THE NIKODYM PROPERTY AND CARDINAL CHARACTERISTICS OF THE CONTINUUM

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Abstract. We present a general method of constructing Boolean algebras with the Nikodym property and of some given cardinalities. The construction is dependent on the values of some classical cardinal characteristics of the continuum. As a result we obtain a consistent example of an infinite Boolean algebra with the Nikodym property and of cardinality strictly less than the continuum $\mathfrak{c}$. It follows that the existence of such an algebra is undecidable by the usual axioms of set theory. Besides, our results shed some new light on the Efimov problem and cofinalities of Boolean algebras.

1. Introduction

Let $\mathcal{A}$ be a Boolean algebra. A sequence of measures $\langle\mu_n\colon\ n\in\omega\rangle$ on $\mathcal{A}$ is *elementwise bounded* if $\sup_{n\in\omega}\big|\mu_n(a)\big|<\infty$ for every $a\in\mathcal{A}$ and it is *uniformly bounded* if $\sup_{n\in\omega}\big\|\mu_n\big\|<\infty$. The Nikodym Boundedness Theorem states that if $\mathcal{A}$ is σ-complete, then every elementwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded. This principle, due to its numerous applications, is one of the most important results in the theory of vector measures, see Diestel and Uhl [13, Section I.3]. Since σ-completeness is rather a strong property of Boolean algebras, Schachermayer [40] made a detailed study of the theorem and introduced the Nikodym property for general Boolean algebras.

Definition 1.1. A Boolean algebra $\mathcal{A}$ has *the Nikodym property* if every elementwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.

Several classes of Boolean algebras have been shown to have the Nikodym property, e.g. algebras with the following properties: *Interpolation Property* (Seever [42]), *property (E)* (Schachermayer [40]), *Subsequential Completeness Property* (Haydon [26]), *property (f)* (Moltó [35]), *Subsequential Interpolation Property* (Freniche [22]), *Weak Subsequential Completeness Property* (Aizpuru [1]). Schachermayer [40] also proved that the algebra $\mathcal{J}$ of Jordan-measurable subsets of the unit interval $[0,1]$ has the Nikodym property — the result was later generalized by Valdivia [45] to higher finite dimensions.

It is an easy exercise to show that an infinite σ-complete Boolean algebra has cardinality at least continuum $\mathfrak{c}$ — in fact, Comfort and Hager [10] showed that if $\mathcal{A}$ is an infinite σ-complete Boolean algebra, then $|\mathcal{A}|=|\mathcal{A}|^\omega$. Koszmider and Shelah [32] showed that if an infinite Boolean algebra $\mathcal{A}$ has *the Weak Subsequential Separation Property* (*the WSSP*), then $\mathcal{A}$ contains an independent family of cardinality $\mathfrak{c}$ and hence $|\mathcal{A}|\geq\mathfrak{c}$. Since the WSSP is more general than all the mentioned above completeness or interpolation properties of Boolean algebras, it follows that infinite Boolean algebras with those properties must be of cardinality at least $\mathfrak{c}$, too. It is also clear that the Jordan algebra $\mathcal{J}$ has size $2^{\mathfrak{c}}$. Thus all known so far examples

2010 *Mathematics Subject Classification.* Primary: 03E17, 03E35, 28A60. Secondary: 46L05, 46J10, 54A20.

The author was supported by the FWF Grant I 2374-N35.

of infinite Boolean algebras with the Nikodym property are of cardinality at least $\mathfrak{c}$ — a natural question hence arises.

Question 1.2. *Does every infinite Boolean algebra with the Nikodym property have cardinality at least continuum* $\mathfrak{c}$*?*

In this paper we show that it is not always the case.

Theorem 7.3. *Assume that* $\operatorname{cof}(\mathcal{N}) \le \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}\big([\kappa]^\omega\big) = \kappa$*. Then, there exists a Boolean algebra* $\mathcal{B}$ *with the Nikodym property and of cardinality* κ*.* □

Corollary 7.4. *Assuming* $\operatorname{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$*, there exists a Boolean algebra with the Nikodym property and of cardinality* $\omega_1 < \mathfrak{c}$*.* □

The construction in Theorem 7.3 utilizes the idea presented in an unpublished correct version of the paper [9] by Ciesielski and Pawlikowski (the published version contains a gap in the proof of Lemma 3). The prime motivation for studying infinite Boolean algebras with the Nikodym property and cardinality less than $\mathfrak{c}$ was the result of Brech [6] who consistently proved the existence of an infinite compact space K of weight strictly less than $\mathfrak{c}$ and such that the Banach space $C(K)$ has the Grothendieck property. A Banach space X has *the Grothendieck property* if every weak* convergent sequence $\langle x_n^* \in X^*\colon\ n \in \omega\rangle$ of bounded functionals on X is weakly convergent in X^*. The Nikodym property of Boolean algebras and the Grothendieck property of Banach spaces are closely related; see Schachermayer [40] for a detailed discussion.

The assumption about the cofinality of the Lebesgue null ideal in Theorem 7.3 is quite natural, e.g. if κ is an uncountable regular cardinal in a certain model of ZFC, then there exists a ZFC extension of this model in which $\operatorname{cof}(\mathcal{N}) = \kappa < \mathfrak{c}$ is true; see e.g. Mejía [34]. Also, note that $\operatorname{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$ holds e.g. in the Sacks model (see Blass [4, Section 11.5]). For a discussion on those cardinal numbers for which the equality $\operatorname{cof}\big([\kappa]^\omega\big) = \kappa$ is satisfied, see e.g. Bartoszyński and Judah [3, Section 1.3.B]. In particular, note that in ZFC it is true that $\operatorname{cof}\big([\omega_n]^\omega\big) = \omega_n$ for every $n < \omega$ ([3, Lemma 1.3.10]). Note also that in view of the result of Comfort and Hager (stating that a σ-complete algebra $\mathcal{A}$ satisfies the equality $|\mathcal{A}| = |\mathcal{A}|^\omega$), Theorem 7.3 significantly enlarges the class of possible cardinalities of Boolean algebras with the Nikodym property.

Since no countable Boolean algebra has the Nikodym property (see Section 3), Theorem 7.3 implies that the minimal size of an infinite Boolean algebra with the Nikodym property is *a cardinal characteristics (a cardinal invariant) of the continuum*. We thus define the following number.

Definition 1.3. *The Nikodym number* $\mathfrak{n}$ is the least possible cardinality of an infinite Boolean algebra with the Nikodym property, i.e.

$$\mathfrak{n} = \min\big\{|\mathcal{A}|\colon\ \mathcal{A} \text{ is infinite and has the Nikodym property}\big\}.$$

We will study properties of $\mathfrak{n}$ throughout the paper. In particular, we are interested in establishing connections of $\mathfrak{n}$ with selected classical characteristics of the continuum (such as the bounding number $\mathfrak{b}$ or the splitting number $\mathfrak{s}$). This reveals — at least to some extent — the combinatorial nature of the Nikodym property.

Proposition 3.4 states that under Martin's axiom all infinite Boolean algebras with the Nikodym property are of cardinality at least $\mathfrak{c}$, i.e. under Martin's axiom $\mathfrak{n} = \mathfrak{c}$. This — together with Theorem 7.3 — implies the following corollary.

Corollary 7.5. *The existence of an infinite Boolean algebra with the Nikodym property and of cardinality strictly less than* $\mathfrak{c}$ *is independent of* ZFC $+ \neg$CH. □

Our result has several consequences. First of all, it gives an example of a Boolean algebra $\mathcal{A}$ with uncountable cofinality $\operatorname{cof}(\mathcal{A})$ and homomorphism type $h(\mathcal{A})$ and thus generalizes the result of Ciesielski and Pawlikowski [9], see Section 8.1.

Corollary 8.3. *Assume that* $\operatorname{cof}(\mathcal{N}) \leq \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}\left([\kappa]^\omega\right) = \kappa$. *Then, there exists a Boolean algebra* $\mathcal{A}$ *such that* $|\mathcal{A}| = \kappa$, $h(\mathcal{A}) \geq \mathfrak{n}$ *and* $\operatorname{cof}(\mathcal{A}) = \omega_1$. □

The second consequence concerns *the Efimov problem* — a long-standing open problem asking whether every infinite compact space contains either a non-trivial convergent sequence or a copy of $\beta\omega$, the Čech-Stone compactification of the set ω of natural numbers. One can find a detailed discussion on the problem in Hart [25]. So far, there have been known several consistent counterexamples to the problem — called *Efimov spaces*, see e.g. Fedorchuk [19, 20, 21], Dow [15] or Dow and Shelah [17]; however no ZFC counterexample is known. We provide another new consistent example of a Efimov space.

Corollary 8.5. *Assume that* $\operatorname{cof}(\mathcal{N}) \leq \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}\left([\kappa]^\omega\right) = \kappa < \mathfrak{c}$. *Then, there exists a Efimov space* K *such that the weight* $w(K) = \kappa$ *and for every infinite closed subset* L *of* K *we have* $w(L) \geq \mathfrak{n}$. □

The Nikodym property of Boolean algebras can be generalized to C*-algebras in the following way. Let $\mathscr{A}$ be a C*-algebra. Recall that an element $p \in \mathscr{A}$ is called *a projection* if $p^* = p = p^2$, where * is the involution on $\mathscr{A}$, and by $\operatorname{Proj}(\mathscr{A})$ denote the set of all projections in $\mathscr{A}$. We say that a sequence $\left\langle x_n^* \in \mathscr{A}^* \colon\ n \in \omega\right\rangle$ of continuous functionals on $\mathscr{A}$ is *projectionally bounded* if $\sup_{n\in\omega} \left|x_n^*(p)\right| < \infty$ for every $p \in \operatorname{Proj}(\mathscr{A})$ and it is *uniformly bounded* if $\sup_{n\in\omega} \left\|x_n^*\right\| < \infty$.

Definition 1.4. A C*-algebra $\mathscr{A}$ has *the Nikodym property* if every projectionally bounded sequence $\left\langle x_n^* \in \mathscr{A}^* \colon\ n \in \omega\right\rangle$ of continuous functionals on $\mathscr{A}$ is uniformly bounded.

This kind of generalization of the Nikodym property has been studied by several authors, e.g. it was proved that all von Neumann algebras (Darst [11]) and all monotone σ-complete C*-algebras (Brooks and Maitland Wright [8]), which are analogons of σ-complete Boolean algebras, have the Nikodym property. Note that all infinite-dimensional commutative von Neumann algebras have the density at least $\mathfrak{c}$ (see Sakai [38, Remark in Section 1.18]). Taking the space $C\left(K_{\mathcal{B}}\right)$ where $\mathcal{B}$ is the Boolean algebra from Theorem 7.3, we obtain an example of a commutative C*-algebra with the Nikodym property and (almost) arbitrary density.

Corollary 8.12. *Assume that* $\operatorname{cof}(\mathcal{N}) \leq \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}\left([\kappa]^\omega\right) = \kappa$. *Then, there exists a commutative C*-algebra* $\mathscr{A}$ *with the Nikodym property and such that* $\operatorname{dens}(\mathscr{A}) = \kappa$, $h(\mathscr{A}) \geq \mathfrak{n}$ *and* $\operatorname{cof}(\mathscr{A}) = \omega_1$. □

The numbers $\operatorname{cof}(\mathscr{A})$ and $h(\mathscr{A})$, called respectively *the cofinality* and *the homomorphism type* of $\mathscr{A}$, are defined in Section 8.3.

The plan of the paper is as follows. We start with Section 2, where we recall some standard notation and terminology. In Section 3 we provide several lower bounds for cardinalities of infinite Boolean algebras with the Nikodym property (i.e. for the number $\mathfrak{n}$) in terms of standard cardinal invariants of the continuum. In Section 4 we prove some facts concerning sequences of measures on a Boolean algebra which are elementwise bounded but not uniformly bounded (we call them *anti-Nikodym sequences*). In Sections 5–7 we present a consistent construction of an infinite Boolean algebra with the Nikodym property and of cardinality strictly less than $\mathfrak{c}$ as well as we analyze the combinatorics of tools used during the construction in terms of standard cardinal invariants of the continuum. Section 8 provides consequences of the result.

1.1. **Acknowledgements.** The author would like to thank his PhD advisor Professor Piotr Koszmider for valuable comments and inspiring discussions which led him to the final results as well as for the assistance during the preparation of the paper.

2. Notation and terminology

2.1. **Sets and spaces.** The notation we use is standard, see e.g. Bartoszyński and Judah [3] or Jech [27].

Let X be a set. χ_X denotes the characteristic function of X. The power set of X is denoted by $\wp(X)$. $|X|$ denotes the cardinality of X. $[X]^{<\omega}$ and $[X]^\kappa$ for a given cardinal number κ denote respectively the set of all finite subsets of X and the set of all subsets of X of cardinality κ; both $[X]^\omega$ and $[X]^{<\omega}$ are always considered as being ordered by inclusion. *Countable* always means infinite countable, i.e. of cardinality ω. If $X \subseteq Y$ we say that X is *cofinite* (*cocountable*) in Y if $|Y \setminus X| < \omega$ ($|Y \setminus X| = \omega$).

A sequence $\langle x_n \colon\ n \in \omega\rangle$ is *non-trivial* if $\big|\{x_n \colon n \in \omega\}\big| = \omega$. By *an increasing sequence* $f \in \omega^\omega$ we always mean a sequence which is strictly increasing, i.e. $f(n) < f(n+1)$ for every $n \in \omega$. If σ is a finite zero-one sequence, i.e. $\sigma \in \{0,1\}^n$ for some $n \in \omega$, then $[\sigma]$ denotes the clopen subset $\big\{t \in 2^\omega \colon\ \sigma \subseteq t\big\}$ of the Cantor set 2^ω. If $t \in 2^\omega$ and $n \in \omega$, then $t \restriction n$ denotes the finite sequence $\langle t(0), \ldots, t(n-1)\rangle$ of length n. If $n, m \in \omega$ and $t \in \omega^n$, then $t^\frown m$ is a sequence $\langle t(0), \ldots, t(n-1), m\rangle$.

A natural number $n \in \omega$ is usually identified with the set of its predecessors, i.e. $n = \{0, 1, \ldots, n-1\}$, so an expression of the form $X \setminus n$, where X is a set, simply means $X \setminus \{0, \ldots, n-1\}$.

$\mathbb{R}_+$ ($\mathbb{R}_-$) denotes the set of all positive (negative) real numbers and sgn is the signum function on $\mathbb{R}$. If z is a complex number, then $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of z, respectively. We will use frequently the following immediate variant of the triangle inequality for the absolute value: $|a+b| \geq |a| - |b|$ (for every $a, b \in \mathbb{R}$).

We will appeal to the following observation repeatedly implied by the Pigeon Hole Principle.

Lemma 2.1. *For given positive integer N, set $\mathscr{A}$ of cardinality $2N$, two non-empty disjoint sets $\mathscr{B}$ and $\mathscr{C}$, and function $f \colon \mathscr{A} \to \mathscr{B} \cup \mathscr{C}$, there is a subset $\mathscr{A}'$ of $\mathscr{A}$ of cardinality N and a set $\mathscr{D} \in \{\mathscr{B}, \mathscr{C}\}$ such that $f(a) \in \mathscr{D}$ for every $a \in \mathscr{A}'$.*

Generally, for given positive $n, N \in \omega$, set $\mathscr{A}$ of cardinality $2^n \cdot N$, two non-empty disjoint sets $\mathscr{B}$ and $\mathscr{C}$ and functions $f_1, \ldots, f_n \colon \mathscr{A} \to \mathscr{B} \cup \mathscr{C}$, there exist a subset $\mathscr{A}'$ of $\mathscr{A}$ of cardinality N and sets $\mathscr{D}_1, \ldots, \mathscr{D}_n \in \{\mathscr{B}, \mathscr{C}\}$ such that $f_i(a) \in \mathscr{D}_i$ for every $a \in \mathscr{A}'$ and $i \leq n$. $\square$

If X is a topological space, then $\mathrm{dens}(X)$ denotes its density and $w(X)$ its weight. If A is a subset of a topological space $(X, \mathcal{T})$, then we write $\overline{A}$ for the closure of A in $(X, \mathcal{T})$.

A *compact* space is always assumed to be Hausdorff. If K is a compact space, then $C(K)$ denotes the Banach space of all continuous complex-valued functions on K endowed with the supremum norm. Recall that $C(K)$ with a complex conjugation as the involution forms a C*-algebra.

A topological space X is *scattered* if every non-empty closed subset of X has an isolated point. A subset of a topological space is *perfect* if it is closed and has no isolated points. A point x in a compact space X is a *$\mathbb{G}_\delta$-point* if x is of countable character in X.

2.1.1. *Martin's axiom.* Concerning Martin's axioms, we will always use the notation and terminology presented in Bartoszyński and Judah [3, Section 1.4.B].

2.2. **Boolean algebras and measures.** Let $\mathcal{A}$ be a Boolean algebra. By $\wedge$, $\vee$ and $\setminus$ we denote the operations of conjuction, disjunction and difference in $\mathcal{A}$, respectively. The zero and unit element in $\mathcal{A}$ are denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively. The Stone space of $\mathcal{A}$ is always denoted by $K_{\mathcal{A}}$. Recall that

$$|\mathcal{A}| = w\big(K_{\mathcal{A}}\big) = \operatorname{dens}\big(C\big(K_{\mathcal{A}}\big)\big),$$

if $\mathcal{A}$ is infinite. For an element a of $\mathcal{A}$ its corresponding clopen subset of $K_{\mathcal{A}}$ is denoted by $[a]$. For a set I of indices, a family $\big\{a_i \in \mathcal{A}:\ i \in I\big\}$ is *an antichain* in $\mathcal{A}$ if $a_i \wedge a_j = \mathbf{0}$ for every $i \neq j \in I$. In particular, if $\mathcal{A}$ is a subalgebra of $\wp(X)$ for some set X, then an antichain is just a sequence of mutually disjoint subsets of X.

FC denotes the algebra consisting of all finite and cofinite subsets of ω. The countable free Boolean algebra is denoted by $Fr(\omega)$ and its completion by $\overline{Fr(\omega)}$.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A function $\mu\colon \mathcal{A} \to \mathbb{K}$ is called *a measure* on $\mathcal{A}$ if it is a finitely additive function of finite variation, i.e.

$$\|\mu\| = \sup\Big\{\big|\mu\big(a_1\big)\big| + \ldots + \big|\mu\big(a_n\big)\big| \colon a_1, \ldots, a_n \in \mathcal{A}\ \&\ \forall i \neq j\colon a_i \wedge a_j = \mathbf{0}\Big\} < \infty.$$

The space of all measures on $\mathcal{A}$ is denoted as $\operatorname{ba}(\mathcal{A})$. Note that $\operatorname{ba}(\mathcal{A})$ with the variation as a norm constitutes a Banach space. If μ is a measure on $\mathcal{A}$, then its unique Borel extension on $K_{\mathcal{A}}$ is denoted also by μ (see Semadeni [43, Chapter 17 and Section 18.7]). Recall the Riesz representation theorem stating that for a compact space K the dual space $C(K)^*$ is isometrically isomorphic to the space $M(K)$ of all complex Radon (σ-additive) measures on K with finite variation (see Rudin [36, Chapters 2 and 6]). On the other hand, for an algebra $\mathcal{A}$, $M\big(K_{\mathcal{A}}\big)$ is isometrically isomorphic to $\operatorname{ba}(\mathcal{A})$.

We call a measure $\mu \in \operatorname{ba}(\mathcal{A})$ *positive* if $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.

If x is a point in the Stone space $K_{\mathcal{A}}$, then by δ_x we denote the Dirac delta concentrated at x, i.e. $\delta_x(A) = \chi_A(x)$.

We also use the following notations frequently. Given a sequence $\big\langle \mu_n:\ n \in \omega\big\rangle$ of measures, we usually denote it for abbreviation by $\overline{\mu}$. Similarly, given k sequences $\big\langle \mu_n^1:\ n \in \omega\big\rangle, \ldots, \big\langle \mu_n^k:\ n \in \omega\big\rangle$ of measures, we will denote them by $\overline{\mu}^1, \ldots, \overline{\mu}^k$, respectively. If μ is a measure on $\mathcal{A}$ and $a \in \mathcal{A}$, then $\mu \restriction a$ is the measure on $\mathcal{A}$ defined by the formula $(\mu \restriction a)(b) = \mu(a \wedge b)$ for every $b \in \mathcal{A}$. $\overline{\mu} \restriction a$ denotes the sequence $\big\langle \mu_n \restriction a:\ n \in \omega\big\rangle$. Similarly, if $\mathcal{A}'$ is a subalgebra of $\mathcal{A}$, then $\overline{\mu} \restriction \mathcal{A}'$ denotes the sequence $\big\langle \mu_n \restriction \mathcal{A}':\ n \in \omega\big\rangle$ of measures restricted to the algebra $\mathcal{A}'$.

2.3. **Cardinal invariants of the continuum.** In this section we recall some classical cardinal invariants (cardinal characteristics) of the continuum. All the necessary and detailed information concerning them can be found in the monograph of Bartoszyński and Judah [3] and the handbook articles of van Douwen [14] and Blass [4].

2.3.1. *Bounded and dominating families.* Let $f, g \in \omega^\omega$. We say that g *dominates* f if there is $N \in \omega$ such that $f(n) < g(n)$ for every $n > N$. A family $\mathcal{F} \subseteq \omega^\omega$ is *unbounded* if for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ not dominated by g. A family $\mathcal{F} \subseteq \omega^\omega$ is *dominating* if for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ dominating g.

Definition 2.2. *The bounding number* $\mathfrak{b}$ is defined as the minimal size of an unbounded family:

$$\mathfrak{b} = \min\big\{|\mathcal{F}|:\ \mathcal{F} \subseteq \omega^\omega \text{ is unbounded}\big\}.$$

Definition 2.3. *The dominating number* $\mathfrak{d}$ is defined as the minimal size of a dominating family:

$$\mathfrak{d} = \min\big\{|\mathcal{F}|:\ \mathcal{F} \subseteq \omega^\omega \text{ is dominating}\big\}.$$

We will need the following simple lemma asserting existence of dominating families of a special kind.

Lemma 2.4. *There exists a dominating family $\mathcal{D}$ of cardinality $\mathfrak{d}$ consisting of increasing functions such that for every increasing sequence $f = \langle f(n) \in \omega\colon\ n \in \omega\rangle$ there exists $g = \langle g(n) \in \omega\colon\ n \in \omega\rangle \in \mathcal{D}$ dominating f with the following additional property:*

$$\text{if } g(n) \geq f(m) \text{ for some } n, m \in \omega, \text{ then } g(n+1) > f(m+1).$$

Proof. Let $\mathcal{D}_0$ be any dominating family of cardinality $\mathfrak{d}$ consisting of increasing functions $f \in \omega^\omega$ such that $f(0) > 0$. Without loss of generality, we may assume that for every $f \in \omega^\omega$ there is $h \in \mathcal{D}_0$ such that $f(n) < h(n)$ and $f(n+1) < h(n)$ for *every* $n \in \omega$. For every $h \in \mathcal{D}_0$ define $g_h \in \omega^\omega$ as follows:

$$g_h(n) = \underbrace{h \circ \ldots \circ h}_{n+1 \text{ times}}(0),$$

i.e. $g_h(n)$ is the $n+1$ iteration of h at 0. Put:

$$\mathcal{D} = \big\{g_h\colon\ h \in \mathcal{D}_0\big\}.$$

Trivially, $\mathcal{D}$ is dominating and consists of increasing functions. Let $f \in \omega^\omega$ be increasing. Take $h \in \mathcal{D}_0$ such that $f(n) < h(n)$ and $f(n+1) < h(n)$ for every $n \in \omega$. The function g_h dominates f and if $g_h(n) \geq f(m)$ for some $n, m \in \omega$, then

$$g_h(n+1) = h(g_h(n)) \geq h(f(m)) \geq h(m) > f(m+1).$$

$\square$

Corollary 2.5. *There exists a dominating family $\mathcal{D}$ of cardinality $\mathfrak{d}$ consisting of increasing functions such that for every increasing sequence $f = \langle f(n) \in \omega\colon\ n \in \omega\rangle$ there exist $g = \langle g(n) \in \omega\colon\ n \in \omega\rangle \in \mathcal{D}$ dominating f and an increasing sequence $\langle n_k \in \omega\colon\ k \in \omega\rangle$ such that:*

$$\big[f\big(n_{2k}\big), f\big(n_{2k+1}\big)\big] \subseteq \big(g(2k), g(2k+2)\big)$$

for every $k \in \omega$.

Proof. Let $\mathcal{D}$ be the family from Lemma 2.4. Let $f \in \omega^\omega$ be increasing and $g \in \mathcal{D}$ dominating f such that:

$$\text{if } g(n) \geq f(m) \text{ for some } n, m \in \omega, \text{ then } g(n+1) > f(m+1).$$

For every $k \in \omega$ let $m_k \in \omega$ be the smallest number such that $f\big(m_k\big) > g(2k)$. Then, $g(2k+1) > f\big(m_k\big)$ and $g(2k+2) > f\big(m_k+1\big)$. Define the sequence $\langle n_k\colon\ k \in \omega\rangle$ as follows: $n_{2k} = m_k$ and $n_{2k+1} = m_k + 1$. $\square$

2.3.2. *Almost disjoint families.* We say that two sets $A, B \in [\omega]^\omega$ are *almost disjoint* if $|A \cap B| < \omega$. A family $\mathcal{F} \subseteq [\omega]^\omega$ is *almost disjoint* if every two distinct elements of $\mathcal{F}$ are almost disjoint.

As the following standard lemma shows, uncountable almost disjoint families can be useful to ensure partial σ-additivity of measures on a Boolean algebra.

Lemma 2.6. *Let $\mathcal{G} = \big\{A_\xi\colon\ \xi < \omega_1\big\}$ be a family of infinite almost disjoint subsets of ω and let $\langle a_n\colon\ n \in \omega\rangle$ be an antichain in some Boolean algebra $\mathcal{A}$. Assume that for every $\xi < \omega_1$ the supremum $\bigvee_{n \in A_\xi} a_n$ belongs to $\mathcal{A}$. Then, for every sequence $\overline{\mu} = \langle \mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}$ there exists $\xi \in \omega_1$ such that for every $k \in \omega$ and $B \subseteq A_\xi$ for which the supremum $\bigvee_{n \in B} a_n$ belongs to $\mathcal{A}$ the following equality holds:*

$$\big|\mu_k\big|\Big(\bigvee_{n \in B} a_n\Big) = \sum_{n \in B} \big|\mu_k\big|\big(a_n\big).$$

Proof. For every $\xi < \omega_1$ let C_ξ be a closed subset of $K_{\mathcal{A}}$ given by the formula

$$C_\xi = \left[\bigvee_{n\in A_\xi} a_n\right] \setminus \bigcup_{n\in A_\xi} [a_n],$$

i.e. C_ξ is a boundary of $\bigcup_{n\in A_\xi}[a_n]$ in $K_{\mathcal{A}}$. It is easy to see that $C_\xi \cap C_\eta = \emptyset$ whenever $\xi \neq \eta$. Hence, for every $k \in \omega$ by the finiteness of the measure $\left|\mu_k\right|$ there exists a cocountable set $F_k \subseteq \omega_1$ such that and $\left|\mu_k\right|\left(C_\xi\right) = 0$ for every $\xi \in F_k$. Let $\xi \in F_k$ and $B \subseteq A_\xi$ be a subset for which $\bigvee_{n\in B} a_n \in \mathcal{A}$. The boundary C_B of $\left[\bigvee_{n\in B} a_n\right]$ in $K_{\mathcal{A}}$ is a subset of C_ξ, hence $\left|\mu_k\right|\left(C_B\right) = 0$ (recall that we identify $\mu_k \in \mathrm{ba}(\mathcal{A})$ with the corresponding measure $\mu_k \in M\left(K_{\mathcal{A}}\right)$). Thus we have:

$$\left|\mu_k\right|\Big(\bigvee_{n\in B} a_n\Big) = \left|\mu_k\right|\Big(\Big[\bigvee_{n\in B} a_n\Big]\Big) = \left|\mu_k\right|\Big(C_B \cup \bigcup_{n\in B}[a_n]\Big) =$$

$$\left|\mu_k\right|\left(C_B\right) + \left|\mu_k\right|\Big(\bigcup_{n\in B}[a_n]\Big) = \sum_{n\in B}\left|\mu_k\right|\left([a_n]\right) = \sum_{n\in B}\left|\mu_k\right|\left(a_n\right).$$

The set $F = \bigcap_{k\in\omega} F_k$ is non-empty (in fact it is also cocountable in ω_1). Take any $\xi \in F$. □

2.3.3. *Splitting families.* Given two sets $A, B \in [\omega]^\omega$ we say that B *splits* A if $|A \cap B| = |A \setminus B| = \omega$. A family $\mathcal{F} \subseteq [\omega]^\omega$ is *splitting* if for every $A \in [\omega]^\omega$ there is $B \in \mathcal{F}$ splitting A.

Definition 2.7. *The splitting number* $\mathfrak{s}$ is defined as the minimal size of a splitting family:

$$\mathfrak{s} = \min\left\{|\mathcal{F}|:\ \mathcal{F} \subseteq [\omega]^\omega \text{ is splitting}\right\}.$$

2.3.4. *Cichoń's diagram.* All the necessary information concerning Cichoń's diagram can be found in the monograph of Bartoszyński and Judah [3, Chapter 2]. In this section we only recall two of its elements.

Definition 2.8. Let $\mathcal{M}$ and $\mathcal{N}$ denote the σ-ideals of all meager and all zero Lebesgue measure subsets of the real line $\mathbb{R}$, respectively.

- *The cofinality of measure* $\mathrm{cof}(\mathcal{N})$ is the minimal size of a base of $\mathcal{N}$:

  $$\mathrm{cof}(\mathcal{N}) = \min\left\{|\mathcal{F}|:\ \mathcal{F} \subseteq \mathcal{N}\ \&\ \forall A \in \mathcal{N}\ \exists B \in \mathcal{F}:\ A \subseteq B\right\}.$$

- *The covering of category* $\mathrm{cov}(\mathcal{M})$ is the minimal size of a covering of $\mathbb{R}$ by meager subsets:

  $$\mathrm{cov}(\mathcal{M}) = \min\left\{|\mathcal{F}|:\ \mathcal{F} \subseteq \mathcal{M}\ \&\ \bigcup\mathcal{F} = \mathbb{R}\right\}.$$

We will need the following characterization of $\mathrm{cov}(\mathcal{M})$ due to Keremedis (cf. also Bartoszyński and Judah [3, Theorem 2.4.5]). Recall that MA_κ(*countable*) denotes Martin's axiom restricted to countable posets and at most κ many dense subsets.

Proposition 2.9 (Keremedis [30, Theorem 1])**.** *Let κ be a cardinal number. Then,* $\mathrm{cov}(\mathcal{M}) > \kappa$ *if and only if* MA_κ(*countable*) *holds.* □

Finally, let us invoke the following characterization of $\mathrm{cof}(\mathcal{N})$ due to Bartoszyński and Judah which appears crucial in the proofs of Lemma 5.1.

Proposition 2.10 (Bartoszyński–Judah [3, Section 2.3.A])**.** *Let $\mathcal{C}$ denote the family of all subsets of ω^ω of the form $\prod_{n\in\omega} T_n$ such that $T_n \in [\omega]^{n+1}$ for all $n \in \omega$. Then*

$$\mathrm{cof}(\mathcal{N}) = \min\left\{|\mathcal{F}|:\ \mathcal{F} \subseteq \mathcal{C}\ \&\ \bigcup\mathcal{F} = \omega^\omega\right\}.$$

Note that the definition of the family $\mathcal{C}$ in Bartoszyński and Judah [3, Definition 2.3.2] is a bit different — it considers sets of the form $T = \prod_{n\in\omega} T_n$ satisfying the condition: $\sum_{n\in\omega} \frac{|T_n|}{n^2} < \infty$. However, it is a folklore fact that we can use sets $T = \prod_{n\in\omega} T_n$ where $T_n \in [\omega]^{n+1}$, cf. Ciesielski and Pawlikowski [9], Gruenhage and Levy [24, Theorem 2.10] or Sobota [44, Proposition 2.2.26].

3. The Nikodym property and cardinal invariants

Since, by the Nikodym theorem, the algebra $\wp(\omega)$ has the Nikodym property, we immediately have the following upper bound: $\mathfrak{n} \leq \mathfrak{c}$. To see that $\mathfrak{n} > \omega$, i.e. no countable Boolean algebra has the Nikodym property, recall the following well-known facts: 1) if the Stone space $K_\mathcal{A}$ of a Boolean algebra $\mathcal{A}$ has a non-trivial convergent sequence, than $\mathcal{A}$ cannot have the Nikodym property (indeed, if $\langle x_n\colon\ n \in \omega\rangle$ is a non-trivial sequence in $K_\mathcal{A}$ converging to x, then the sequence of measures $\langle n(\delta_{x_n} - \delta_x)\colon\ n \in \omega\rangle$ is elementwise bounded but not uniformly bounded), 2) the Stone space of a countable Boolean algebra is homeomorphic to an infinite closed subset of the Cantor set 2^ω, hence it contains a non-trivial convergent sequence.

We generalize the above fact as follows (see also Sobota [44, Section 2.3]).

Proposition 3.1. $\mathfrak{n} \geq \max(\mathfrak{s}, \mathrm{cov}(\mathcal{M}))$. □

Proof. Let $K_\mathcal{A}$ be the Stone space of an infinite Boolean algebra $\mathcal{A}$ with the Nikodym property. Then, $K_\mathcal{A}$ does not contain any non-trivial convergent sequences and hence is not scattered.

Booth [5] proved that the splitting number $\mathfrak{s}$ is equal to the minimal weight of an infinite compact space which is not sequentially compact. Since every infinite sequentially compact space contains a non-trivial convergent sequence, it follows that $K_\mathcal{A}$ is not sequentially compact and hence $w\big(K_\mathcal{A}\big) \geq \mathfrak{s}$.

Geschke [23, Theorem 2.1] showed that if K is an infinite non-scattered compact space such that $w(K) < \mathrm{cov}(\mathcal{M})$, then K contains a perfect set with a $\mathbb{G}_\delta$-point (in the relative topology) and hence K contains a non-trivial convergent sequence. This implies that $w\big(K_\mathcal{A}\big) \geq \mathrm{cov}(\mathcal{M})$. □

Proposition 3.2. $\mathfrak{n} \geq \mathfrak{b}$.

Proof. Recall that a locally convex space X is *barrelled* if every convex, balanced, absorbing and closed subset (*a barrel*) of X is a neighborhood of zero in X (see Kelley and Namioka [29, pp. 104–105]). Saxon and Sánchez Ruiz [39] proved that every barrelled metrizable space has dimension at least $\mathfrak{b}$. On the other hand, it follows from Schaefer [41, Section IV.1 and Theorem IV.5.2] that a Boolean algebra $\mathcal{A}$ has the Nikodym property if and only if the normed space $B_s(\mathcal{A})$ (with the supremum norm) of all simple functions on $K_\mathcal{A}$ is barrelled. The (algebraic) dimension of $B_s(\mathcal{A})$ is equal to $|\mathcal{A}|$. □

(Remark: In Section 8.3 we present Proposition 8.8 which is a version of Proposition 3.2 for C*-algebras.)

Corollary 3.3. $\mathfrak{n} \geq \max\big(\mathfrak{s}, \mathfrak{b}, \mathrm{cov}(\mathcal{M})\big)$. □

Note that values of the cardinal invariants $\mathfrak{s}$, $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{M})$ are independent of each other (see e.g. Mejía [34, Section 4] and Brendle and Fischer [7, Section 4]).

Proposition 2.9 implies the following corollary.

Corollary 3.4. *Under* MA_κ*(countable), every infinite Boolean algebra with the Nikodym property is of cardinality at least* $\mathfrak{c}$. □

A natural candidate for bounding $\mathfrak{n}$ from below is the dominating number $\mathfrak{d}$, for which we also have $\mathfrak{d} \geq \max\big(\mathfrak{s}, \mathfrak{b}, \operatorname{cov}(\mathcal{M})\big)$.

Question 3.5. *Is the equality* $\mathfrak{n} = \mathfrak{d}$ *true?*

Note that the inequality $\mathfrak{d} > \max\big(\mathfrak{s}, \mathfrak{b}, \operatorname{cov}(\mathcal{M})\big)$ is relatively consistent, see Blass [4, Section 11.9] or Mejía [33, Theorem 13].

Let us conclude the section with the following remarks.

Remark 3.6. Proposition 8.2 asserts that the cofinality of $\mathfrak{n}$ is uncountable. However, note that adding one Cohen real over a model which contains a given algebra, adds a convergent sequence to its Stone space (see Dow and Fremlin [16, Introduction]), so as every algebra of size smaller than $\mathfrak{c}$ has a Cohen real over it in the Cohen model, we conclude that $\mathfrak{n} = \mathfrak{c}$ in this model. On the other hand, note that in this model the cofinality $\operatorname{cf}(\mathfrak{c}) = \operatorname{cf}(\mathfrak{n})$ may be an arbitrary uncountable cardinal number κ.

Corollary 3.7. *The regularity of* $\mathfrak{n}$ *is undecidable in* ZFC. □

Remark 3.8. It follows that for every infinite Boolean algebra $\mathcal{A}$ with the Nikodym property there exists a homomorphism $\varphi : \mathcal{A} \to \overline{Fr(\omega)}$ such that $Fr(\omega) \subseteq \varphi[\mathcal{A}] \subseteq \overline{Fr(\omega)}$. Indeed, since $K_{\mathcal{A}}$ is not scattered, $\mathcal{A}$ is not superatomic and hence $Fr(\omega)$ is a subalgebra of $\mathcal{A}$. The existence of the homomorphism φ follows from the Sikorski Extension Theorem.

Schachermayer proved that the class of Boolean algebras with the Nikodym property is closed under homomorphic images ([40, Proposition 2.11]), hence $\varphi[\mathcal{A}]$ has also the Nikodym property. This implies that to seek an algebra with the Nikodym property and of cardinality $\mathfrak{n}$, it is enough to study subalgebras of $\overline{Fr(\omega)}$ containing $Fr(\omega)$. This also suggests that Boolean algebras with the Nikodym property are in a sense "complete" (cf. Theorem 6.9).

4. Anti-Nikodym sequences of measures

Let $\mathcal{A}$ be a Boolean algebra.

Definition 4.1. Let $\overline{\mu} = \langle \mu_n \colon n \in \omega \rangle$ be a sequence of measures on $\mathcal{A}$ and let $a \in \mathcal{A}$. We say that *a is big for $\overline{\mu}$* if for every $\rho > 0$ there exist $b \leq a$ and $n \in \omega$ such that $\big|\mu_n(b)\big| > \rho$.

The following proposition follows immediately from the definition of big elements.

Proposition 4.2. *For a given sequence $\overline{\mu} = \langle \mu_n \colon n \in \omega \rangle$ of measures on $\mathcal{A}$ and an element $a \in \mathcal{A}$, the following are equivalent:*

(1) a is big for $\overline{\mu}$;
(2) $\sup_{n\in\omega} \sup_{b\leq a} \big|\mu_n(b)\big| = \infty$;
(3) $\overline{\mu} \restriction a$ is not uniformly bounded, i.e. $\sup_{n\in\omega} \big\|\mu_n \restriction a\big\| = \infty$;
(4) every $b \in \mathcal{A}$ such that $b \geq a$ is big for $\overline{\mu}$. □

Definition 4.3. A sequence $\overline{\mu}$ of measures on $\mathcal{A}$ is called *anti-Nikodym on $a \in \mathcal{A}$*, if $\overline{\mu} \restriction a$ is elementwise bounded and a is big for $\overline{\mu}$. We say simply that $\overline{\mu}$ is *anti-Nikodym* if it is anti-Nikodym on $\mathbf{1}$.

Lemma 4.4. *If a sequence $\overline{\mu} = \langle \mu_n \colon n \in \omega \rangle$ of measures on $\mathcal{A}$ is anti-Nikodym on some element $a \in \mathcal{A}$, then there exists a point $t \in [a]$ such that $\overline{\mu}$ is anti-Nikodym on every neighborhood of t in $K_{\mathcal{A}}$.*

The point t will be called *a Nikodym concentration point of the sequence $\overline{\mu}$.*

Proof. Assume that for every point $t \in [a]$ there exists a neighborhood $[a_t] \subseteq K_{\mathcal{A}}$ of t such that $\overline{\mu} \restriction a_t$ is uniformly bounded. Then, by compactness of $[a]$ there exist $t_1, \ldots, t_n \in [a]$ such that $a \leq a_{t_1} \vee \ldots \vee a_{t_m}$. This in turn implies that

$$\sup_{n\in\omega} \left\|\mu_n \restriction a\right\| = \sup_{n\in\omega} \left|\mu_n\right|(a) \leq \sup_{n\in\omega} \left|\mu_n\right|\left(a_{t_1}\right) + \ldots + \sup_{n\in\omega} \left|\mu_n\right|\left(a_{t_m}\right) =$$

$$\sup_{n\in\omega} \left\|\mu_n \restriction a_{t_1}\right\| + \ldots + \sup_{n\in\omega} \left\|\mu_n \restriction a_{t_m}\right\| < \infty,$$

which is a contradiction with Proposition 4.2 (3), since a is big for $\overline{\mu}$. □

(Note that in the above proof we did not used the elementwise boundedness of $\overline{\mu}$.)

Let $\overline{\mu}$ be an anti-Nikodym sequence of measures on $\mathcal{A}$ and $t \in K_{\mathcal{A}}$ be its Nikodym concentration point. It is immediate that t cannot be isolated in $K_{\mathcal{A}}$. Aizpuru [2, page 4] observed even more — if $K_{\mathcal{A}}$ does not contain any non-trivial convergent sequences, then t cannot be a P-point in $K_{\mathcal{A}}$. Recall that a point x in a topological space X is *a P-point* if its filter of neighborhoods is closed under countable intersection, i.e. if $\langle U_n\colon\ n \in \omega\rangle$ is a sequence of open neighborhoods of x, then $\bigcap_{n\in\omega} U_n$ has the non-empty interior. Recall also that *the support* $\operatorname{supp}(\mu)$ of a measure $\mu \in C\left(K_{\mathcal{A}}\right)^*$ is the smallest closed subset of K such that $|\mu|(U) = 0$ for every open set U disjoint with $\operatorname{supp}(\mu)$. The next proposition shows that the assumption about the non-existence of non-trivial convergent sequences in $K_{\mathcal{A}}$ is redundant in Aizpuru's result.

Proposition 4.5. *Let $\overline{\mu} = \langle\mu_n\colon\ n \in \omega\rangle$ be an anti-Nikodym sequence of measures on $\mathcal{A}$ and $t \in K_{\mathcal{A}}$ be its Nikodym concentration point. Then, t is not a P-point in $K_{\mathcal{A}}$.*

Proof. First note that if $x \in K_{\mathcal{A}}$ is a P-point and $Y \subseteq K_{\mathcal{A}} \setminus \{x\}$ satisfies the countable chain condition, then $x \notin \overline{Y}$. Indeed, let $\mathcal{U}$ be any maximal family of pairwise disjoint open sets in $K_{\mathcal{A}}$ such that $x \notin \overline{U}$ and $U \cap Y \neq \emptyset$ for every $U \in \mathcal{U}$. Then, $|\mathcal{U}| \leq \omega$ and hence there exists an open neighborhood V of x in $K_{\mathcal{A}}$ such that $V \cap \overline{U} = \emptyset$ for every $U \in \mathcal{U}$. By maximality of $\mathcal{U}$ it follows that $V \cap Y = \emptyset$, and so $x \notin \overline{Y}$.

Recall that for every measure $\mu \in C\left(K_{\mathcal{A}}\right)^*$ its support $\operatorname{supp}(\mu)$ satisfies the countable chain condition. Hence, it is easy to see that the set

$$Y = \bigcup_{n\in\omega} \operatorname{supp}(\mu_n) \setminus \{t\}$$

also satisfies this condition.

Assume that t is a P-point in $K_{\mathcal{A}}$. Then, there exists an open neighborhood V of t in $K_{\mathcal{A}}$ such that $V \cap Y = \emptyset$. It follows that for every $n \in \omega$ we have that

$$\mu_n \restriction V = \mu_n \restriction \{t\} = \alpha_n \delta_t$$

for some $\alpha_n \in \mathbb{C}$. Since $\overline{\mu}$ is anti-Nikodym, the set $\left\{\alpha_n\ :\ n \in \omega\right\}$ is bounded, and hence $\sup_{n\in\omega} \left\|\mu_n \restriction V\right\| < \infty$, which contradicts the fact that t is a Nikodym concentration point of $\overline{\mu}$. □

Let $T(\overline{\mu})$ be the set of all Nikodym concentration points of an anti-Nikodym sequence $\overline{\mu}$. It is immediate that $T(\overline{\mu})$ is closed in $K_{\mathcal{A}}$. The next proposition shows that in the case of the free countable Boolean algebra $Fr(\omega)$, $T(\overline{\mu})$ can be in fact *any* closed subset of its Stone space 2^ω.

Proposition 4.6. *Let $F \subseteq 2^\omega$ be a closed set. Then, there exists an anti-Nikodym sequence $\overline{\mu}$ of measures on $Fr(\omega)$ for which $T(\overline{\mu}) = F$.*

Proof. We first consider the case when F is perfect. We shall define an anti-Nikodym sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures such that $\operatorname{supp}(\mu_n) \subseteq F$ for every $n \in \omega$. Thus, since F is homeomorphic to 2^ω, without any loss of generality we may assume that $F = 2^\omega$. Denote $\mathcal{A} = Fr(\omega)$.

Define an anti-Nikodym sequence $\overline{\mu} = \langle \mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}$ as follows. Fix $n \in \omega$. For $\sigma \in 2^n$ let $x(\sigma)$ and $y(\sigma)$ be two arbitrary distinct points from the clopen $[\sigma]$. Put:

$$\mu_n = \sum_{\sigma\in 2^n} \big(\delta_{x(\sigma)} - \delta_{y(\sigma)}\big).$$

Then $\|\mu_n\| = 2^{n+1}$. On the other hand, if $a \in \mathcal{A}$, then by compactness of $[a]$ there is $N \in \omega$ such that $x(\sigma) \in [a]$ if and only if $y(\sigma) \in [a]$, for every $n > N$ and $\sigma \in 2^n$; so $\mu_n(a) = 0$. Hence, $\overline{\mu}$ is anti-Nikodym.

Let $t \in 2^\omega$ and assume that $t \in [\sigma]$ for some $\sigma \in 2^{<\omega}$. For every $\tau \in 2^{<\omega}$ we have $x(\sigma^\frown\tau), y(\sigma^\frown\tau) \in [\sigma]$, hence:

$$\sup_{n\in\omega} |\mu_n|\big(\,[\sigma]\,\big) \geq 2^{|\tau|+1}.$$

Thus, $\sup_{n\in\omega} \big\|\mu_n \restriction [\sigma]\,\big\| = \infty$, and hence t is a Nikodym concentration point of $\overline{\mu}$.

Let now $F = P \cup D$, where P is perfect, D is scattered and $P \cap D = \emptyset$. Let $\alpha = |D|$; then $\alpha \leq \omega$. Let $\langle N_\xi \in [\omega]^\omega :\ \xi \leq \alpha\rangle$ be a partition of ω. For every $\xi \leq \alpha$, enumerate $N_\xi = \langle n(\xi,k)\colon\ k \in \omega\rangle$. Write also $D = \langle x_\xi\colon\ \xi < \alpha\rangle$. For every $\xi < \alpha$, let $\langle x^\xi_n \in 2^\omega\colon\ n \in \omega\rangle$ be a non-trivial sequence converging to x_ξ such that the diameter $\operatorname{diam}\{x^\xi_n\colon\ n \in \omega\} < 1/2^{\xi+1}$; then $\lim_{k\to\infty} x^\xi_{n(\xi,k)} = x_\xi$, too. For every $k \in \omega$ and $\xi < \alpha$ put:

$$\mu_{n(\xi,k)} = n(\xi,k)\cdot\Big(\delta_{x^\xi_{n(\xi,k)}} - \delta_{x_\xi}\Big).$$

Then, x_ξ is a Nikodym concentration point of the anti-Nikodym sequence $\langle \mu_{n(\xi,k)}\colon\ k \in \omega\rangle$.

Let $\overline{\nu} = \langle \nu_n\colon\ n \in \omega\rangle$ be an anti-Nikodym sequence such that $T(\overline{\nu}) = P$ and $\operatorname{supp}(\nu_n) \subseteq P$ for every $n \in \omega$, e.g. the one described above. For every $k \in \omega$ put:

$$\mu_{n(\alpha,k)} = \nu_{n(\alpha,k)}.$$

Then, $T\big(\langle \mu_{n(\alpha,k)}\colon\ k \in \omega\rangle\big) = P$.

Finally, since the diameters of the sequences $\langle x^\xi_n\colon\ n \in \omega\rangle$ converge to 0 and the Hausdorff distance between two disjoint non-empty clopen subsets of 2^ω is always positive, $\overline{\mu} = \langle \mu_n\colon\ n \in \omega\rangle$ is elementwise bounded and hence anti-Nikodym. Obviously, $T(\overline{\mu}) = P \cup D = F$. $\square$

Lemma 4.7. *Let $\overline{\mu} = \langle \mu_n\colon\ n \in \omega\rangle$ be an anti-Nikodym sequence on some $a \in \mathcal{A}$ and let $t \in [a]$ be a Nikodym concentration point of $\overline{\mu}$. Then, for every positive real number ρ there exists an element $b \in \mathcal{A}$ with the following properties:*

- *$b \leq a$ and $t \in [a \setminus b]$,*
- *there exists $n \in \omega$ such that $|\mu_n(b)| > \rho$.*

Proof. Since $\overline{\mu}$ is anti-Nikodym on a, there exist $c \leq a$ and $n \in \omega$ such that

$$|\mu_n(c)| > \sup_{m\in\omega} |\mu_m(a)| + \rho$$

and hence

$$|\mu_n(a \setminus c)| = |\mu_n(c) - \mu_n(a)| \geq |\mu_n(c)| - |\mu_n(a)| > \rho.$$

If $t \in [c]$, then put $b = a \setminus c$, otherwise put $b = c$. $\square$

The last lemma of this section is a simple application of the triangle inequality.

Lemma 4.8. *Let μ be a measure on $\mathcal{A}$ and let $a \in \mathcal{A}$ be such an element that $\big|\mu(a)\big| > \rho$ for some positive number ρ. If $a_1, \ldots, a_n$ are such mutually disjoint elements of $\mathcal{A}$ that $a = a_1 \vee \ldots \vee a_n$, then there is $1 \leq j \leq n$ for which $\big|\mu\big(a_j\big)\big| > \rho/n$.* □

5. $\operatorname{cof}(\mathcal{N})$ and antichains

The main motivation for this section is following. A crucial step in Darst's *reductio ad absurdum* proof of the Nikodym Boundedness Theorem (see Diestel and Uhl [13, Theorem I.3.1 and page 33]) is for a given anti-Nikodym sequence $\langle\mu_n\colon\ n \in \omega\rangle$ of measures on a σ-complete Boolean algebra $\mathcal{A}$ to construct an antichain $\langle a_n \in \mathcal{A}\colon\ n \in \omega\rangle$ and an increasing sequence $\langle n(k)\colon\ k \in \omega\rangle$ of natural numbers such that for every $k \in \omega$ the following inequality holds:

$$\big|\mu_{n(k)}\big(a_k\big)\big| > \sum_{i=0}^{k-1}\big|\mu_{n(k)}\big(a_i\big)\big| + k + 1.$$

The existence of such an antichain allows us to obtain an element $a \in \mathcal{A}$ for which $\sup_{n\in\omega}\big|\mu_n(a)\big| = \infty$, contradicting the fact that $\overline{\mu}$ is elementwise bounded.

If $\overline{\mu}$ is anti-Nikodym on a Boolean algebra $\mathcal{A}$, then to construct an antichain described in the previous paragraph it is enough to use Lemma 4.7 — no special properties of $\mathcal{A}$ are required. The main result of this section — Lemma 5.1 — states that in the case of the free countable Boolean algebra $Fr(\omega)$, the antichain can be chosen from a certain fixed family of $\operatorname{cof}(\mathcal{N})$ many antichains. This observation will appear crucial for the proof of the main theorems of this paper — Theorems 7.2 and 7.3.

Lemma 5.1. *If $\operatorname{cof}(\mathcal{N}) = \kappa$ for some cardinal number κ, then there exists a family $\big\{\langle a_n^\gamma \in Fr(\omega)\colon\ n \in \omega\rangle\colon\ \gamma < \kappa\big\}$ of κ many antichains in the free countable Boolean algebra $Fr(\omega)$ with the following property:*

> *for every anti-Nikodym sequence of real-valued measures $\langle\mu_n\colon\ n \in \omega\rangle$ on $Fr(\omega)$ there exist $\gamma < \kappa$ and an increasing sequence $\langle n(k)\colon\ k \in \omega\rangle$ of natural numbers such that for every $k \in \omega$ the following inequality is satisfied:*

$$\big|\mu_{n(k)}\big(a_k^\gamma\big)\big| > \sum_{i=0}^{k-1}\big|\mu_{n(k)}\big(a_i^\gamma\big)\big| + k + 1.$$

Before we provide the proof of Lemma 5.1, we will prove several auxiliary technical results. To the end of the section we shall need the following four sequences $K, L, M, N \in \omega^\omega$:

- $K(n) = (2^n \cdot 3)^n$,
- $L(n) = K(n)^{K(n)\cdot 2\cdot 3^{n+1}}$,
- $N(n) = L(0) + L(1) + \ldots + L(n) + n + 1$,
- $M(n) = \big(2^{3^n}\big)^{n+1} \cdot 2^{n+1}$.

Note that in the definition of $N(n)$ the last summand is just $n + 1$, not $L(n + 1)$.

Lemma 5.2. *Let $\mathcal{A}$ be a Boolean algebra. Fix an integer $k \geq 3$. Let $d_1, \ldots, d_k, e \in \mathcal{A}$ be mutually disjoint and let μ be a real-valued measure on $\mathcal{A}$. Assume that ρ is such positive real number that $\big|\mu(e)\big| > 2\rho$ and $\big|\mu\left(e \vee d_j\right)\big| \leq \rho$ for every $j = 1, \ldots, k$. Then:*

$$\Big|\mu\Big(e \vee \bigvee_{j=1}^{k} d_j\Big)\Big| > \rho.$$

Proof. Since for every $j = 1, \ldots, k$ we have that $\big|\mu\left(e \vee d_j\right)\big| \le \rho$, it follows that:

$$\big|\mu(d_j)\big| = \big|\mu(e) - \mu(e \vee d_j)\big| \ge \big|\mu(e)\big| - \big|\mu(e \vee d_j)\big| > 2\rho - \rho = \rho$$

and

$$\Big|\big|\mu(d_k)\big| - \big|\mu(e)\big|\Big| \le \big|\mu(d_k) + \mu(e)\big| = \big|\mu(e \vee d_k)\big| \le \rho,$$

so

$$\big|\mu(d_k)\big| - \big|\mu(e)\big| \ge -\rho.$$

Moreover, since $\big|\mu(e)\big| > 2\rho$ and $\big|\mu\left(e \vee d_j\right)\big| \le \rho$, the sign of every $\mu(d_j)$ must be opposite to the sign of $\mu(e)$, i.e. there exists $s \in \{-1, +1\}$ such that

$$\operatorname{sgn} \mu(d_j) = s = -\operatorname{sgn} \mu(e)$$

for every $j = 1, \ldots, k$. Having in mind that $k \ge 3$, we immediately obtain the following:

$$\Big|\mu\Big(e \vee \bigvee_{j=1}^{k} d_j\Big)\Big| = \Big|\mu(e) + \sum_{j=1}^{k} \mu(d_j)\Big| \ge \Big|\sum_{j=1}^{k} \mu(d_j)\Big| - \big|\mu(e)\big| =$$

$$\sum_{j=1}^{k-1} \big|\mu(d_j)\big| + \Big(\big|\mu(d_k)\big| - \big|\mu(e)\big|\Big) > (k-1)\rho - \rho = (k-2)\rho \ge \rho,$$

where the middle equality follows from the fact that all $\mu(d_j)$'s have the same sign s. $\square$

Lemma 5.3. *Let $\mathcal{A}$ be a Boolean algebra. Fix $k \in \omega$. Let $d_1, \ldots, d_{K(k)}, e \in \mathcal{A}$ be mutually disjoint and let $\mu_1, \ldots, \mu_k$ be real-valued measures on $\mathcal{A}$. Assume that $\rho_1, \ldots, \rho_k$ are such positive real numbers that $\big|\mu_i(e)\big| > 2\rho_i$ for every $i \le k$. Then, there exists a non-empty set $E \subseteq \big\{1, \ldots, K(k)\big\}$ satisfying for every $i \le k$ the following inequality:*

$$\Big|\mu_i\Big(e \vee \bigvee_{j \in E} d_j\Big)\Big| > \rho_i.$$

The cardinality of the set E is a power of 3, i.e. $|E| = 3^{k'}$ for some $0 \le k' \le k$.

Proof. Let $A = \big\{\mu_1, \ldots, \mu_k\big\}$ and $J = \big\{1, \ldots, K(k)\big\}$.

Let us say that a measure $\mu_i \in A$ is *good* for a set $E \subseteq J$ if:

$$\Big|\mu_i\Big(e \vee \bigvee_{j \in E} d_j\Big)\Big| > \rho_i,$$

and *bad* otherwise, i.e.

$$\Big|\mu_i\Big(e \vee \bigvee_{j \in E} d_j\Big)\Big| \le \rho_i.$$

Then, our aim is to find a non-empty set $E \subseteq J$ such that every measure $\mu_i \in A$ is good for E.

We will do it in at most $k+1$ steps. More precisely, we will construct the following two sequences of length m for some $1 \le m \le k+1$:

(i) a sequence $J_1, \ldots, J_m \subseteq \wp(J)$ of families of *candidates* for the set E such that:
  - (i.1) $|J_l| = (2^k)^{k-l} \cdot 3^{k-l+1}$ for every $1 \le l < m$,
  - (i.2) sets from J_{l+1} will be unions of three distinct sets from J_l for every $1 \le l < m$;

(ii) a sequence $B_1, \ldots, B_m \subseteq A$ of disjoint sets of measures satisfying the following properties:

(ii.1) if $1 \le l < m$, then each $\mu_i \in B_l$ will be bad for every $E \in J_l$,
(ii.2) if $1 \le l \le m$, then each $\mu_i \in A \setminus \bigcup_{r=1}^{l} B_r$ will be good for every $E \in J_l$,
(ii.3) the set B_m will be the only empty set in the sequence $B_1, \ldots, B_m$, i.e. there will be no bad measures for any $E \in J_m$.

Let $l < m$. Notice that it may happen that a measure $\mu_i \in A$ is good for every $E \in J_l$ and simultaneously bad for every $E \in J_{l+1}$. However, due to Lemma 5.2 if μ_i is bad for every $E \in J_l$, then it is good for every $E \in J_{l'}$ where $l < l' \le m$. Thus, as the set E we can (and we will!) take any element of J_m.

We start as follows. Let $J' = \big\{\{j\}\colon\ j \in J\big\}$. By Lemma 2.1 (with $n = k$, $\mathscr{A} = J'$, $\mathscr{B} = \{0\}$, $\mathscr{C} = \{1\}$ and f_i defined as follows: $f_i\big(\{j\}\big) = 0$ if μ_i is good for $\{j\}$, $f_i\big(\{j\}\big) = 1$ otherwise), there exists $J_1 \subseteq J'$ of cardinality $K(k)/2^k = (2^k)^{k-1} \cdot 3^k$ such that the following sets

$$X = \big\{\mu_i \in A\colon\ \mu_i \text{ is good for every } \{j\} \in J_1\big\} \quad \text{and}$$

$$Y = \big\{\mu_i \in A\colon\ \mu_i \text{ is bad for every } \{j\} \in J_1\big\}$$

constitute a partition of A (i.e. are disjoint and their union is A). Put $B_1 = Y$.

Let us now assume that for some $1 \le l \le k$ we have found the sets $J_1, \ldots, J_l$ with the properties (i.1)-(i.2) and the sets $B_1, \ldots, B_l$ with the properties (ii.1)-(ii.3).

If $B_l = \emptyset$, then $m = l$ and we are done.

If $B_l \neq \emptyset$ and $A = \bigcup_{r=1}^{l} B_r$, then $m = l + 1$ — hence put $B_m = \emptyset$ and $J_m = \big\{E_1 \cup E_2 \cup E_3\big\}$ for some distinct $E_1, E_2, E_3 \in J_l$.

If neither of the above two cases holds, then note that $l < k$ (since $|A| = k$, $B_1, \ldots, B_l \neq \emptyset$ and $A \neq \bigcup_{r=1}^{l} B_r$) and let J' be any partition of J_l into 3-element sets and let J'' be a set of unions of elements of J', i.e.

$$J'' = \big\{E_1 \cup E_2 \cup E_3\colon\ \{E_1, E_2, E_3\} \in J'\big\}.$$

The cardinality of J'' is $(2^k)^{k-l} \cdot 3^{k-l}$ (recall that $|J_l| = (2^k)^{k-l} \cdot 3^{k-l+1}$), hence by Lemma 2.1 (cf. the first step), there exists $J_{l+1} \subseteq J''$ of cardinality $(2^k)^{k-l-1} \cdot 3^{k-l}$ such that the following sets

$$X = \Big\{\mu_i \in A \setminus \bigcup_{r=1}^{l} B_r\colon\ \mu_i \text{ is good for every } E \in J_{l+1}\Big\} \quad \text{and}$$

$$Y = \Big\{\mu_i \in A \setminus \bigcup_{r=1}^{l} B_r\colon\ \mu_i \text{ is bad for every } E \in J_{l+1}\Big\}$$

constitute a partition of $A \setminus \bigcup_{r=1}^{l} B_r$. Put $B_{l+1} = Y$, and proceed to the next step (i.e. the $(l+1)$-th step where $l + 1 \le k$).

Note that elements of J_1 are singletons and that for every $1 < l \le m$ elements of J_l are unions of three distinct elements of J_{l-1}. Hence, for every $1 \le l \le m$ and $E \in J_l$ the cardinality $|E|$ is a power of 3, precisely $|E| = 3^{l-1}$, so finally $k' = m - 1$. $\square$

Lemma 5.4. *Let $\mathcal{A}$ be a Boolean algebra. Fix $1 \le k \in \omega$ and let $2 \le l \le k+1$. Let $d_1, \ldots, d_{L(l-1)}, e \in \mathcal{A}$ be mutually disjoint and let $\mu_1, \ldots, \mu_{l-1}$ and $\nu_1, \ldots, \nu_{L(l-1)}$ be real-valued measures on $\mathcal{A}$. Assume that $\rho_1, \ldots, \rho_l$ are such positive real numbers that:*

(i) $\big|\nu_j\big(d_j\big)\big| > \Big(2^{3^k}\Big)^{k+1} \cdot 2^{k+1} \cdot \rho_l$ *for every* $j = 1, \ldots, L(l-1)$,

(ii) $\big|\mu_i(e)\big| > \Big(2^{3^k}\Big)^{k+2-l} \cdot 2^{k+2-l} \cdot \rho_i$ *for every* $i = 1, \ldots, l-1$.

Then, there exist a (possibly empty) set $F \subseteq \{1, \ldots, L(l-1)\}$ *and an index* $j_0 \in \{1, \ldots, L(l-1)\}$ *such that for every pair* $(\mu, \rho) \in \{(\mu_1, \rho_1), \ldots, (\mu_{l-1}, \rho_{l-1}), (\nu_{j_0}, \rho_l)\}$ *it holds:*

$$\Big|\mu\Big(e \vee \bigvee_{j\in F} d_j\Big)\Big| > \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho.$$

Proof. We start with a trivial case when there simply exists $j_0 \in \{1, \ldots, L(l-1)\}$ such that

$$\big|\nu_{j_0}(e)\big| > \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l.$$

We put then $F = \emptyset$ and finish the proof.

Let us thus assume that for every $j \in \{1, \ldots, L(l-1)\}$ we have

$$(*) \qquad \big|\nu_j(e)\big| \leq \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l.$$

In this case we will find the non-empty set F as follows. By Lemma 5.3, we know that there exists a set $E \subseteq \{1, \ldots, L(l-1)\}$ of cardinality 3^m for some $0 \leq m \leq l-1$ such that the following inequality holds for every $i < l$ (we divided by 2 the middle factor from the inequality (ii)):

$$\Big|\mu_i\Big(e \vee \bigvee_{j\in E} d_j\Big)\Big| > \Big(2^{3^k}\Big)^{k+2-l} \cdot 2^{k+1-l} \cdot \rho_i.$$

Let $\mathcal{E}$ be a collection of all such sets $E \subseteq \{1, \ldots, L(l-1)\}$.

We start again with a trivial case when there is $E \in \mathcal{E}$ and an element $j_0 \in E$ for which

$$\Big|\nu_{j_0}\Big(e \vee \bigvee_{j\in E} d_j\Big)\Big| > \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l;$$

we then put $F = E$ and finish the proof. If it is not the case, i.e. for every $E \in \mathcal{E}$ and each element $j_0 \in E$ it holds that

$$(**) \qquad \Big|\nu_{j_0}\Big(e \vee \bigvee_{j\in E} d_j\Big)\Big| \leq \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l,$$

then we need to construct a set F and find j_0 in another way.

Let us first make the following observations concerning elements of $\mathcal{E}$.

(1) No $E \in \mathcal{E}$ can be a singleton, since otherwise by (i) and $(*)$ for $E = \{j_0\}$ we would have:

$$\Big|\nu_{j_0}\Big(e \vee \bigvee_{j\in E} d_j\Big)\Big| = \big|\nu_{j_0}(e) + \nu_{j_0}(d_{j_0})\big| \geq \big|\nu_{j_0}(d_{j_0})\big| - \big|\nu_{j_0}(e)\big| >$$

$$\Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1} \cdot \rho_l - \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l > \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l,$$

contradicting $(**)$ (recall that $l \geq 2$).

(2) For every $E \in \mathcal{E}$ and $j_0 \in E$ we have:

$$\Big|\nu_{j_0}\Big(\bigvee_{j\in E\setminus\{j_0\}} d_j\Big)\Big| > \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+2-l} \cdot \rho_l,$$

since by $(**)$, (i) and $(*)$:

$$\Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l \geq \Big|\nu_{j_0}\Big(e \vee \bigvee_{j\in E} d_j\Big)\Big| = \Big|\nu_{j_0}(e) + \nu_{j_0}(d_{j_0}) + \nu_{j_0}\Big(\bigvee_{j\in E\setminus\{j_0\}} d_j\Big)\Big| \geq$$

$$\left|\nu_{j_0}\big(d_{j_0}\big)\right| - \left|\nu_{j_0}(e)\right| - \left|\nu_{j_0}\Big(\bigvee_{j\in E\setminus\{j_0\}} d_j\Big)\right| >$$

$$\left(2^{3^k}\right)^{k+1-l} \cdot 2^{k+1} \cdot \rho_l - \left(2^{3^k}\right)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l - \left|\nu_{j_0}\Big(\bigvee_{j\in E\setminus\{j_0\}} d_j\Big)\right|,$$

and after swapping the upper left and bottom right elements recall again that $l \geq 2$.

(3) Since $1 < |E| \leq 3^{l-1}$ for every $E \in \mathcal{E}$, by the observation (2) and Lemma 4.8, for every $E \in \mathcal{E}$ and $j_0 \in E$ there is $j \in E \setminus \{j_0\}$ such that

$$\left|\nu_{j_0}\big(d_j\big)\right| > \left(2^{3^k}\right)^{k+1-l} \cdot 2^{k+2-l} \cdot \rho_l/3^{l-1}.$$

We construct the set F and choose an index j_0 in several steps. The idea of the procedure is following. In the first step we find many disjoint sets $E \in \mathcal{E}$ and then we pick one arbitrary index j_E^1 from each set E. Then, in the second step we construct new sets $E \in \mathcal{E}$ containing only indices j_E^1 from the previous step and from each of those new sets E's we pick one new arbitrary index j_E^2. We repeat this pyramid-like procedure until we are able to obtain only one set $E \in \mathcal{E}$ and hence pick only one index j_E^m — this index is the sought final index j_0! In each step, this index has belonged to some unique E — by the observations (2) and (3) we can choose from each such E another index j_E such that $\left|\nu_{j_0}\big(d_{j_E}\big)\right|$ is big. The set F will be the union of certain sets $E \in \mathcal{E}$ consisting only of those ultimately picked j_E's.

Let us start the construction of F. Since $L(l-1) = K(l-1)^{K(l-1)\cdot 2\cdot 3^{l-1}}$, by Lemma 5.3, there are $K(l-1)^{K(l-1)\cdot 2\cdot 3^{l-1}-1}$ pairwise disjoint subsets

$$E_1^1, \ldots, E^1_{K(l-1)^{K(l-1)\cdot 2\cdot 3^{l-1}-1}} \in \mathcal{E}.$$

From each E_r^1 choose arbitrarily one j_r^1. By Lemma 5.3, there are $K(l-1)^{K(l-1)\cdot 2\cdot 3^{l-1}-2}$ pairwise disjoint subsets

$$E_1^2, \ldots, E^2_{K(l-1)^{K(l-1)\cdot 2\cdot 3^{l-1}-2}} \in \mathcal{E}$$

made of j_r^1's. From each E_r^2 pick one j_r^2. Again, by Lemma 5.3, there are $K(l-1)^{K(l-1)\cdot 2\cdot 3^{l-1}-3}$ pairwise disjoint subsets

$$E_1^3, \ldots, E^3_{K(l-1)^{K(l-1)\cdot 2\cdot 3^{l-1}-3}} \in \mathcal{E}$$

made of j_r^2's. Continue in this manner until the $K(l-1)\cdot 2\cdot 3^{l-1}$-th step, when you obtain only one subset

$$E_1^{K(l-1)\cdot 2\cdot 3^{l-1}} \in \mathcal{E}.$$

Take any $j_0 \in E_1^{K(l-1)\cdot 2\cdot 3^{l-1}}$ — this is the sought index.

For every $1 \leq r \leq K(l-1)\cdot 2\cdot 3^{l-1}$, there is unique s for which $j_0 \in E_s^r$. Appealing to the observation (3), from each such E_s^r take $j_r \neq j_0$ for which

$$(***) \quad \left|\nu_{j_0}\big(d_{j_r}\big)\right| > \left(2^{3^k}\right)^{k+1-l} \cdot 2^{k+2-l} \cdot \rho_l/3^{l-1}.$$

Let $D = \left\{j_1, \ldots, j_{K(l-1)\cdot 2\cdot 3^{l-1}}\right\}$. Since $|D| = K(l-1)\cdot 2\cdot 3^{l-1}$, by Lemma 2.1, there is a subset D' of D of cardinality $K(l-1)\cdot 3^{l-1}$ and a number $\varsigma \in \{-1, +1\}$ such that the equality

$$\operatorname{sgn}\Big(\nu_{j_0}\big(d_{j_r}\big)\Big) = \varsigma$$

is satisfied for every $j_r \in D'$. Now use Lemma 5.3 inductively to obtain (one after another) 3^{l-1} pairwise disjoint subsets $E_1, \ldots, E_{3^{l-1}}$ of D' such that

$$\left|\mu_i\Big(e \vee \bigvee_{r=1}^{s} \bigvee_{j\in E_r} d_j\Big)\right| > 2^{3^{l-1}-s} \cdot \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_i \geq \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_i$$

for every $s = 1, \ldots, 3^{l-1}$ and every $i = 1, \ldots, l-1$. Besides, by $(*)$ and $(***)$ we have:

$$\left|\nu_{j_0}\Big(e \vee \bigvee_{r=1}^{3^{l-1}} \bigvee_{j\in E_r} d_j\Big)\right| = \left|\nu_{j_0}(e) + \nu_{j_0}\Big(\bigvee_{r=1}^{3^{l-1}} \bigvee_{j\in E_r} d_j\Big)\right| \geq$$

$$\left|\nu_{j_0}\Big(\bigvee_{r=1}^{3^{l-1}} \bigvee_{j\in E_r} d_j\Big)\right| - \left|\nu_{j_0}(e)\right| = \sum_{r=1}^{3^{l-1}} \sum_{j\in E_r} \left|\nu_{j_0}(d_j)\right| - \left|\nu_{j_0}(e)\right| >$$

$$3^{l-1} \cdot \left(\Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+2-l} \cdot \rho_l / 3^{l-1}\right) - \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l \geq \Big(2^{3^k}\Big)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_l,$$

where the second equality follows from the fact that all $\nu_{j_0}(d_j)$'s have the same sign ς. Put

$$F = \bigvee_{r=1}^{3^{l-1}} \bigvee_{j\in E_r} d_j.$$

□

An important remark. Note that in the proofs of Lemmas 5.2–5.4 we did not use the full information about measures μ, μ_i or ν_j — what we in fact only used were the logic values of inequalites of type $|\mu(d)| > \rho$, where μ was a measure on a Boolean algebra $\mathcal{A}$, d an element of $\mathcal{A}$ and ρ a positive real number. This observation will be of important meaning in the next proof where we will appeal to Lemma 5.4 using only those logic values — the information about them will be carried by the functions $Z^n_{j,m}$.

Proof of Lemma 5.1. The idea of the proof is as follows. In the first step, for every anti-Nikodym sequence of real-valued measures $\langle \mu_n\colon\ n \in \omega\rangle$ on $Fr(\omega)$ we construct an infinite antichain $\langle b_k \in Fr(\omega)\colon\ k \in \omega\rangle$ and an increasing sequence $\langle n(k) \in \omega\colon\ k \in \omega\rangle$ such that the value $\left|\mu_{n(k)}(b_k)\right|$ is "large" for every $k \in \omega$. We divide the antichain $\langle b_k\colon\ k \in \omega\rangle$ into infinitely many finite consecutive sequences such that for every $n \in \omega$ the n-th collection is of the form $\langle b^n_1, \ldots, b^n_{N(n)}\rangle$ (so $\langle b_0\rangle = \langle b^0_1\rangle, \langle b_1, \ldots, b_{N(0)+N(1)}\rangle = \langle b^1_1, \ldots, b^1_{N(1)}\rangle$ etc.). For every $n \in \omega$, let $\langle \mu^n_1, \ldots, \mu^n_{N(n)}\rangle$ be the collection of the measures from the sequence $\langle \mu_{n(k)}\colon\ k \in \omega\rangle$ corresponding to the sequence $\langle b^n_1, \ldots, b^n_{N(n)}\rangle$, i.e. the value $\left|\mu^n_i(b^n_i)\right|$ is "large" for every $i = 1, \ldots, N(n)$. The Stone space $K_{Fr(\omega)}$ may be identified with the Cantor set 2^ω, while $Fr(\omega)$ itself can be seen as the algebra of clopen subsets of $K_{Fr(\omega)}$ — this allows us to think about clopen sets of $K_{Fr(\omega)}$ as being described with finite zero-one sequences. Thus, by an appropriate "translating" procedure we can describe *a pair* consisting of the sequence of collections $\langle\langle b^n_1, \ldots, b^n_{N(n)}\rangle\colon\ n \in \omega\rangle$ and the sequences of values $\langle \left|\mu_{n(k)}(b_l)\right|\colon\ k, l \in \omega\rangle$ as an element of 2^ω or more generally ω^ω.

In the second step, using the Bartoszyński-Judah theorem (Proposition 2.10), we cover ω^ω with $\mathrm{cof}(\mathcal{N})$ many products of the form $T = \prod_{n\in\omega} T_n$ where each $T_n \in [\omega]^{n+1}$. Take one such product T. Let us say for a moment that T *covers* an anti-Nikodym sequence $\langle \mu_n\colon\ n \in \omega\rangle$ if it covers the result of the "translating" procedure described in the previous paragraph applied to $\langle \mu_n\colon\ n \in \omega\rangle$. The procedure is conducted in such a way that if a sequence $\langle \mu_n\colon\ n \in \omega\rangle$ is covered by T, then for every $n \in \omega$ the set T_n contains an element of ω corresponding to the pair φ_n consisting of the n-th collection $\langle b_1^n, \ldots, b_{N(n)}^n\rangle$ and "the information" $\langle z_{i,j}^n\colon\ i, j = 1, \ldots, N(n)\rangle$ stating for every i, j whether $\left|\mu_i^n\big(b_j^n\big)\right|$ is "large". Since $\left|T_n\right| = n + 1$, every sequence $\langle \mu_n\colon\ n \in \omega\rangle$ covered by T must have its n-th pair φ_n being exactly one of at most $n + 1$ many different n-th pairs whose corresponding elements of ω are contained in T_n — in other words, there are only $n + 1$ possible values of the n-th pair for all sequences of measures covered by T. Based on those at most $n + 1$ different n-th pairs, using only their collections $\langle b_1^n, \ldots, b_{N(n)}^n\rangle$ of elements of the algebra $Fr(\omega)$ and the information written in corresponding collections $\langle z_{i,j}^n\colon\ i, j = 1, \ldots, N(n)\rangle$, with an aid of Lemma 5.4 we construct the n-th element a_n of an antichain $\langle a_n\colon\ n \in \omega\rangle$ — this antichain is the sought antichain for all anti-Nikodym sequences of measures covered by T.

(Note that the above description is only an intuitive image of the "translating" procedure and the construction. Similarly, the notation used above is a simplified version of the one used in the proof.)

Let us conduct the proof. Let $\mathcal{D} \subseteq \omega^\omega$ be a dominating family of cardinality $\mathfrak{d}$ described in Corollary 2.5. As mentioned above, we see the Stone space $K_{Fr(\omega)}$ of $Fr(\omega)$ as the Cantor space 2^ω and $Fr(\omega)$ as the algebra of clopen subsets of 2^ω (which are described with finite zero-one sequences).

1. Step. Let $\overline{\mu} = \langle \mu_n\colon\ n \in \omega\rangle$ be an anti-Nikodym sequence of real-valued measures on $Fr(\omega)$. By Lemma 4.4, there exists a Nikodym concentration point $t \in 2^\omega$ of $\overline{\mu}$.

By Lemma 4.7, there exists $b_1^0 \in Fr(\omega)$ and a number $m(0,1) \in \omega$ such that $\left|\mu_{m(0,1)}\big(b_1^0\big)\right| > 1$ and $t \notin \big[b_1^0\big]$. Find $g(0) \in \omega$ such that $g(0) > 0$ and $[t \restriction g(0)] \cap \big[b_1^0\big] = \emptyset$.

Now, inductively use Lemma 4.7 to obtain the following:

- an increasing sequence of natural numbers $g = \langle g(n)\colon\ n \in \omega\rangle$ and sets $\mathcal{G}_n$ of all zero-one sequences of length $g(n)$, i.e. $\mathcal{G}_n = \{0,1\}^{g(n)}$,
- for each $n > 0$ a sequence of length $N(n)$ of natural numbers

  $$m(n,1) < \ldots < m(n, N(n)) \in \omega$$

  such that $m(n-1, N(n-1)) < m(n,1)$,
- for each $n > 0$ a sequence of length $N(n)$ of pairwise disjoint elements $b_1^n, \ldots, b_{N(n)}^n \in Fr(\omega)$ such that

$$(*) \qquad [b_1^n], \ldots, \Big[b_{N(n)}^n\Big] \subseteq [t \restriction g(n-1)] \quad \text{and} \quad [t \restriction g(n)] \cap \Big[b_1^n \vee \ldots \vee b_{N(n)}^n\Big] = \emptyset$$

  and for every $j = 1, \ldots, N(n)$ the following two conditions hold:
  - for every $\sigma \in \mathcal{G}_n$ either $[\sigma] \subseteq \big[b_j^n\big]$ or $[\sigma] \cap \big[b_j^n\big] = \emptyset$,
  - the following inequality is satisfied:

$$(**) \qquad \left|\mu_{m(n,j)}\big(b_j^n\big)\right| > \big(N(n)+1\big)^n \cdot M(n) \cdot \rho_n,$$

  where

$$\rho_n = \sum_{F \subseteq \mathcal{G}_{n-1}} \sup_m \left|\mu_m\left(\bigvee_{\sigma \in F} [\sigma]\right)\right| + n + 1.$$

  (The suprema are finite by the elementwise boundedness of $\overline{\mu}$.)

Due to the properties of the family $\mathcal{D}$ stated in Corollary 2.5, we may assume that g is an element of $\mathcal{D}$.

Note that for every element b_j^n there is a subset $F_j^n \subseteq \mathcal{G}_n$ such that $b_j^n = \bigvee_{\sigma\in F_j^n}[\sigma]$. Also, for every $m = 1,\dots,M(n)$ and every $j = 1,\dots,N(n)$ define a sequence $Z_{j,m}^n \in \{0,1\}^{\wp(\mathcal{G}_n)}$ in the following way:

$$Z_{j,m}^n(F) = \begin{cases} 1 & \text{if} \quad \left|\mu_{m(n,j)}\left(\bigvee_{\sigma\in F}[\sigma]\right)\right| > m\cdot\rho_n, \\ 0 & \text{if} \quad \left|\mu_{m(n,j)}\left(\bigvee_{\sigma\in F}[\sigma]\right)\right| \le m\cdot\rho_n \end{cases}$$

for every $F \subseteq \mathcal{G}_n$. The information carried by functions $Z_{j,m}^n$'s is all the necessary data we need to construct the antichain in the next step (e.g. this data will be sufficient for invoking Lemma 5.4) — we will not need the full information about exact values of measures $\mu_{m(n,j)}$'s on elements $\bigvee_{\sigma\in F}[\sigma]$ for $F \subseteq \mathcal{G}_n$, cf. the important remark just before the proof.

Now, for each $n\in\omega$ put

$$\varphi_n(\overline{\mu}) = \left\langle \left\langle \left\langle m(n,j);\ F_j^n;\ \left\langle Z_{j,1}^n,\dots,Z_{j,M(n)}^n\right\rangle \right\rangle \colon\ j=1,\dots,N(n)\right\rangle;\ t\restriction g(n)\right\rangle,$$

so the first coordinate of $\varphi_n(\overline{\mu})$ contains the sequence of length $N(n)$ whose elements comprise the following information:

- an index $m(n,j)$ of a measure $\mu_{m(n,j)}$ from the sequence $\overline{\mu}$ which is "large" on the element $b_j^n = \bigvee_{\sigma\in F_j^n}[\sigma]$ of $Fr(\omega)$,
- "estimations" of values of $\mu_{m(n,j)}$ on clopens given by sequences from $\mathcal{G}_n$.

Then, the sequence $\varphi(\overline{\mu}) = \left\langle \varphi_n(\overline{\mu})\colon\ n\in\omega\right\rangle$ is an element of the following product:

$$\mathcal{P}_g \;=\; \prod_{n\in\omega}\left(\left(\omega\times\wp(\mathcal{G}_n)\times\left(\{0,1\}^{\wp(\mathcal{G}_n)}\right)^{M(n)}\right)^{N(n)}\times\mathcal{G}_n\right).$$

This way, for every anti-Nikodym sequence $\overline{\mu}$ on $Fr(\omega)$ we can find a corresponding element of $\mathcal{P}_g$ for some $g\in\mathcal{D}$. Note that since the choice of the element b and the natural number n in Lemma 4.7 is in general not unique, for any $\overline{\mu}$ we may find many corresponding elements in different $\mathcal{P}_g$'s as well as many anti-Nikodym sequences may have the same corresponding element of one $\mathcal{P}_g$.

2. Step. Fix $g\in\mathcal{D}$. Due to the Bartoszyński–Judah characterization of $\operatorname{cof}(\mathcal{N})$ (Proposition 2.10) there exists a covering of $\mathcal{P}_g$ with κ many sets T of the form $\prod_{n\in\omega}T_n$, where for each $n\in\omega$

$$T_n \;\subseteq\; \left(\omega\times\wp(\mathcal{G}_n)\times\left(\{0,1\}^{\wp(\mathcal{G}_n)}\right)^{M(n)}\right)^{N(n)}\times\mathcal{G}_n$$

and $\left|T_n\right| = n+1$.

Let T be as above and denote by T^* the set of all anti-Nikodym sequences $\overline{\mu}$ of real-valued measures on $Fr(\omega)$ for which $\varphi(\overline{\mu})\in T$. Since $\mathfrak{d}\le\operatorname{cof}(\mathcal{N})$, it is enough to construct an antichain $\left\langle a_n\in Fr(\omega)\colon\ n\in\omega\right\rangle$ such that for every $\overline{\mu} = \left\langle\mu_n\colon\ n\in\omega\right\rangle\in T^*$ there exists an increasing sequence $\left\langle n(k)\colon\ k\in\omega\right\rangle$ of natural numbers for which the following inequality holds:

$$\left|\mu_{n(k)}\left(a_k\right)\right| > \sum_{i=0}^{k-1}\left|\mu_{n(k)}\left(a_i\right)\right| + k + 1;$$

this will finish the proof the lemma.

Fix thus $k \in \omega$ and take any such anti-Nikodym sequences $\overline{\mu}^1, \ldots, \overline{\mu}^{k+1} \in T^*$ that if for an anti-Nikodym sequence $\overline{\mu} \in T^*$ we have $\varphi_k(\overline{\mu}) \in T_k$, then there is $i \leq k+1$ such that $\varphi_k(\overline{\mu}) = \varphi_k\big(\overline{\mu}^i\big)$. Let $t_1, \ldots, t_{k+1} \in 2^\omega$ be Nikodym concentration points of respective $\overline{\mu}^1 = \big\langle \mu_n^1 \colon\ n \in \omega\big\rangle, \ldots, \overline{\mu}^{k+1} = \big\langle \mu_n^{k+1} \colon\ n \in \omega\big\rangle$ such that

$$\varphi_k\big(\overline{\mu}^i\big) = \Bigg\langle \Bigg\langle \Big\langle m_i(k,j);\ F_j^{i,k};\ \big\langle Z_{j,1}^{i,k}, \ldots, Z_{j,M(k)}^{i,k}\big\rangle\Big\rangle \colon\ j = 1, \ldots, N(k)\Bigg\rangle;\ t_i \restriction g(k)\Bigg\rangle \in T_k$$

for each $i = 1, \ldots, k+1$. Put

$$b_j^{i,k} = \bigvee_{\sigma \in F_j^{i,k}} [\sigma]$$

for every $i = 1, \ldots, k+1$ and $j = 1, \ldots, N(k)$.

Before we proceed with the construction of the k-th element of the antichain, we need to do some preparation to obtain "basic bricks" from which we will construct the k-th element of the antichain. Those "bricks" will be obtained from elements $b_j^{i,k}$ (as their mutual intersections). First, put

$$q_k = \bigvee_{i=1}^{k+1} \bigvee_{j=0}^{N(k)} b_j^{i,k}.$$

Next, for every $i = 1, \ldots, k+1$ put

$$\mathcal{Q}_k^i = \Big\{ b_j^{i,k} \colon\ 1 \leq j \leq N(k) \Big\} \qquad \text{if} \qquad q_k \setminus \bigvee_{j=1}^{N(k)} b_j^{i,k} = \mathbf{0},$$

or

$$\mathcal{Q}_k^i = \Big\{ b_j^{i,k} \colon\ 1 \leq j \leq N(k) \Big\} \cup \left\{ q_k \setminus \bigvee_{j=1}^{N(k)} b_j^{i,k} \right\},$$

otherwise. The collections $\mathcal{Q}_k^i$ are partitions of q_k, i.e. their elements are pairwise disjoint and $q_k = \bigvee \mathcal{Q}_k^i$ for every $i = 1, \ldots, k+1$. Let then $\mathcal{R}_k$ be the coarsest partition of q_k finer than every $\mathcal{Q}_k^i$, i.e.

$$\mathcal{R}_k = \left\{ \bigwedge_{i=1}^{k+1} b_i \colon\ b_1 \in \mathcal{Q}_k^1, \ldots, b_{k+1} \in \mathcal{Q}_k^{k+1} \right\} \setminus \{\mathbf{0}\}.$$

Note that for every $i = 1, \ldots, k+1$ and $j = 1, \ldots, N(k)$ there are at most $\big(N(k)+1\big)^k$ elements below $b_j^{i,k}$ in $\mathcal{R}_k$. Hence, by Lemma 4.8, for every $b_j^{i,k}$ there exists $c_j^{i,k} \in \mathcal{R}_k$ such that $c_j^{i,k} \leq b_j^{i,k}$ and (we divided $(**)$ by $N(k)^k$)

$$\big|\mu_{m_i(k,j)}^i\left(c_j^{i,k}\right)\big| > M(k) \cdot \rho_{i,k},$$

where:

$$\rho_{i,k} = \sum_{F \subseteq \mathcal{G}_{k-1}} \sup_m \big|\mu_m^i\left(\bigvee_{\sigma \in F} [\sigma]\right)\big| + k + 1.$$

For every $i = 1, \ldots, k+1$ put then

$$\mathcal{R}_k^i = \left\{ c_j^{i,k} \colon\ 1 \leq j \leq N(k)\ \&\ \Big([t_m \restriction g(k)] \cap \left[c_j^{i,k}\right] = \emptyset \text{ for every } 1 \leq m \leq k+1 \Big) \right\};$$

note that $\big|\mathcal{R}_k^i\big| \geq L(0) + L(1) + \ldots + L(k)$ (this is why we have needed the summand $k+1$ in the definition of $N(k)$!). Finally, let $\mathcal{S}_k^1, \ldots, \mathcal{S}_k^{k+1}$ be such mutually disjoint families that $\mathcal{S}_k^i \subseteq \mathcal{R}_k^i$ and $\left|\mathcal{S}_k^i\right| = L(i-1)$ for every $i = 1, \ldots, k+1$. Write $\mathcal{S}_k^i = \left\{d_1^{i,k}, \ldots, d_{L(i-1)}^{i,k}\right\}$. The elements $d_j^{i,k}$ are "basic bricks" which we will use to construct the k-th element of the antichain.

We shall proceed by induction on $l = 1, \ldots, k+1$ to construct an increasing sequence $e_1^k \le e_2^k \le \ldots \le e_{k+1}^k$ in $Fr(\omega)$ and to find indices $j_1, \ldots, j_{k+1} \in \left\{1, \ldots, N(k)\right\}$ such that for every $l = 2, \ldots, k+1$:

- there is a (possibly empty) set $F_l \subseteq \left\{1, \ldots, L(l-1)\right\}$ such that
$$e_l^k \setminus e_{l-1}^k = \bigvee_{j \in F_l} d_j^{l,k} \le \bigvee \mathcal{S}_l^k,$$
- $\left|\mu_{m_i(k,j_i)}^i\left(e_l^k\right)\right| > \left(2^{3^k}\right)^{k+1-l} \cdot 2^{k+1-l} \cdot \rho_{i,k}$ for every $i = 1, \ldots, l$.

(Hint: e_{k+1}^k will be the element a_k of the sought antichain.)

Let j_1 be such index that $c_{j_1}^{1,k} = d_1^{1,k}$ and put $e_1^k = d_1^{1,k}$. Fix $l \in \{2, \ldots, k+1\}$ and assume that we have found required element e_{l-1}^k and indices $j_i \in \left\{1, \ldots, N(k)\right\}$ for $i = 1, \ldots, l-1$. Now, appeal to Lemma 5.4 with:

- $e = e_{l-1}^k$,
- $\mu_i = \mu_{m_i(k,j_i)}^i$ and $\rho_i = \rho_{i,k}$ for $i = 1, \ldots, l-1$,
- $\nu_j = \mu_{m_l(k,j)}^l$ and $d_j = d_j^{l,k}$ for $j = 1, \ldots, L(l-1)$,

to obtain a set $F_l \subseteq \left\{1, \ldots, L(l-1)\right\}$ and an index $j_0 \in \left\{1, \ldots, L(l-1)\right\}$. Define j_l as an index for which $c_{j_l}^{l,k} = d_{j_0}^{l,k}$ and put:
$$e_l^k = e_{l-1}^k \vee \bigvee_{j \in F_l} d_j^{l,k}.$$
Note again that in the proofs of Lemmas 5.2–5.4 when considering values of measures $\mu_i = \mu_{m_i(k,j_i)}^i$ we only appeal to information carried by the functions $Z_{j,m}^{i,k}$'s.

Correctness of the construction. Assume now that for every $k \in \omega$ the element e_{k+1}^k has been constructed as described above and put $a_k = e_{k+1}^k$. We first show that $\left\langle a_k\colon\ k \in \omega\right\rangle$ is an antichain. Let thus $k < l \in \omega$. Let $\overline{\mu}^1, \ldots, \overline{\mu}^{k+1} \in T^*$ and $\overline{\nu}^1, \ldots, \overline{\nu}^{l+1} \in T^*$ be such that for every $\overline{\mu} \in T^*$ there are $i \le k+1$ and $j \le l+1$ such that $\varphi_k(\overline{\mu}) = \varphi_k(\overline{\mu}^i)$ and $\varphi_l(\overline{\mu}) = \varphi_l(\overline{\nu}^j)$. Let $t_1, \ldots, t_{l+1} \in 2^\omega$ be Nikodym concentration points of $\overline{\nu}^1, \ldots, \overline{\nu}^{l+1}$, respectively, such that $t_j \restriction g(l)$ is the second coordinate of the pair $\varphi_l(\overline{\nu}^j)$ for every $j \le l+1$. Note that since $k < l$, for every $i \le k+1$ there is $j \le l+1$ such that $t_j \restriction g(k)$ is the second coordinate of the pair $\varphi_k(\overline{\mu}^i)$. Now notice that a_k is a sum of elements from the families $\mathcal{R}_k^1, \ldots, \mathcal{R}_k^{k+1}$ — by their definitions those elements are disjoint with $\bigcup_{j=1}^{l+1} [t_j \restriction g(k)]$. On the other hand, the element a_l is built up from elements from the families $\mathcal{R}_l^1, \ldots, \mathcal{R}_l^{l+1} l$ — by $(*)$ in the definition of the procedure φ from Step 1 those elements are in turn subsets of $\bigcup_{j=1}^{l+1} [t_j \restriction g(k)]$. This implies that $a_k \wedge a_l = \mathbf{0}$.

Let $\left\langle \mu_n\colon\ n \in \omega\right\rangle \in T^*$ and fix $k \in \omega$. According to the construction of $e_{k+1}^k = a_k$ there is a number $i_0 \le N(n)$ such that
$$\left|\mu_{m(k,i_0)}\left(a_k\right)\right| > \sum_{i=0}^{k-1} \left|\mu_{m(k,i_0)}\left(a_i\right)\right| + k + 1.$$
Put $n(k) = m(k, i_0)$. Since $m(n-1, N(n-1)) < m(n, 1)$ for every $n \in \omega$, $\left\langle n(k)\colon\ k \in \omega\right\rangle$ is increasing. This finishes the proof of the lemma. □

6. Two other Nikodym cardinal invariants

6.1. **The anti-Nikodym numbers.** Lemma 5.1 states that there are $\operatorname{cof}(\mathcal{N})$ many antichains $\langle a_n^\gamma \in Fr(\omega)\colon\ n \in \omega\rangle$ such that for a given anti-Nikodym sequence $\langle \mu_n\colon\ n \in \omega\rangle$ on $Fr(\omega)$ there exist $\gamma < \operatorname{cof}(\mathcal{N})$ and an increasing sequence $\langle n(k)\colon\ k \in \omega\rangle$ such that for every $k \in \omega$ the following inequality is satisfied:

$$\left|\mu_{n(k)}\left(a_k^\gamma\right)\right| > \sum_{i=0}^{k-1}\left|\mu_{n(k)}\left(a_i^\gamma\right)\right| + k + 1.$$

We now shortly argue that the number $\operatorname{cof}(\mathcal{N})$ actually works for any countable Boolean algebra — let us thus introduce the following definitions.

Definition 6.1. Let κ be a cardinal number. We say that a Boolean algebra $\mathcal{A}$ has *the κ-anti-Nikodym property* if there exists a family $\left\{\langle a_n^\gamma \in \mathcal{A}\colon\ n \in \omega\rangle\colon\ \gamma < \kappa\right\}$ of κ many antichains in $\mathcal{A}$ with the following property:

> for every anti-Nikodym sequence of real-valued measures $\langle \mu_n\colon\ n \in \omega\rangle$ on $\mathcal{A}$ there exist $\gamma < \kappa$ and an increasing sequence $\langle n_k\colon\ k \in \omega\rangle$ of natural numbers such that for every $k \in \omega$ the following inequality is satisfied:
>
> $$\left|\mu_{n_k}\left(a_k^\gamma\right)\right| > \sum_{i=0}^{k-1}\left|\mu_{n_k}\left(a_i^\gamma\right)\right| + k + 1.$$

Definition 6.2. *The anti-Nikodym number* $\mathfrak{n}_a(\mathcal{A})$ for an algebra $\mathcal{A}$ is defined as follows:

$$\mathfrak{n}_a(\mathcal{A}) = \min\left\{\kappa\colon\ \mathcal{A} \text{ has the } \kappa\text{-anti-Nikodym property}\right\}.$$

Thus, by Lemma 5.1, we immediately have that $\mathfrak{n}_a(Fr(\omega)) \leq \operatorname{cof}(\mathcal{N})$. The next proposition shows that if an algebra $\mathcal{B}$ is a quotient of an algebra $\mathcal{A}$, then the anti-Nikodym number of $\mathcal{B}$ is not greater than that of $\mathcal{A}$.

Proposition 6.3. *Let $\mathcal{A}, \mathcal{B}$ be Boolean algebras and $h : \mathcal{A} \to \mathcal{B}$ an epimorphism. Then, $\mathfrak{n}_a(\mathcal{A}) \geq \mathfrak{n}_a(\mathcal{B})$.*

Proof. First, note that if $\langle \mu_n\colon\ n \in \omega\rangle$ is an anti-Nikodym sequence of measures on $\mathcal{B}$, then $\langle \mu_n \circ h\colon\ n \in \omega\rangle$ is anti-Nikodym on $\mathcal{A}$ (since $\langle \mu_n \circ h\colon\ n \in \omega\rangle$ is trivially elementwise bounded and $\|\mu \circ h\| \geq \|\mu\|$ for any measure μ on $\mathcal{B}$ by surjectivity of h).

Now, if $A = \left\{\langle a_n^\gamma \in \mathcal{A}\colon\ n \in \omega\rangle\colon\ \gamma < \mathfrak{n}_a(\mathcal{A})\right\}$ is a family witnessing that $\mathcal{A}$ has the $\mathfrak{n}_a(\mathcal{A})$-anti-Nikodym property, then $B = \left\{\langle h\left(a_n^\gamma\right) \in \mathcal{B}\colon\ n \in \omega\rangle\colon\ \gamma < \mathfrak{n}_a(\mathcal{A})\right\}$ is a family witnessing that $\mathcal{B}$ has the $\mathfrak{n}_a(\mathcal{A})$-anti-Nikodym property. □

Since the algebra FC is a quotient of every countable Boolean algebra $\mathcal{A}$ (since the Stone space $K_{\mathcal{A}}$ contains a non-trivial convergent sequence), it follows that $\mathfrak{n}_a(FC) \leq \mathfrak{n}_a(\mathcal{A})$ for every countable $\mathcal{A}$. Similarly, every countable Boolean algebra $\mathcal{A}$ is a quotient of the free countable algebra $Fr(\omega)$, hence $\mathfrak{n}_a(Fr(\omega)) \geq \mathfrak{n}_a(\mathcal{A})$.

Corollary 6.4. *For every countable Boolean algebra $\mathcal{A}$, the following inequalities hold:*

$$\mathfrak{n}_a(FC) \leq \mathfrak{n}_a(\mathcal{A}) \leq \mathfrak{n}_a(Fr(\omega)).$$

□

Since every countable Boolean algebra is a subalgebra of $Fr(\omega)$ and hence of $\wp(\omega)$, we immediately obtain the following result.

Corollary 6.5.

$$\mathfrak{n}_a(FC) = \min\big\{\mathfrak{n}_a(\mathcal{A})\colon\ \mathcal{A} \subseteq \wp(\omega) \textit{ is countable}\big\}.$$

$$\mathfrak{n}_a(Fr(\omega)) = \sup\big\{\mathfrak{n}_a(\mathcal{A})\colon\ \mathcal{A} \subseteq \wp(\omega) \textit{ is countable}\big\}.$$

□

For abbreviation, let us denote $\mathfrak{n}_a$ for $\mathfrak{n}_a(Fr(\omega))$. The number $\mathfrak{n}_a$ is called *the anti-Nikodym number.*

The next proposition gives a lower bound for $\mathfrak{n}_a(FC)$ and thus for $\mathfrak{n}_a$.

Proposition 6.6. $\mathfrak{b} \leq \mathfrak{n}_a(FC)$.

Proof. We see the Stone space of the algebra FC as the space $\omega \cup \{\infty\}$ (i.e. the one-point compactification of ω with ∞ being the only non-isolated point). Let $\big\{\big\langle a_n^\gamma \in FC\colon\ n \in \omega\big\rangle\colon\ \gamma < \kappa\big\}$ be a family of antichains for some $\kappa < \mathfrak{b}$. Since $\big\langle a_n^\gamma \in FC\colon\ n \in \omega\big\rangle$ is an infinite antichain for every $\gamma < \kappa$, there is no $n \in \omega$ such that $\infty \in [a_n^\gamma]$ and hence for every $n \in \omega$ the set $[a_n^\gamma]$ is finite. For every $\gamma < \kappa$ and $n \in \omega$ put

$$M_n^\gamma = \max\big\{m\colon m \in [a_n^\gamma]\,\big\}.$$

Since $\kappa < \mathfrak{b}$, there is an increasing sequence $\big\langle M_n \in \omega\colon\ n \in \omega\big\rangle$ dominating strictly every $\big\langle M_n^\gamma\colon\ n \in \omega\big\rangle$. For every $n \in \omega$ define the measure μ_n as follows:

$$\mu_n = n\left(\delta_{M_n} - \delta_\infty\right).$$

The sequence $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ is anti-Nikodym. Fix an increasing sequence $\big\langle n_k \in \omega\colon\ k \in \omega\big\rangle$ and let $\gamma < \kappa$. There is $N \in \omega$ such that for every $k > N$ we have $M_k^\gamma < M_k$ and since $\big\langle n_k\colon\ k \in \omega\big\rangle$ is increasing we have:

$$\mu_{n_k}\big(a_k^\gamma\big) = 0$$

for every $k > N$. □

Corollary 6.7.

(1) $\mathfrak{b} \leq \mathfrak{n}_a \leq \operatorname{cof}(\mathcal{N})$.
(2) Under Martin's axiom, $\mathfrak{n}_a = \mathfrak{c}$. □

6.2. **The Nikodym extracting number.** In this section we introduce a new notion of completeness of Boolean algebras, namely the Nikodym completeness, which fully characterizes the Nikodym property. Based on this, we present and estimate a new cardinal invariant called the Nikodym extracting number, which will be related in the next section to the Nikodym number $\mathfrak{n}$.

Definition 6.8. A Boolean algebra $\mathcal{A}$ is *Nikodym complete* if for every elementwise bounded sequence $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ of measures on $\mathcal{A}$ and every antichain $\big\langle a_n\colon\ n \in \omega\big\rangle$ in $\mathcal{A}$, there are $a \in \mathcal{A}$, an increasing sequence $\big\langle n_k \in \omega\colon\ k \in \omega\big\rangle$ and $\rho > 0$ such that $a_{n_k} \leq a$ and

$$\left|\mu_{n_k}\right|\Big(a \setminus \bigvee_{i=0}^{k} a_i\Big) < \rho$$

for every $k \in \omega$.

Theorem 6.9. *A Boolean algebra $\mathcal{A}$ has the Nikodym property if and only if $\mathcal{A}$ is Nikodym complete.*

Proof. Assume $\mathcal{A}$ is Nikodym complete but does not have the Nikodym property. There exists a elementwise bounded sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}$ which is not uniformly bounded. By Lemma 4.7 there exist an antichain $\langle a_k\colon\ k \in \omega\rangle$ in $\mathcal{A}$ and an increasing sequence $\langle n_k \in \omega\colon\ k \in \omega\rangle$ satisfying for every $k \in \omega$ the following inequality:

$$\big|\mu_{n_k}(a_k)\big| > \sum_{i=0}^{k-1} \big|\mu_{n_k}(a_i)\big| + k.$$

By the Nikodym completeness of $\mathcal{A}$, there are $a \in \mathcal{A}$, an increasing sequence $\langle n_{k_l}\colon\ l \in \omega\rangle \in \omega$ and $\rho > 0$ such that $a_{k_l} \le a$ and

$$\big|\mu_{n_{k_l}}\big|\Big(a \setminus \bigvee_{i=0}^{l} a_{k_i}\Big) < \rho$$

for every $l \in \omega$. We now have:

$$\big|\mu_{n_{k_l}}(a)\big| = \big|\sum_{i<l} \mu_{n_{k_l}}(a_{k_i}) + \mu_{n_{k_l}}(a_{k_l}) + \mu_{n_{k_l}}\Big(a \setminus \bigvee_{i=0}^{l} a_{k_i}\Big)\big| \ge$$

$$\big|\mu_{n_{k_l}}(a_{k_l})\big| - \sum_{i<l} \big|\mu_{n_{k_l}}(a_{k_i})\big| - \big|\mu_{n_{k_l}}\big|\Big(a \setminus \bigvee_{i=0}^{l} a_{k_i}\Big)\big| > k_l - \rho,$$

hence

$$\sup_{n\in\omega} \big|\mu_n(a)\big| \ge \sup_{l\in\omega} \big|\mu_{n_{k_l}}(a)\big| \ge \sup_{l\in\omega} \big(k_l - \rho\big) = \infty,$$

contradicting the elementwise boundedness of $\langle \mu_n\colon\ n \in \omega\rangle$. Thus, $\mathcal{A}$ has the Nikodym property.

Assume now that $\mathcal{A}$ has the Nikodym property but is not Nikodym complete. There exist a pointwise bounded sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}$ (hence uniformly bounded!) and an antichain $\langle a_n\colon\ n \in \omega\rangle$ in $\mathcal{A}$ such that for every $a \in \mathcal{A}$, increasing $\langle n_k \in \omega\colon\ k \in \omega\rangle$ and $\rho > 0$ there is $k \in \omega$ for which either $a_{n_k} \not\le a$, or $\big|\mu_{n_k}\big|\Big(a \setminus \bigvee_{i=0}^{k} a_{n_i}\Big) \ge \rho$ (or both). Put $a = \mathbf{1}$ and $n_k = k$ for every $k \in \omega$. Then, for every $\rho_N = N \in \omega$ there exists $k_N \in \omega$ such that:

$$\rho_N \le \big|\mu_{n_{k_N}}\big|\Big(a \setminus \bigvee_{i=0}^{k_N} a_{n_i}\Big) \le \big\|\mu_{n_{k_N}}\big\|,$$

so $\sup_{n\in\omega} \big\|\mu_n\big\| = \infty$, a contradiction with the Nikodym property. Thus, $\mathcal{A}$ must be Nikodym complete. □

We now introduce a property of subfamilies of $[\omega]^\omega$ strongly related to the Nikodym completeness.

Definition 6.10. Given $\mathcal{F} \subseteq [\omega]^\omega$, we say that an antichain $\langle a_n\colon\ n \in \omega\rangle$ in a Boolean algebra $\mathcal{A}$ is *$\mathcal{F}$-complete in $\mathcal{A}$* if $\bigvee_{n\in A} a_n \in \mathcal{A}$ for every $A \in \mathcal{F}$.

Note that $\mathcal{A}$ is σ-complete if and only if every antichain in $\mathcal{A}$ is $[\omega]^\omega$-complete.

Definition 6.11. A family $\mathcal{F} \subseteq [\omega]^\omega$ is *Nikodym extracting* if for every algebra $\mathcal{A}$ the following condition holds:

for every sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of positive measures on $\mathcal{A}$ and every $\mathcal{F}$-complete antichain $\langle a_n\in\mathcal{A}\colon\ n\in\omega\rangle$ in $\mathcal{A}$, there is $A\in\mathcal{F}$ such that the following inequality is satisfied:

$$\mu_n\Big(\bigvee_{\substack{k\in A\\ k>n}} a_k\Big)<1$$

for every $n\in A$.

The following proposition is a part of Darst's proof that σ-complete Boolean algebras have the Nikodym property.

Proposition 6.12 (Darst [11, page 474])**.** $[\omega]^\omega$ *is Nikodym extracting.*

Proof. Let $\langle\mu_n\colon\ n\in\omega\rangle$ be a sequence of positive measures on a Boolean algebra $\mathcal{A}$ and $\langle a_n\colon\ n\in\omega\rangle$ an $[\omega]^\omega$-complete antichain in $\mathcal{A}$.

Let $\omega=\bigcup_{k\in\omega}N^1_k$ be a partition of ω into infinite sets. Put $n_1=1$. By boundedness of μ_{n_1} we have:

$$\sum_{k\in\omega}\mu_{n_1}\Big(\bigvee_{n\in N^1_k}a_n\Big)\le\mu_{n_1}\Big(\bigvee_{n\in\omega}a_n\Big)\le\mu_{n_1}(\mathbf{1})<\infty,$$

so there is $l_1\in\omega$ such that $n_1<\min N^1_{l_1}$ and

$$\mu_{n_1}\Big(\bigvee_{n\in N^1_{l_1}}a_n\Big)<1.$$

Now, let $N^1_{l_1}=\bigcup_{k\in\omega}N^2_k$ be a partition of $N^1_{l_1}$ into infinite sets. Put $n_2=\min N^1_{l_1}$ and repeat the same procedure as for n_1 to obtain $l_2\in\omega$ such that $n_2<\min N^2_{l_2}$ and

$$\mu_{n_2}\Big(\bigvee_{n\in N^2_{l_2}}a_n\Big)<1.$$

Repeat the above procedure to obtain infinite sequences $\omega=N^0_{l_0}\supset N^1_{l_1}\supset N^2_{l_2}\supset\ldots$ and $1=n_1<n_2<n_3<\ldots$, where $n_k\in N^{k-1}_{l_{k-1}}\setminus N^k_{l_k}$, such that:

$$\mu_{n_k}\Big(\bigvee_{n\in N^k_{l_k}}a_n\Big)<1$$

for every $k\in\omega$. Put $A=\{n_k\colon\ k\in\omega\}$. We have:

$$\mu_{n_k}\Big(\bigvee_{\substack{m\in\omega\\ m>k}}a_{n_m}\Big)\le\mu_{n_k}\Big(\bigvee_{n\in N^k_{l_k}}a_n\Big)<1,$$

for every $k\in\omega$. □

Definition 6.13. *The Nikodym extracting number* $\mathfrak{n}_e$ is defined as follows:

$$\mathfrak{n}_e=\min\big\{|\mathcal{F}|\colon\ \mathcal{F}\subseteq[\omega]^\omega\ \text{ is Nikodym extracting}\big\}.$$

Proposition 6.12 yields that $\mathfrak{n}_e\le\mathfrak{c}$. However, this result can be strenghtened.

Proposition 6.14. $\mathfrak{n}_e\le\mathfrak{d}$.

Proof. Let $\mathcal{D}=\langle g_\eta\colon\ \eta<\mathfrak{d}\rangle$ be a dominating family in ω^ω consisting of increasing functions and $\mathcal{G}=\langle A_\xi\colon\ \xi<\omega_1\rangle$ be a family of infinite almost disjoint subsets of ω. For every $\xi<\kappa$ enumerate A_ξ as $\langle m(\xi,n)\colon\ n\in\omega\rangle$ and for every $\eta<\mathfrak{d}$ let $A_{\xi\eta}=\langle m(\xi,n^\eta_k)\colon\ k\in\omega\rangle$ be a subsequence of A_ξ such that for every $k\in\omega$

$$m\big(\xi,n^\eta_{k+1}\big)>g_\eta\big(m(\xi,n^\eta_k)\big).$$

Put:

$$\mathcal{F} = \{A_\xi\colon\ \xi < \omega_1\} \cup \{A_{\xi\eta}\colon\ \xi < \omega_1, \eta < \mathfrak{d}\}.$$

Then $|\mathcal{F}| = \mathfrak{d}$. We will show that $\mathcal{F}$ is Nikodym extracting.

Let $\langle \mu_n\colon\ n \in \omega\rangle$ be a sequence of positive measures on a Boolean algebra $\mathcal{A}$ and $\langle a_n\colon\ n \in \omega\rangle$ an $\mathcal{F}$-complete antichain in $\mathcal{A}$. As $\mathcal{G}$ is uncountable, by Lemma 2.6, there exists $\xi < \omega_1$ such that for every $k \in \omega$

$$\mu_k\Big(\bigvee_{n\in A_\xi} a_n\Big) = \sum_{n\in A_\xi} \mu_k(a_n).$$

It follows that there exists $f \in \omega^\omega$ such that for every $k \in \omega$

$$\sum_{\substack{n>f(k)\\ n\in A_\xi}} \mu_k(a_n) < 1.$$

Let $\eta < \mathfrak{d}$ be such that $f(n) < g_\eta(n)$ for every $n \in \omega$. By $\mathcal{F}$-completeness of $\langle a_n\colon\ n \in \omega\rangle$, we have for every $k \in \omega$

$$\mu_k\Big(\bigvee_{n\in A_{\xi\eta}} a_n\Big) = \sum_{n\in A_{\xi\eta}} \mu_k(a_n).$$

For every $k \in \omega$ put $l(k) = m(\xi, n_k^\eta)$, i.e. $A_{\xi\eta} = \langle l(k)\colon\ k \in \omega\rangle$. Fix $k \in \omega$. Since $l(k+1) > g_\eta(l(k)) > f(l(k))$, we have

$$\mu_{l(k)}\Big(\bigvee_{\substack{n>l(k)\\ n\in A_{\xi\eta}}} a_n\Big) = \sum_{\substack{n>l(k)\\ n\in A_{\xi\eta}}} \mu_{l(k)}(a_n) = \sum_{\substack{n>g_\eta(l(k))\\ n\in A_{\xi\eta}}} \mu_{l(k)}(a_n) < 1.$$

□

Proposition 6.15. *Let κ be a cardinal number. Assuming* MA_κ*(countable),* $\mathfrak{n}_e > \kappa$.

Proof. The proof has two steps.

The first step. Let us say for a moment that a family $\mathcal{F} \subseteq [\omega]^\omega$ has *the property (*)* if there is no antichain $\langle a_n\colon\ n \in \omega\rangle \subseteq [\omega]^2$ (*antichain of pairs*) such that for every $A \in \mathcal{F}$ there is $n \in \omega$ such that $a_n \subseteq A$. Define the following auxiliary cardinal invariant:

$$\lambda = \min\{|\mathcal{F}|\colon\ \text{non-empty } \mathcal{F} \subseteq [\omega]^\omega \text{ has the property (*)}\}.$$

It is easy to see that $\lambda \le \mathfrak{c}$. We prove that assuming MA_κ(*countable*) the inequality $\lambda > \kappa$ holds.

Define a poset $\mathbb{P}$ as follows:

$$\mathbb{P} = \{(a_1, \ldots, a_n)\colon\ n \in \omega,\ a_1, \ldots, a_n \in [\omega]^2 \text{ mutually disjoint}\},$$

where $(a_1, \ldots, a_n) \le (b_1, \ldots, b_m)$ if $n \ge m$ and $a_i = b_i$ for every $i \le m$. Then, $\mathbb{P}$ is countable.

Let $\mathcal{F} \subseteq [\omega]^\omega$ be an arbitrary family such that $|\mathcal{F}| \le \kappa$. We shall show that $\mathcal{F}$ does not have the property (*) which will imply that $\kappa < \lambda$. For every $A \in \mathcal{F}$ and every $n \in \omega$ put:

$$D_A = \{(a_1, \ldots, a_m) \in \mathbb{P}\colon\ a_m \subseteq A\},$$

$$E_n = \{(a_1, \ldots, a_k) \in \mathbb{P}\colon\ k \ge n\}.$$

D_A's and E_n's are dense in $\mathbb{P}$. By MA_κ(*countable*), there exists a $\mathbb{P}$-generic ultrafilter G intersecting every D_A and every E_n. Put $g = \bigcup G$. By properties of G, the sequence g witnesses that $\mathcal{F}$ does not have the property (*).

The second step. Let $\mathcal{F} \subseteq [\omega]^\omega$ be such that $|\mathcal{F}| < \lambda$. We now prove that $\mathcal{F}$ is not Nikodym extracting by constructing a sequence of positive measures $\langle \mu_k \colon\ k \in \omega \rangle$ on $\wp(\omega)$ and an antichain $\langle b_n \colon\ n \in \omega \rangle$ in $\wp(\omega)$ such that for every $A \in \mathcal{F}$ and some $k \in A$ the inequality

$$\mu_k\Big(\bigvee_{\substack{m \in A \\ m > k}} b_m \Big) < 1$$

is not satisfied.

Since $|\mathcal{F}| < \lambda$, there exists an antichain of pairs $\langle a_n \colon\ n \in \omega \rangle$ such that for every $A \in \mathcal{F}$ there exists $n \in \omega$ for which $a_n \subseteq A$. Write $a_n = \{k_n, l_n\}$, where $k_n < l_n$. For every $k \in \omega$ define $\mu_k \in \mathrm{ba}(\wp(\omega))$ as follows:

$$\mu_k = \begin{cases} \delta_{l_n}, & \text{if } k = k_n, \\ 0, & \text{otherwise,} \end{cases}$$

and put $b_n = \{n\}$. The antichain $\langle b_n \colon\ n \in \omega \rangle$ is $[\omega]^\omega$-complete in $\wp(\omega)$.

Take any $A \in \mathcal{F}$. Then $a_n = \{k_n, l_n\} \subseteq A$ for some n. We have:

$$\mu_{k_n}\Big(\bigcup_{\substack{m \in A \\ m > k_n}} b_m \Big) = \delta_{l_n}\big(b_{l_n}\big) = \delta_{l_n}\big(\{l_n\}\big) = 1.$$

This proves that $\mathcal{F}$ is not a Nikodym extracting family. Thus, under $\mathrm{MA}_\kappa(\mathit{countable})$, $\mathfrak{n}_e \geq \lambda > \kappa$. □

By Proposition 2.9, we immediately obtain the following corollary.

Corollary 6.16.
(1) $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{n}_e \leq \mathfrak{d}$.
(2) Under Martin's axiom, $\mathfrak{n}_e = \mathfrak{c}$. □

7. The main construction

In this section for every κ such that $\mathrm{cof}\big([\kappa]^\omega\big) = \kappa \geq \max(\mathfrak{n}_a, \mathfrak{n}_e)$ we present a construction of a Boolean algebra with the Nikodym property and of cardinality κ.

Lemma 7.1. *Let $\kappa > 0$ be a cardinal such that $\mathrm{cof}\big([\kappa]^\omega\big) = \kappa$. Then, $\mathrm{cf}(\kappa) > \omega$.*

Proof. (Let α be a cardinal and X a set. If $f \colon \alpha \to X$ is a function and $\xi < \alpha$ (i.e. $\xi \in \alpha$), then $f(\xi)$ denotes as usual the value of f at ξ, and $f[\xi]$ stands for the image of the set $\xi = \{\eta \colon\ \eta < \xi\}$ through f, i.e. $f[\xi] = \{f(\eta) \colon\ \eta < \xi\}$.)

Assume $\mathrm{cf}(\kappa) = \omega$. Let $f \colon \kappa \to [\kappa]^\omega$ be a function. We shall show that the range $\mathrm{ran}\, f$ is not cofinal in $[\kappa]^\omega$.

Let $\langle \alpha_n \colon\ n \in \omega \rangle$ be a strictly increasing sequence of infinite cardinals less than κ such that $\lim_{n \to \infty} \alpha_n = \kappa$. Since for every $n \in \omega$ we have:

$$\Big| \bigcup f\big[\alpha_n\big] \Big| \leq \alpha_n \cdot \omega = \alpha_n < \alpha_{n+1},$$

we can pick

$$\xi_n \in \alpha_{n+1} \setminus \bigcup f\big[\alpha_n\big].$$

Put:

$$A = \big\{\xi_n \colon\ n \in \omega\big\}.$$

Then, $A \in [\kappa]^\omega$, but there is no $\xi < \kappa$ such that $A \subseteq f(\xi)$. Indeed, assume there is such ξ. Let $n \in \omega$ be such that $\xi \in \alpha_n$. Then, $f(\xi) \in f\big[\alpha_n\big]$, so

$$\xi_n \in A \subseteq f(\xi) \subseteq \bigcup f\big[\alpha_n\big],$$

but $\xi_n \notin \bigcup f\left[\alpha_n\right]$ — a contradiction. $\square$

Theorem 7.2. *Assume that* $\max(\mathfrak{n}_a, \mathfrak{n}_e) \leq \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}\left([\kappa]^\omega\right) = \kappa$. *Then, there exists a Boolean algebra* $\mathcal{B}$ *with the Nikodym property and of cardinality* κ.

Proof. The constructed algebra will be a subalgebra of $\wp(\kappa)$ (if $\kappa < \mathfrak{c}$, then we can conduct the construction in $\wp(\omega)$). Let $\mathcal{G} \subseteq [\omega]^\omega$ be a Nikodym extracting family such that $|\mathcal{G}| = \mathfrak{n}_e$. Enumerate $\mathcal{G} = \left\langle A_\xi\colon\ \xi < \mathfrak{n}_e\right\rangle$.

We shall now construct by induction an increasing sequence $\left\langle \mathcal{B}_\zeta\colon\ \zeta \leq \omega_1\right\rangle$ of subalgebras of $\wp(\kappa)$ such that $|\mathcal{B}_\zeta| = \kappa$ for each $\zeta \leq \kappa$ and $\mathcal{B} = \mathcal{B}_{\omega_1}$ has the Nikodym property. Thus, let $\mathcal{B}_0$ be any subalgebra of $\wp(\kappa)$ of cardinality κ and for a limit ordinal $\lambda \leq \omega_1$ put $\mathcal{B}_\lambda = \bigcup_{\zeta<\lambda}\mathcal{B}_\zeta$. Let $\zeta < \omega_1$ and assume $\mathcal{B}_\zeta$ is constructed. We extend $\mathcal{B}_\zeta$ to $\mathcal{B}_{\zeta+1}$ as follows.

Since $\operatorname{cof}\left([\kappa]^\omega\right) = \kappa$, there exists a cofinal family $\mathcal{F}_\zeta \subseteq [\mathcal{B}_\zeta]^\omega$ of cardinality κ. We may assume that elements of $\mathcal{F}_\zeta$ are Boolean algebras, hence any element $\mathcal{A}$ of $\mathcal{F}_\zeta$ has the $\mathfrak{n}_a$-anti-Nikodym property. So let $\mathcal{A} \in \mathcal{F}_\zeta$ and let $\left\{\left\langle a_n^\gamma\colon\ n \in \omega\right\rangle\colon\ \gamma < \mathfrak{n}_a\right\}$ be a family of antichains in $\mathcal{A}$ witnessing this property. For every $\gamma < \mathfrak{n}_a$ and $\xi < \mathfrak{n}_e$ define:

$$b_\xi^\gamma = \bigvee_{n \in A_\xi} a_n^\gamma.$$

and put:

$$\Phi(\mathcal{A}) = \left\{b_\xi^\gamma\colon\ \gamma < \mathfrak{n}_a,\ \xi < \mathfrak{n}_e\right\}.$$

We define the algebra $\mathcal{B}_{\zeta+1}$ as an algebra generated by the set

$$\mathcal{B}_\zeta \cup \bigcup_{\mathcal{A}\in\mathcal{F}_\zeta} \Phi(\mathcal{A}).$$

It is immediate that $\left|\mathcal{B}_{\zeta+1}\right| = \kappa$.

We shall first prove that there are no anti-Nikodym sequences of real-valued measures on $\mathcal{B}$. So, for the sake of contradiction, assume there is an anti-Nikodym sequence $\overline{\mu} = \left\langle \mu_n\colon\ n \in \omega\right\rangle$ of real-valued measures on $\mathcal{B}$. By inductive use of Lemma 4.7, there exist an antichain $\left\langle a_k\colon\ k \in \omega\right\rangle$ in $\mathcal{B}$ and a sequence $\left\langle m(k)\colon\ k \in \omega\right\rangle$ of natural numbers such that $\left|\mu_{m(k)}\left(a_k\right)\right| > k$ for every $k \in \omega$. By Lemma 7.1, there exists $\zeta < \omega_1$ such that $\left\langle a_k\colon\ k \in \omega\right\rangle \subseteq \mathcal{B}_\zeta$ and hence there exists $\mathcal{A} \in \mathcal{F}_\zeta$ for which $\left\langle a_k\colon\ k \in \omega\right\rangle \subseteq \mathcal{A}$. Let $\left\{\left\langle a_n^\gamma \in \mathcal{A}\colon\ n \in \omega\right\rangle\colon\ \gamma < \mathfrak{n}_a\right\}$ be a family of antichains witnessing that $\mathcal{A}$ has the $\mathfrak{n}_a$-anti-Nikodym property. As $\overline{\mu} \restriction \mathcal{A}$ is anti-Nikodym on $\mathcal{A}$, there exist $\gamma < \mathfrak{n}_a$ and an increasing sequence $\left\langle n(k)\colon\ k \in \omega\right\rangle$ of natural numbers such that for every $k \in \omega$

$$\left|\mu_{n(k)}\left(a_k^\gamma\right)\right| > \sum_{i=0}^{k-1}\left|\mu_{n(k)}\left(a_i^\gamma\right)\right| + k + 1.$$

Since $\left\langle a_n^\gamma\colon\ n \in \omega\right\rangle$ is $\mathcal{G}$-complete in $\mathcal{B}$, there exists $\xi < \mathfrak{n}_e$ such that for every $k \in A_\xi$ we have:

$$\left|\mu_{n(k)}\right|\Big(\bigvee_{\substack{i>k\\ i\in A_\xi}} a_i^\gamma\Big) < 1,$$

and hence for every $k \in A_\xi$ we obtain the following sequence of inequalities:

$$\left|\mu_{n(k)}\left(b_\xi^\gamma\right)\right| = \Big|\sum_{\substack{i<k\\ i\in A_\xi}} \mu_{n(k)}\left(a_i^\gamma\right) + \mu_{n(k)}\left(a_k^\gamma\right) + \mu_{n(k)}\Big(\bigvee_{\substack{i>k\\ i\in A_\xi}} a_i^\gamma\Big)\Big| \geq$$

$$\left|\mu_{n(k)}\left(a_k^\gamma\right)\right| - \sum_{\substack{i<k\\ i\in A_\xi}} \left|\mu_{n(k)}\left(a_i^\gamma\right)\right| - \left|\mu_{n(k)}\Big(\bigvee_{\substack{i>k\\ i\in A_\xi}} a_i^\gamma\Big)\right| >$$

$$\Big(\sum_{i<k} \left|\mu_{n(k)}\left(a_i^\gamma\right)\right| + k + 1\Big) - \sum_{\substack{i<k\\ i\in A_\xi}} \left|\mu_{n(k)}\left(a_i^\gamma\right)\right| - \left|\mu_{n(k)}\right|\Big(\bigvee_{\substack{i>k\\ i\in A_\xi}} a_i^\gamma\Big) >$$

$$k+1-1=k.$$

Thus, we have found an element $b = b_\xi^\gamma$ of $\mathcal{B}$ such that

$$\sup_n \left|\mu_n(b)\right| \geq \sup_k \left|\mu_{n(k)}(b)\right| = \infty,$$

which contradicts the fact that $\overline{\mu}$ is anti-Nikodym on $\mathcal{B}$.

Finally, notice that if there are no anti-Nikodym sequences of real-valued measures on $\mathcal{B}$, then there are not any anti-Nikodym sequences of complex-valued measures on $\mathcal{B}$ as well. To see this, assume there is an anti-Nikodym sequence $\langle \mu_n \colon\ n \in \omega\rangle$ of complex-valued measures. Since it is elementwise bounded, its real and imaginary parts, $\langle\, \mathrm{Re}\,(\mu_n) :\ n \in \omega\rangle$ and $\langle\, \mathrm{Im}\,(\mu_n) :\ n \in \omega\rangle$ respectively, are also elementwise bounded. As these are sequences of real-valued measures, they are uniformly bounded. The triangle inequality implies that $\langle \mu_n \colon\ n \in \omega\rangle$ is also uniformly bounded.

Since $\mathcal{B}$ has no anti-Nikodym sequence, $\mathcal{B}$ has the Nikodym property. $\square$

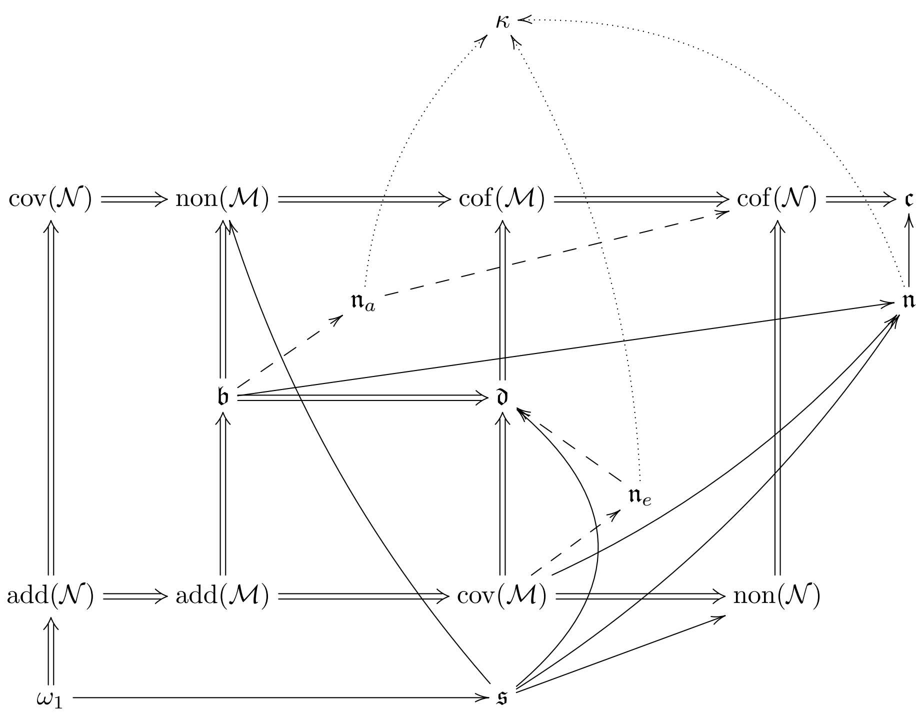

Figure 1. A diagram to Theorems 7.2 and 7.3 (note that we assume that $\mathrm{cof}\left([\kappa]^\omega\right) = \kappa$). Cichoń's diagram is marked with double arrows. A single continuous arrow means that the inequality may be consistently strict while a dashed arrow means that we do not know that.

In terms of $\mathfrak{n}$, Theorem 7.2 states that if $\max(\mathfrak{n}_a, \mathfrak{n}_e) \leq \kappa$ for a cardinal number κ such that $\operatorname{cof}\left([\kappa]^\omega\right) = \kappa$, then $\mathfrak{n} \leq \kappa$. Note that since $\operatorname{cof}(\mathcal{N}) \geq \max(\mathfrak{n}_a, \mathfrak{n}_e)$, as a corollary we obtain the theorem announced in the introductory section.

Theorem 7.3. *Assume that* $\operatorname{cof}(\mathcal{N}) \leq \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}\left([\kappa]^\omega\right) = \kappa$*. Then, there exists a Boolean algebra* $\mathcal{B}$ *with the Nikodym property and of cardinality* κ*.* □

Note that in the above construction, we did not assume that $\kappa < \mathfrak{c}$ — this assumption is only necessary to obtain an algebra of cardinality strictly less than continuum; e.g. as $\operatorname{cof}\left([\omega_1]^\omega\right) = \omega_1$, we have immediately the following corollary.

Corollary 7.4. *Assuming* $\operatorname{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$*, there exists a Boolean algebra with the Nikodym property and of cardinality* $\omega_1 < \mathfrak{c}$*.* □

Note that the assumption $\operatorname{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$ is satisfied e.g. in the Sacks model, see Blass [4, Section 11.5].

Corollary 7.5. *The existence of an infinite Boolean algebra with the Nikodym property and of cardinality strictly less than* $\mathfrak{c}$ *is independent of* ZFC + ¬CH. □

8. Consequences and open problems

In the previous section, under the assumption that $\operatorname{cof}(\mathcal{N}) \leq \kappa$ for κ such that $\operatorname{cof}\left([\kappa]^\omega\right) = \kappa$, we have constructed an example of a Boolean algebra with the Nikodym property and of cardinality κ. Since the assumption that $\operatorname{cof}\left([\operatorname{cof}(\mathcal{N})]^\omega\right) = \operatorname{cof}(\mathcal{N}) < \mathfrak{c}$ is satisfied in certain models of ZFC, we obtain that a system of the inequalities $\mathfrak{n} \leq \operatorname{cof}(\mathcal{N}) < \mathfrak{c}$ is relatively consistent. In connection to Question 3.5 we ask the following.

Question 8.1. *Is the equality* $\mathfrak{n} = \operatorname{cof}(\mathcal{N})$ *true?*

Mejía [33, Section 6.1] proved that for any regular uncountable cardinal κ in some model of ZFC, there is a ZFC extension of this model in which

$$\omega_1 < \operatorname{cov}(\mathcal{M}) = \operatorname{cof}(\mathcal{N}) = \kappa < \mathfrak{c}.$$

Since $\operatorname{cov}(\mathcal{M}) \leq \mathfrak{n}$ by Proposition 3.1, this implies that it is relatively consistent that

$$\omega_1 < \operatorname{cov}(\mathcal{M}) = \mathfrak{n} = \operatorname{cof}(\mathcal{N}) < \mathfrak{c}.$$

8.1. **Cofinalities of Boolean algebras.**

If $\mathcal{A}$ is an infinite Boolean algebra, then *the cofinality* $\operatorname{cof}(\mathcal{A})$ of $\mathcal{A}$ is the least cardinal number κ such that $\mathcal{A}$ is a union of strictly increasing sequence of length κ of subalgebras of $\mathcal{A}$ and *the homomorphism type* $h(\mathcal{A})$ of $\mathcal{A}$ is the least cardinality of an infinite homomorphic image of $\mathcal{A}$. Koppelberg [31] proved that for every infinite Boolean algebra $\mathcal{A}$ the following inequalities hold:

$$\omega \leq \operatorname{cof}(\mathcal{A}) \leq h(\mathcal{A}) \leq \mathfrak{c}.$$

Moreover, under Martin's axiom, she proved that $\operatorname{cof}(\mathcal{A}) = h(\mathcal{A}) = \omega$ whenever $\omega \leq |\mathcal{A}| < \mathfrak{c}$.

Just and Koszmider [28] showed that the assumption of Martin's axiom in Koppelberg's result is important — they constructed a model of ZFC (being a variant of the Sacks model) in which there is a Boolean algebra $\mathcal{A}$ such that

$$\omega < |\mathcal{A}| = h(\mathcal{A}) = \operatorname{cof}(\mathcal{A}) = \omega_1 < \mathfrak{c}.$$

Ciesielski and Pawlikowski [9] improved the result of Just and Koszmider and proved the existence of such an algebra assuming only that $\operatorname{cof}(\mathcal{N}) = \omega_1$ (which again holds e.g. in the Sacks model).

Schachermayer [40, Proposition 4.6] proved that if an infinite Boolean algebra $\mathcal{A}$ has the Nikodym property, then $\operatorname{cof}(\mathcal{A}) > \omega$. Since for every infinite subset $\mathcal{F} \subseteq \mathcal{A}$ the algebra generated by $\mathcal{F}$ has the same cardinality as $\mathcal{F}$, this yields the following immediate consequence concerning the cofinality of $\mathfrak{n}$, already announced in Remark 3.6.

Proposition 8.2. $\operatorname{cf}(\mathfrak{n}) > \omega$. □

Schachermayer also proved that the class of Boolean algebras with the Nikodym property is closed under homomorphic images ([40, Proposition 2.11]). This immediately implies that for every infinite Boolean algebra $\mathcal{A}$ with the Nikodym property it holds that $h(\mathcal{A}) \geq \mathfrak{n}$. Moreover, since for the algebra $\mathcal{B}$ constructed in the proof of Theorem 7.3 we have $\operatorname{cof}(\mathcal{B}) \leq \omega_1$, we immediately obtain the following generalization of the above-mentioned result of Pawlikowski and Ciesielski.

Corollary 8.3. *Assume that* $\operatorname{cof}(\mathcal{N}) \leq \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}\big([\kappa]^\omega\big) = \kappa$. *Then, there exists a Boolean algebra* $\mathcal{A}$ *such that* $|\mathcal{A}| = \kappa$, $h(\mathcal{A}) \geq \mathfrak{n}$ *and* $\operatorname{cof}(\mathcal{A}) = \omega_1$. □

8.2. **The Efimov problem.** Recall that the problem reads as follows.

Problem 8.4 (Efimov [18])**.** *Does every infinite compact space contain either a non-trivial convergent sequence or a copy of* $\beta\omega$*, the Čech–Stone compactification of* ω*?*

Thus, an infinite compact space without any non-trivial convergent sequences and any copy of $\beta\omega$ is called *a Efimov space.*

Note that for a given Boolean algebra $\mathcal{A}$, if its Stone space $K_\mathcal{A}$ contains a non-trivial convergent sequence, then $h(\mathcal{A}) = \omega$. Indeed, if $\langle x_n \in K_\mathcal{A}\colon\ n \in \omega\rangle$ is non-trivial and converges to $x \in K_\mathcal{A}$, then $\{x_n\colon\ n \in \omega\} \cup \{x\}$ is a closed countable subset of $K_\mathcal{A}$, and hence by the Stone duality $h(\mathcal{A}) = \omega$. Also recall that $w\big(K_\mathcal{A}\big) = |\mathcal{A}|$ and $w(\beta\omega) = \mathfrak{c}$. Thus, the Stone spaces of the Boolean algebras constructed by Just and Koszmider and by Ciesielski and Pawlikowski, mentioned in the previous section, are also Efimov spaces. Similarly, if in Theorem 7.3 we assume that $\kappa < \mathfrak{c}$, then the Stone space $K_\mathcal{B}$ of the constructed algebra $\mathcal{B}$ is a Efimov space as well. Moreover, due to the Stone duality, $K_\mathcal{B}$ has an additional feature: its every infinite closed subset has large weight.

Corollary 8.5. *Assume that* $\operatorname{cof}(\mathcal{N}) \leq \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}\big([\kappa]^\omega\big) = \kappa < \mathfrak{c}$. *Then, there exists a Efimov space* K *such that the weight* $w(K) = \kappa$ *and for every infinite closed subset* L *of* K *we have* $w(L) \geq \mathfrak{n}$. □

By Geschke's result (see the proof of Proposition 3.1) and Proposition 2.9, given a cardinal number κ under Martin's axiom $\mathrm{MA}_\kappa(\textit{countable})$ every Efimov space has weight strictly greater than κ.

Corollary 8.6. *Let* κ *be a cardinal number. Assume* $\mathrm{MA}_\kappa(\textit{countable})$*. Then, every Efimov space* K *has weight* $w(K) > \kappa$*. In particular, if* L *is a closed infinite subset of a Efimov space, then* $w(L) > \kappa$*.* □

Corollary 8.7. *Assuming Martin's axiom, every Efimov space is of weight at least* $\mathfrak{c}$*. In particular, if* L *is a closed infinite subset of a Efimov space, then* $w(L) \geq \mathfrak{c}$*.* □

Note that Dow and Shelah [17] proved that assuming Martin's axiom there do exist Efimov spaces of weight $\mathfrak{c}$.

8.3. **The Nikodym property for C*-algebras.** In the introductory section of this paper the generalization of the Nikodym property to C*-algebras in terms of sequences of functionals bounded on projections was presented (Definition 1.4).

Note that this generalization coincides with the Nikodym property for Boolean algebras in the case of $\mathscr{A} = C\big(K_{\mathcal{A}}\big)$, where $\mathcal{A}$ is a Boolean algebra. A simple use of the Hahn–Banach theorem (Rudin [37, Theorem 3.5]) shows that if a C*-algebra $\mathscr{A}$ has the Nikodym property, then its set of projections $\operatorname{Proj}(\mathscr{A})$ must be linearly dense in $\mathscr{A}$. This implies e.g. that if a compact space K has a non-degenerate connected component, then $C(K)$ does not have the Nikodym property. On the other hand, since for every $n \in \omega$ the C*-algebra $B(\mathbb{C}^n)$ of bounded operators on the n-dimensional Hilbert space $\mathbb{C}^n$ contains a Hamel basis consisting of orthogonal projections, by the Banach–Steinhaus theorem $B(\mathbb{C}^n)$ has the Nikodym property.

The next proposition, which is a version of Proposition 3.2 for C*-algebras, yields that the set $\operatorname{Proj}(\mathscr{A})$ cannot be too small if $\mathscr{A}$ is infinite-dimensional.

Proposition 8.8. *If an infinite-dimensional C*-algebra $\mathscr{A}$ has the Nikodym property, then $|\operatorname{Proj}(\mathscr{A})| \geq \mathfrak{b}$.*

Proof. The assertion can be proved in the exactly same way as Proposition 3.2 replacing only the space $B_s(\mathcal{A})$ with the space $\operatorname{span}\operatorname{Proj}(\mathscr{A})$. However, we present below a completely different approach to the proof.

Recall that the Josefson–Nissenzweig theorem states that for every infinite-dimensional Banach space X there exists a weak* null sequence of norm-one functionals in X^* (see Diestel [12, Chapter XII]). Let $\big\langle x_n^* \in \mathscr{A}^*\colon\ n \in \omega\big\rangle$ be such a sequence. For every $p \in \operatorname{Proj}(\mathscr{A})$ define a sequence $c_p \in \mathbb{R}_+^\omega$ as follows:

$$c_p(n) = \big|x_n^*(p)\big| + 1/n$$

(the $1/n$ summand is contributed to shift c_p away from 0). Then, $\lim_{n\to\infty} c_p(n) = 0$.

Assume that $|\operatorname{Proj}(\mathscr{A})| < \mathfrak{b}$. It is not difficult to see that if $\mathcal{F} \subseteq \mathbb{R}_+^\omega$ is a family of sequences converging to 0 of cardinality strictly less than $\mathfrak{b}$, then there is $c \in \mathbb{R}_+^\omega$ converging to 0 and dominating every element of $\mathcal{F}$. Thus, since $|\operatorname{Proj}(\mathscr{A})| < \mathfrak{b}$, there is such an element $c \in \mathbb{R}_+^\omega$ for the family $\mathcal{F} = \big\{c_p\colon\ p \in \operatorname{Proj}(\mathscr{A})\big\}$. Now, define a sequence $\big\langle y_n^* \in \mathscr{A}^*\colon\ n \in \omega\big\rangle$ as follows:

$$y_n^* = \frac{1}{c(n)} x_n^*.$$

Then $\|y_n^*\|$ tends to ∞, but $\big\langle y_n^*(p)\colon\ n \in \omega\big\rangle$ is bounded for every $p \in \operatorname{Proj}(\mathscr{A})$, which proves that $\mathscr{A}$ does not have the Nikodym property. □

Corollary 8.9. *Under Martin's axiom, every infinite-dimensional C*-algebra with the Nikodym property has at least $\mathfrak{c}$ many projections.* □

Remark 8.10. Let us note that if H is a separable infinite-dimensional Hilbert space (e.g. ℓ_2), then the von Neumann algebra $B(H)$ of all bounded operators on H has the Nikodym property, since due to Darst [11] all von Neumann algebras do. On the other hand, it can be shown that the subalgebra $K(H) \subseteq B(H)$ of all compact operators on H has $\mathfrak{c}$ many projections (since it contains the algebra $B\big(\mathbb{C}^2\big)$) yet not the Nikodym property.

For an infinite-dimensional C*-algebra $\mathscr{A}$ its *cofinality* $\operatorname{cof}(\mathscr{A})$ is the least cardinal κ for which there exists a strictly increasing sequence $\big\langle \mathscr{A}_\xi\colon\ \xi < \kappa\big\rangle$ of C*-subalgebras of $\mathscr{A}$ such that $\bigcup_{\xi<\kappa} \mathscr{A}_\xi$ is dense in $\mathscr{A}$, and *the homomorphism type* $h(\mathscr{A})$ of $\mathscr{A}$ is the minimal density of an infinite-dimensional *-homomorphic image of $\mathscr{A}$. For an infinite Boolean algebra $\mathcal{A}$, Geschke [23, Lemma 1.3] proved that $\operatorname{cof}\big(C\big(K_{\mathcal{A}}\big)\big) = \operatorname{cof}(\mathcal{A})$. A similar result also holds for the homomorphism type.

Lemma 8.11. *For an infinite Boolean algebra $\mathcal{A}$, $h\big(C\big(K_{\mathcal{A}}\big)\big) = h(\mathcal{A})$.*

Proof. Let $\mathcal{B}$ be an infinite Boolean algebra such that there exists a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then by the Stone duality $K_{\mathcal{B}} \subseteq K_{\mathcal{A}}$. Due to the Tietze theorem the *-homomorphism $T\colon C\big(K_{\mathcal{A}}\big) \to C\big(K_{\mathcal{B}}\big)$ given by the formula $T(f) = f \restriction K_{\mathcal{B}}$ is surjective. Since dens $\big(C\big(K_{\mathcal{B}}\big)\big) = |\mathcal{B}|$, $h\big(C\big(K_{\mathcal{A}}\big)\big) \leq h(\mathcal{A})$.

Let now Y be an infinite-dimensional C*-algebra such that there exists a surjective *-homomorphism $T\colon C(K_{\mathcal{A}}) \to Y$. Since T is a homomorphism, Y must be commutative and hence of the form $Y = C\big(K_{\mathcal{B}}\big)$ for some Boolean algebra $\mathcal{B}$ (note that a *-homomorphism transforms projections onto projections). T induces a homomorphism $\varphi_T\colon \mathcal{A} \to \mathcal{B}$ such that for given $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have $\varphi_T(a) = b$ if and only if $T\left(\chi_{[a]}\right) = \chi_{[b]}$. This implies that $h\big(C\big(K_{\mathcal{A}}\big)\big) \geq h(\mathcal{A})$. $\square$

By Theorem 7.3, Corollary 8.3 and Lemma 8.11, we obtain the following result.

Corollary 8.12. *Assume that* $\operatorname{cof}(\mathcal{N}) \leq \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}\left([\kappa]^\omega\right) = \kappa$. *Then, there exists a commutative C*-algebra* $\mathscr{A}$ *with the Nikodym property and such that* $\operatorname{dens}(\mathscr{A}) = \kappa$, $h(\mathscr{A}) \geq \mathfrak{n}$ *and* $\operatorname{cof}(\mathscr{A}) = \omega_1$. $\square$

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