Preserving splitting families

Diego A. Mejía

diego.mejia@shizuoka.ac.jp

Shizuoka University

Joint work with Martin Goldstern, Jakob Kellner, and Saharon Shelah

University of Vienna

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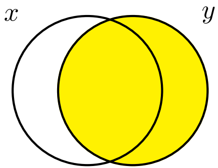
Splitting

For $x, y \in [\omega]^{\aleph_0}$, *$x$ splits y* iff $x \cap y$ and $y \smallsetminus x$ are infinite.



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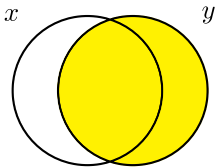

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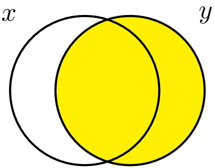

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The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family.

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Here $\mathfrak{b}(\mathbf{R}) = \mathfrak{b}(\mathbf{R}_{\mathrm{sp}}) = \mathfrak{s}$ and $\mathfrak{d}(\mathbf{R}) = \mathfrak{d}(\mathbf{R}_{\mathrm{sp}}) = \mathfrak{r}$. (Actually $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}_{\mathrm{sp}}$).

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Judah & Shelah (1988)

Under CH, any FS (finite support) iteration of Suslin ccc posets forces that $[\omega]^{\aleph_0} \cap V$ is a splitting family.

Our approach

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Force splitting families that can be preserved after a large class of FS iterations.

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To force splitting families:

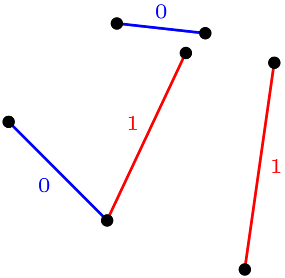
We use Hechler-type forcings of the form $\mathbb{G}_{\mathbf{B}}$ for some 2-*labeled graph* $\mathbf{B}$.

2-graphs

Definition (2-graph)

A 2-*labeled graph* (2-graph) is a triplet $\mathbf{B} = \langle B, R_0, R_1 \rangle$ such that

1. each $\langle B, R_i \rangle$ is a simple graph ($i \in \{0, 1\}$),
2. $R_0 \cap R_1 = \emptyset$.



Good colorings

A coloring $\eta : B \to \{0, 1\}$ *respects* **B** if

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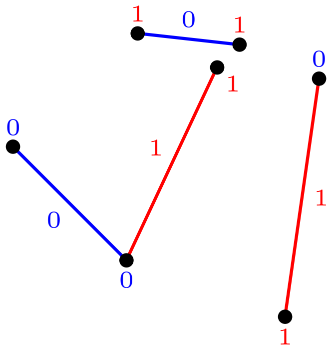

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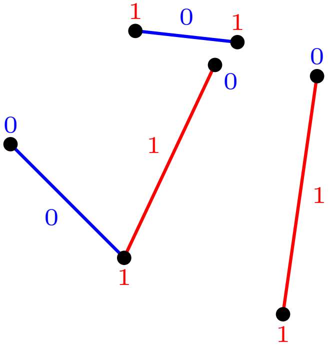

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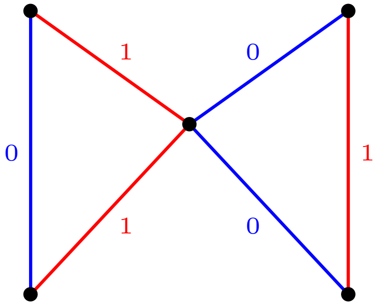

2-graph without a good coloring

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Remark

If $\mathbf{B}$ is suitable, then for any $a \in \mathbf{B}$ and $i \in \{0,1\}$, there is some R_i-clique of size $\aleph_1$ containing a.

Existence of suitable graphs

Theorem (Goldstern & Kellner & M. & Shelah (GKMS))

There exists a suitable 2-graph in ZFC.

The forcing $\mathbb{G}_{\mathbf{B}}$

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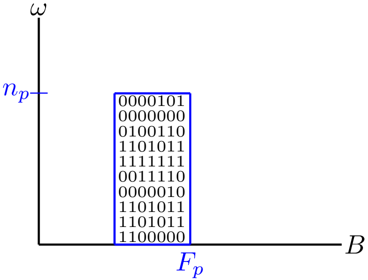

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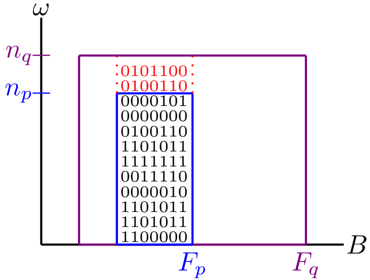
- **Order:** $q \leq p$ iff $p \subseteq q$ and, for any $i \in n_q \setminus n_p$, the partial coloring $q(\cdot, i) : F_p \to \{0,1\}$ respects $\mathbf{B}$.



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Remark

For $A \subseteq B$, $\mathbb{G}_{\mathbf{B}\restriction A}$ may not be a complete subposet of $\mathbb{G}_{\mathbf{B}}$.

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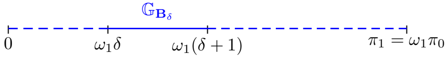

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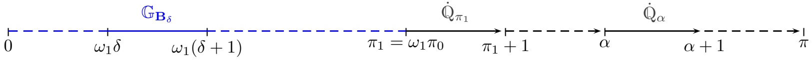
3. $\mathbb{P}_\pi$ is obtained by a FS iteration of ccc posets $\langle \dot{\mathbb{Q}}_\alpha : \pi_1 \leq \alpha < \pi \rangle$ after $\mathbb{P}_{\pi_1}$.



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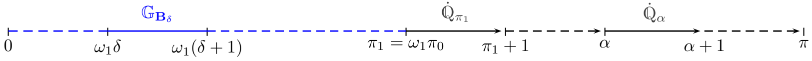

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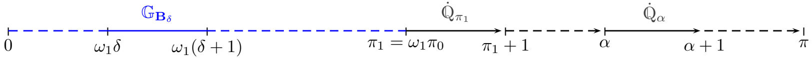

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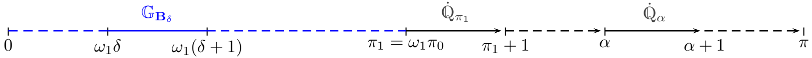

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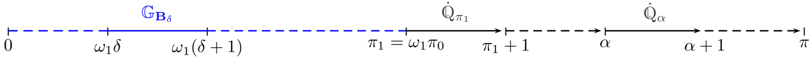

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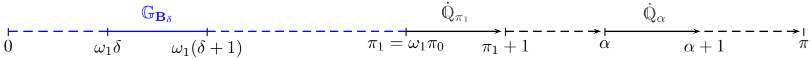

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5. If $\pi_1 < \gamma \leq \pi$ is limit, $\langle \hat{h}_\alpha : \pi_1 \leq \alpha < \gamma \rangle$ is an increasing sequence and each $\hat{h}_\alpha$ is an automorphism on $\mathbb{P}_\alpha$, then $\hat{h}_\gamma := \bigcup_{\alpha<\gamma} \hat{h}_\alpha$ is an automorphism on $\mathbb{P}_\gamma$.

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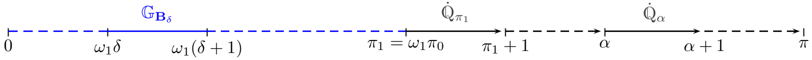

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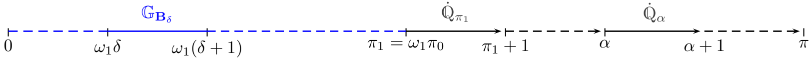

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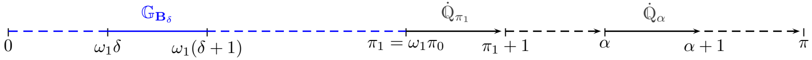
2. $\mathbb{P}_\pi$ is *appropriate* if every good automorphism is compatible with $\mathbb{P}_\pi$.



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Lemma

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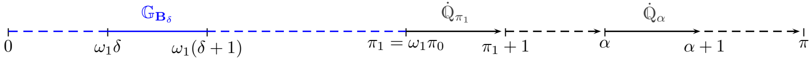
Likewise for $p \in \mathbb{P}_\pi$.

Main result

Definition

Say that $\mathbb{P}_\pi$ is *λ-nice* if, for any $p \in \mathbb{P}_\pi$,

$$|\{\delta < \pi_0 : H(p) \cap B_\delta \neq \emptyset\}| < \lambda.$$



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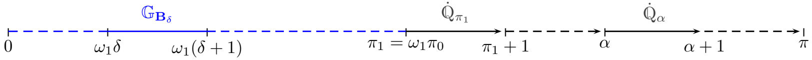
Theorem (GKMS)

Assume λ regular, $\omega_1 \leq \lambda \leq \pi_0$. If $\mathbb{P}_\pi$ is λ-nice and appropriate then it forces $\lambda \preceq_{\mathrm{T}} \mathbf{R}_{\mathrm{sp}}$ witnessed by the "splitting family" $\{c_{\omega_1\delta} : \delta < \lambda\}$.



Proof

Assume some $p \in \mathbb{P}$ forces the contrary, so there is some $\mathbb{P}$-name $\dot{y} \in [\omega]^{\aleph_0}$ such that

$$p \Vdash |\{\delta < \lambda : c_{\omega_1 \delta} R_{\mathrm{sp}} \dot{y}\}| = \lambda.$$

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Find $F \in [\lambda]^\lambda$, $n_0 < \omega$, $e \in \{0,1\}$ and $\{p_\delta : \delta \in F\}$ s.t.

$$p_\delta \leq p,\ \omega_1\delta \in \mathrm{supp}(p_\delta),\ \text{and } p_\delta \Vdash c_{\omega_1\delta}{\restriction}(\dot{y} \smallsetminus n_0) = e.$$

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$$\exists \delta_0 \in F(B_{\delta_0} \cap (H(p) \cup H(\dot{y})) = \emptyset).$$

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For $b \in U$ there is a good automorphism $h^b : \pi_1 \to \pi_1$ such that $h^b\restriction(\pi_1 \smallsetminus B_{\delta_0})$ is the identity and $h^b(a) = b$.

Proof (cont.)

By the Lemma, $\hat{h}^b_\pi(p) = p$ and $\hat{h}^b_\pi(\dot{y}) = \dot{y}$, so $p'_b := \hat{h}^b_\pi(p_{\delta_0}) \leq p$, $\hat{h}^b_\pi(c_{\omega_1\delta_0}) = \hat{h}^b_\pi(c_a) = c_b$ and

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Since U is uncountable, by ccc there are $b \neq d \in U$ such that p'_b, p'_d are compatible, so some q extends them and

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But $bR_{\delta_0,e}d$, which contradicts that q forces that

$$\begin{array}{rcl} F_q & \to & \{0,1\} \\ u & \mapsto & c_u(k) \end{array}$$

respects $\mathbf{B}_{\delta_0}$ for all $k \geq n_{q(\delta_0)}$.

Applications

Theorem (GKMS)

Assume GCH, $\lambda_1 \leq \ldots \leq \lambda_5$ are successor cardinals, and $\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \lambda_{\mathfrak{s}} \leq \lambda_3$ are regular cardinals.

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Precedent: [GMS16]→[GKS19]→[GKMS20]

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[Kellner & Latif & Tonti 18]→[GKS19]→[GKMS20]×2

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[Kellner & Latif & Shelah 19]$\to$[GKMS20]$\times 2$

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Can we modify our applications to force $\mathfrak{m}(\mathrm{ccc}) > \aleph_1$?