was assumed to be closed under isomorphisms, it follows that $(A, P) \in \mathcal{K}$, as we had to show. □

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The following conditions (of Symbiosis) are equivalent:

2. *Every $\Delta_1(R)$-class closed under isomorphisms is $\Delta(\mathcal{L})$-axiomatizable.*

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Examples of Symbiosis

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$\mathcal{L}_{\omega\omega}(\mathsf{I})$ and Cd are symbiotic.

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Proof.

We already saw that $\models_{\mathcal{L}_{\omega\omega}(\mathsf{I})}$ is $\Delta_1(Cd)$.

For the converse, it suffices to prove that $\mathbb{Q}_{Cd}$ is $\Delta(\mathcal{L}_{\omega\omega}(\mathsf{I}))$. We have $(N, E) \in \mathbb{Q}_R$ iff

1. E is wellfounded
2. $(N, E) \models \mathrm{ZFC}^*$
3. For $(M, \in) \cong (N, E)$ we have $M \models Cd(\alpha)$ iff $Cd(\alpha)$

For 3, note that $M \models Cd(\alpha)$ iff $M \models_{\mathcal{L}_{\omega\omega}(\mathsf{I})} \neg\exists x < \alpha \mathsf{I} yz(y \in x)(z \in \alpha)$

So 2 + 3 hold iff

$$(N, E) \models_{\mathcal{L}_{\omega\omega}(\mathsf{I})} \mathrm{ZFC}^* \wedge \forall\alpha\, (\alpha \text{ is a cardinal } \leftrightarrow \neg\exists x < \alpha\, \mathsf{I} yz\, (yEx)\, (zE\alpha)$$

which is an $\mathcal{L}_{\omega\omega}(\mathsf{I})$-sentence.

□

Examples of Symbiosis

Theorem (Bagaria & Väänänen)

$\mathcal{L}_{\omega\omega}(\mathsf{I})$ and Cd are symbiotic.

Proof.

It remains to take care of ①.

- (N, E) is **ill-founded** iff there exists X such that X has no E-minimal element. Add a new predicate X and consider $K^* = \{(N, E, X) \mid (N, E, X) \models (X \text{ has no } E\text{-minimal element})\}$ (which can be expressed in FOL). Then (N, E) is ill-founded iff $(N, E) = \mathcal{M}\restriction\tau_E$ for some $\mathcal{M} \in \mathcal{K}^*$. So being ill-founded is $\Sigma(\mathcal{L}_{\omega\omega})$, thus being well-founded is $\Pi(\mathcal{L}_{\omega\omega})$, so also $\Pi(\mathcal{L}_{\omega\omega}(I))$.
- "Lindström's trick": $(X, <)$ is **well-founded** iff there are sets A_a for every $a \in X$ such that $a < b$ iff $|A_a| < |A_b|$. So add a new sort and new binary relation between two sorts. Consider the class $\mathcal{K}^*$ of structures $\mathcal{M} = (M, A, E, R)$ such that

$$\mathcal{M} \models \forall a, b \in M \; (a < b \to |R(a, .)| < |R(b, .)|)$$

This can be expressed in $\mathcal{L}_{\omega\omega}(\mathsf{I})$. So (N, E) is well-founded iff it is the restriction of a model in $\mathcal{K}^*$. □

More symbiosis

Actually, an even easier proof shows the following:

Theorem

$\mathcal{L}_{\omega\omega}(\mathrm{WO})$ is symbiotic to $\varnothing$ (empty predicate, i.e., just Δ_1-sentences).

Here $\mathcal{L}_{\omega\omega}(\mathrm{WO})$ is the logic with a generalized quantifier expressing that something is a well-order.

More symbiosis

Theorem

$\mathcal{L}^2$ is symbiotic with PwSt.

Proof.

1. The relation $\models_{\mathcal{L}^2}$ is absolute for sufficiently large V_α. Moreover, being V_α is $\Delta_1(PwSt)$-definable. Therefore $\mathcal{A} \models_{\mathcal{L}^2} \phi$
   $\Leftrightarrow \; \exists V_\alpha \, (\mathcal{A} \in V_\alpha \wedge V_\alpha \models (\mathcal{A} \models \phi))$
   $\Leftrightarrow \; \forall V_\alpha \, (\mathcal{A} \in V_\alpha \rightarrow V_\alpha \models (\mathcal{A} \models \phi))$.

More symbiosis

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2. * To show: $\mathbb{Q}_{PwSt}$ is $\Delta(\mathcal{L}^2)$. But this is easy since in full $\mathcal{L}^2$ we can define the **true power set**, i.e., there is a $\mathcal{L}^2$-sentence $\phi(x, y)$ such that $(M, \in) \models \phi(x, y)$ iff $y = \mathcal{P}(x)$.

□

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Theorem

$\mathcal{L}^2$ is symbiotic with PwSt.

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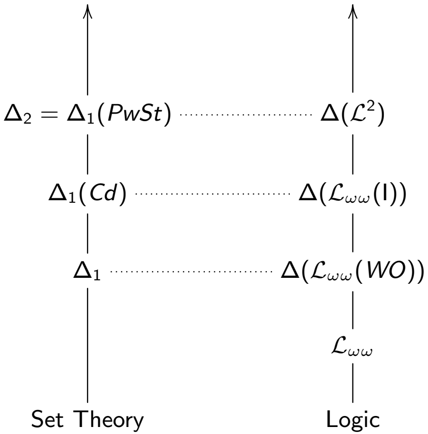
□

Remark: In fact, $\Delta_1(PwSt) = \Delta_2$. This is because Δ_2-formulas are absolute for $\mathcal{H}_\theta$ and "being $\mathcal{H}_\theta$" can also be defined in a $\Delta_1(PwSt)$-way.

Symbiosis


$\Delta_2 = \Delta_1(PwSt)$
$\Delta(\mathcal{L}^2)$
$\Delta_1(Cd)$
$\Delta(\mathcal{L}_{\omega\omega}(\mathsf{I}))$
Δ_1
$\Delta(\mathcal{L}_{\omega\omega}(WO))$
$\mathcal{L}_{\omega\omega}$
Set Theory
Logic

Downward Löwenheim-Skolem

Application of Symbiosis: downward Löwenheim-Skolem (one of many possible versions) and downward reflection.

Definition

The **downwards Löwenheim-Skolem number** of $\mathcal{L}$ is the least κ such that if $\mathcal{A} \models_{\mathcal{L}} \phi$ then there is a sub-structure $\mathcal{B} \subseteq \mathcal{A}$ s.t. $|B| < \kappa$ and $\mathcal{B} \models_{\mathcal{L}} \phi$. Notation: $\mathrm{DLST}(\mathcal{L}) = \kappa$

Definition

The **downward structural reflection number** for a predicate R is the least κ such that if $\mathcal{K}$ is a $\Sigma_1(R)$-class of τ-structures (for fixed τ), then for every $\mathcal{A} \in K$ there is an elementary sub-structure $\mathcal{B} \preceq \mathcal{A}$ such that $|B| < \kappa$ and $\mathcal{B} \in \mathcal{K}$. Notation: $\mathrm{DSR}(R) = \kappa$

Theorem (Bagaria-Väänänen 2015)

Suppose $\mathcal{L}$ and R are symbiotic. Then $\mathrm{DLST}(\mathcal{L}) = \kappa$ iff $\mathrm{DSR}(R) = \kappa$.

Downward Löwenheim-Skolem

Theorem (Bagaria-Väänänen 2015)

Suppose $\mathcal{L}$ and R are symbiotic. Then $\mathsf{DLST}(\mathcal{L}) = \kappa$ *iff* $\mathsf{DSR}(R) = \kappa$.

Proof.

- $\Leftarrow$ is immediate: let ϕ be an $\mathcal{L}$-sentence and $\mathcal{A} \models \phi$. By condition (1) of Symbiosis, $\mathrm{Mod}(\phi)$ is a $\Delta_1(R)$-class, in particular, a $\Sigma_1(R)$-class. So $\mathcal{A} \models \phi \Rightarrow \mathcal{A} \in \mathrm{Mod}(\phi) \Rightarrow \exists \mathcal{B} \preceq \mathcal{A}$ with $|B| \leq \kappa$ and $\mathcal{B} \in \mathrm{Mod}(\phi) \Rightarrow \mathcal{B} \models \phi$.
- $\Rightarrow$ If we just wanted to prove downwards reflection for $\Delta_1(R)$ classes and without elementarity, we could use a direct proof. But this result is stronger. The main idea is: reflection for Σ_1-classes holds in ZFC!

□

Downward Löwenheim-Skolem

Proof.

Let $\mathcal{K}$ be a $\Sigma_1(R)$-class, let $\mathcal{A} \in \mathcal{K}$, and let Φ be the defining formula.

Let $\mathcal{K}^*$ be the class of models (N, E, c) such that (N, E) is isomorphic to an R-correct ZFC*-model $(M, \in)$ satisfying $(M, \in) \models \Phi(c)$. This is defined using $\mathbb{Q}_R$, so by condition (2) of Symbiosis $\mathcal{K}^*$ is $\Delta(\mathcal{L})$-axiomatizable. Therefore there exists ϕ in an extended language, such that $(N, E, c) \in \mathcal{K}^*$ iff $(N, E, c, \dots) \models \phi$.

Let $\mathcal{H}_\theta$ be sufficiently large so that $\mathcal{A} \in \mathcal{H}_\theta$ and $\mathcal{H}_\theta \models \Phi(\mathcal{A})$. Then $(\mathcal{H}_\theta, \in, \mathcal{A}) \in \mathcal{K}^*$, so some extension $(\mathcal{H}_\theta, \in, \mathcal{A}, \dots) \models \phi$. Using DLST$(\mathcal{L})$, there is $(N, \in, \mathcal{A}, \dots) \subseteq (\mathcal{H}_\theta, \in, A, \dots)$ such that $(N, \in, B, \dots) \models \phi$ and $|N| < \kappa$. Thus $(N, \in \mathcal{A}) \in \mathcal{K}^*$, and since $\mathcal{K}^*$ is closed under isomorphisms, also the transitive collapse $(M, \in, \bar{\mathcal{A}})$ of $(N, \in, \mathcal{A})$ is in $\mathcal{K}^*$. But then $(M, \in) \models \Phi(\bar{\mathcal{A}})$, and $(M, \in)$ was R-correct, so by **upwards $\Sigma_1(R)$-absoluteness**, $\Phi(\bar{\mathcal{A}})$ is true, and $|\bar{\mathcal{A}}| \leq |M| < \kappa$.

To show that, additionally, $\bar{\mathcal{A}} \preceq \mathcal{A}$, use a more complicated argument by adding **Skolem functions** to the models in $\mathcal{K}^*$. □

Large Cardinal strength

Application:

Theorem

DSR($PwSt$) $= \kappa$ *iff* κ *is the first supercompact cardinal.*

Proof.

It is known that $DLST(\mathcal{L}^2) = \kappa$ iff κ is the first supercompact (Magidor). So by Symbiosis between $\mathcal{L}^2$ and $PwSt$, the same holds for DSR($\mathcal{L}^2$). □

Downward Löwenheim-Skolem

Definition

The **strict downwards Löwenheim-Skolem number** of $\mathcal{L}$ is the least κ such that if $\mathcal{A} \models_{\mathcal{L}} \phi$ and $|\mathcal{A}| = \kappa$, then there is a sub-structure $\mathcal{B} \subseteq \mathcal{A}$ s.t. $|B| < \kappa$ and $\mathcal{B} \models_{\mathcal{L}} \phi$.
Notation: $\text{DLST}^{-}(\mathcal{L}) = \kappa$

Definition

The **strict downward structural reflection number** for a predicate R is the least κ such that if $\mathcal{K}$ is a $\Sigma_1(R)$-class of τ-structures (for fixed τ), then for every $\mathcal{A} \in K$ such that $|\mathcal{A}| = \kappa$, there is an elementary sub-structure $\mathcal{B} \preceq \mathcal{A}$ such that $|B| < \kappa$ and $\mathcal{B} \in \mathcal{K}$.
Notation: $\text{DSR}^{-}(R) = \kappa$

Theorem (Bagaria-Väänänen 2015)

Suppose $\mathcal{L}$ and R are symbiotic. Then $\text{DLST}^{-}(\mathcal{L}) = \kappa$ iff $\text{DSR}^{-}(R) = \kappa$.

Large Cardinal Strength

Theorem (Bagaria-Väänänen)

$\mathrm{DLST}^-(\mathcal{L}_{\omega\omega}(\mathrm{I})) = \kappa$ *iff* $\mathrm{DSR}^-(Cd) = \kappa$ *iff* κ *is weakly inaccessible.*

The proof is:

1. $\mathrm{DLST}^-(\mathcal{L}_{\omega\omega}(\mathrm{I})) = \kappa \quad \Rightarrow \quad \kappa$ weakly inaccessible.
2. κ weakly inaccessible $\quad \Rightarrow \quad \mathrm{DLST}^-(\mathcal{L}_{\omega\omega}(\mathrm{I})) = \kappa$.
3. The theorem follows from Symbiosis between $\mathcal{L}_{\omega\omega}(\mathrm{I})$ and Cd.

Large Cardinal Strength

Theorem (Bagaria-Väänänen)

$\mathrm{DLST}^-(\mathcal{L}_{\omega\omega}(\mathrm{I})) = \kappa$ *iff* $\mathrm{DSR}^-(Cd) = \kappa$ *iff* κ *is weakly inaccessible.*

The proof is:

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2. κ weakly inaccessible $\quad \Rightarrow \quad \mathrm{DLST}^-(\mathcal{L}_{\omega\omega}(\mathrm{I})) = \kappa$.
3. The theorem follows from Symbiosis between $\mathcal{L}_{\omega\omega}(\mathrm{I})$ and Cd.

Another example:

1. W^{Reg} = the generalized quantifier expressing that $<$ is a well-order of order-type a regular cardinal.
2. Reg = the set-theoretic predicate "α is a regular cardinal"
3. $\mathcal{L}_{\omega\omega}(\mathrm{I}, W^{Reg})$ and Reg are symbiotic.

Theorem (Bagaria-Väänänen)

$\mathrm{DLST}^-(\mathcal{L}_{\omega\omega}(\mathrm{I}, W^{Reg})) = \kappa$ *iff* $\mathrm{DSR}^-(Reg) = \kappa$ *iff* κ *is weakly Mahlo.*

Large Cardinal Strength

We also have the basic case:

Corollary

$\mathrm{DLST}(\mathcal{L}_{\omega\omega}(\mathrm{WO})) = \omega.$

Proof.

Recall that $\mathcal{L}_{\omega\omega}(\mathrm{WO})$ is symbiotic with $\varnothing$. But downwards structural reflection for Σ_1 classes is true in ZFC. □

Other properties of logic

Now, let's look at other properties (work in progress).

Originally, we were interested in the question of **compactness** of strong logics $\mathcal{L}$.

Question

Is there a set-theoretic reflection principle for $\Sigma_1(R)$-classes, which could be related to **compactness of** $\mathcal{L}$, for symbiotic $\mathcal{L}$ and R?

Compactness is related to **upwards** Löwenheim-Skolem principles. Therefore it's natural to look at **upwards** reflection principles.

Upwards Löwenheim-Skolem and reflection

Again, one can consider various definitions.

Definition

The **upwards Löwenheim-Skolem number** of $\mathcal{L}$ is the least κ such that if $\mathcal{A} \models_{\mathcal{L}} \phi$ and $|\mathcal{A}| \geq \kappa$, then for every $\kappa' > \kappa$ there is a super-structure $\mathcal{B} \supseteq \mathcal{A}$ with $|B| \geq \kappa'$ and $\mathcal{B} \models_{\mathcal{L}} \phi$. Notation: $\mathrm{ULST}(\mathcal{L}) = \kappa$.

Remarks:

1. One may replace "super-structure" by "elementary extension". This may (sometimes) give equivalent definitions.
2. The **Hanf number** is the same but without the requirement of "super-structure". Note that the Hanf number is always defined (by diagonalization) in ZFC, but $\mathrm{ULST}(\mathcal{L})$ usually implies Large Cardinals.
3. There are possible variations, e.g., for sets of sentences instead of just ϕ, or requiring that $\mathcal{B}$ is an elementary extension, or even an $\mathcal{L}$-elementary extension, etc.

Compactness $\Rightarrow$ upwards Löwenheim-Skolem, but not (always) vice versa.

Upwards reflection

First attempt: "for every $\Sigma_1(R)$-class $\mathcal{K}$ of τ-structures, if there is $\mathcal{A} \in \mathcal{K}$ with $|A| \geq \kappa$, then for every $\kappa' > \kappa$ there is $\mathcal{B} \in \mathcal{K}$ with $\mathcal{A} \preceq \mathcal{B}$ and $|B| \geq \kappa'$."

Upwards reflection

First attempt: "for every $\Sigma_1(R)$-class $\mathcal{K}$ of τ-structures, if there is $\mathcal{A} \in \mathcal{K}$ with $|A| \geq \kappa$, then for every $\kappa' > \kappa$ there is $\mathcal{B} \in \mathcal{K}$ with $\mathcal{A} \preceq \mathcal{B}$ and $|B| \geq \kappa'$."

But there are several problems.

1. We must be careful about the size of the language τ.
2. Symbiosis relies on the Δ-operator. While the Δ-operator preserves **downwards** LST, it does not, in general, preserve **upwards** LST.

Solution: **bounded** version of everything

- We need something called the **bounded** Δ-operator (which Väänänen had already introduced)
- But then we must also adept the set-theoretic notion of a Σ_1-formula to a **bounded** version.
- This requires the new concept: **bounded** Symbiosis.
- The reflection principle must also be bounded.

Bounded Δ

Definition

A class $\mathcal{K}$ of τ-structures is $\Sigma^B(\mathcal{L})$-**axiomatisable** if there is ϕ in some extended language τ', such that

1. $\mathcal{K} = \{\mathcal{A} \mid \exists \mathcal{B}\,(\mathcal{B} \models \phi \text{ and } \mathcal{A} = \mathcal{B}{\restriction}\tau)\}$ and
2. for all $\mathcal{A}$ there exists a cardinal $\lambda_{\mathcal{A}}$, such that for any τ'-structure $\mathcal{B}$: if $\mathcal{B} \models \phi$ and $\mathcal{A} = \mathcal{B}{\restriction}\tau$ then $|\mathcal{B}| \leq \lambda_{\mathcal{A}}$.

$\mathcal{K}$ is $\Delta^B(\mathcal{L}^*)$-*axiomatisable* if both $\mathcal{K}$ and its complement are $\Sigma^B(\mathcal{L}^*)$-axiomatisable.

Idea: there is a **bound** on the size by which we need to extend the model.

Väänänen 1980:

- for many logics $\mathcal{L}$ we have $\Delta(\mathcal{L}) = \Delta^B(\mathcal{L})$.
- for some logics, this is consistently false.
- Δ^B preserves the Hanf number of $\mathcal{L}$.

Σ_1^B formula relation

Well . . . since we changed Δ to Δ^B we also need a corresponding change on the set theory side!

Definition

A formula $\phi(x)$ in set theory is **definably bounding** if for some Δ_0 formula ψ:

$$\forall x(\phi(x) \leftrightarrow \exists y(\psi(x,y) \wedge \rho(y) < F(\rho(x))$$

where F is a so-called **definable bounding function**. This essentially means (modulo some technicalities) that the class

$$\{(A,B) \mid F(|A|) \geq |B|\}$$

is FOL-definable.

If R is a predicate, then $\Sigma_1^B(R)$ and $\Delta_1^B(R)$ is defined in the same way, but with an additional predicate symbol R.

Bounded Symbiosis

Definition (Galeotti-K-Väänänen)

$\mathcal{L}$ and R are **bouned-symbiotic** if

1. $\models_{\mathcal{L}}$ is $\Delta^B_1(R)$, and
2. Every $\Delta^B_1(R)$-class closed under isomorphisms is $\Delta^B(\mathcal{L})$-axiomatizable.

Lemma (Galeotti-K-Väänänen)

All known examples of pairs $\mathcal{L}$ and R which are symbiotic, are in fact bounded symbiotic.

Upwards structural reflection

We also need a corresponding version of upwards structural reflection.

In addition to bounding, we must also put restrictions on the size of vocabularies.

Definition

Let τ be a vocabulary of size λ. The **upwards structural reflection number** for R is the least κ such that for every $\Sigma^B_1(R)$-class $\mathcal{K}$ of τ-structures, if there is $\mathcal{A} \in \mathcal{K}$ with $|A| \geq \kappa$, then for every $\kappa' > \kappa$ there is $\mathcal{B} \in \mathcal{K}$ with $\mathcal{A} \preceq \mathcal{B}$ and $|B| \geq \kappa'$. Notation: $\mathrm{USR}_\lambda(R) = \kappa$.

Main result

Theorem (Galeotti-K-Väänänen)

Suppose $\mathcal{L}$ and R are bounded-symbiotic. Then $\mathrm{ULST}_\omega(\mathcal{L}) = \kappa$ *iff* $\mathrm{USR}_\omega(R) = \kappa$.

The bound ω can be replaced by λ if λ satisfies suitable definability conditions.

Remarks:

- Since we consider restricted vocabularies, we also need to restrict the ULST principle accordingly.
- This result **cannot** hold for arbitrary languages, because for $\lambda \geq \kappa$, $\mathrm{USR}_\lambda(R)$ is always false, while $\mathrm{ULST}_\lambda(\mathcal{L})$ may be true!

One application (still in progress)

As an application, we provide lower and upper bounds for $\mathrm{ULST}(\mathcal{L}^2)$.

Lemma (Galeotti-K-Väänänen)

If κ is an **extendible** *cardinal, then $\mathrm{USR}_\omega(PwSt) \leq \kappa$. By the main theorem, also $\mathrm{ULST}_\omega(\mathcal{L}^2) \leq \kappa$.*

Lemma (Galeotti-K-Väänänen)

*If $\mathrm{ULST}_\omega(\mathcal{L}^2) = \kappa$ then there is an n-***extendible** *cardinal, for every n.*

Conjecture

$\mathrm{ULST}_\omega(\mathcal{L}^2) = \kappa$ iff κ is extendible.

Connections between logic and set theory

Logic	Set Theory	Need
Downward-LST	Downward-SR	Symbiosis (Bagaria-Väänänen)
Upward-LST	Upward-SR	Bounded Symbiosis (Galeotti-K-Väänänen)
$\Uparrow$ Compactness	$\Uparrow$ "Every well-order can be extended to a longer one, within the same class"	???

Thank You!

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