F_σ-ideals, colorings, and representation in Banach spaces

J. Lopez-Abad

Introduction (I)

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This talk is based on joint work with Víctor Olmos (as part of his PhD thesis) and C. Uzcátegui.

Structure of the talk

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3. Representing non-pathological F_σ-ideals on spaces of continuous functions.
4. Coloring ideals; examples of pathological and non-pathological c-coloring ideals; universal d-colorings; tall colorings; the block sequence coloring and c_0.

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3. *Pathological ideals.*

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A is a "pseudo-union" of the A_n's

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using a lower semicontinuous submeasure φ. This provides a classification parallel to F_σ-ideals into summable, non-pathological, and pathological types.

$\mathcal{B}$ and $\mathcal{C}$-representation

L. Drewnowski and I. Labuda defined ideals associated with sequences in Banach spaces:

$$\mathcal{C}(\mathbf{x}) = \{A \subseteq \mathbb{N} : \sum_{n \in A} x_n \text{ is unconditionally convergent}\},$$

and studied the conditions under which these ideals are F_σ-ideals.
Drewnowski and Labuda's result states that a Banach space does not contain c_0 exactly when every ideal $\mathcal{C}(\mathbf{x})$ is F_σ.

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$$\mathcal{B}(\mathbf{x}) = \{A \subseteq \mathbb{N} : \sum_{n \in A} x_n \text{ is weakly-unconditionally convergent}\}$$

$$= \{A \subseteq \mathbb{N} : \sup_{F \subseteq A,\, F \text{ finite}} \|\sum_{n \in F} x_n\| < \infty\}$$

$\mathcal{B}$-representation

Martínez et al. characterized $\mathcal{B}(\mathbf{x})$-ideals as non-pathological F_σ-ideals: Given $\mathcal{I} = \mathrm{FIN}(\varphi)$ with φ a non pathological l.s.c. submeasure $\varphi = \sup_k \varphi_k$ we define for each n the sequence

$$x_n := (\varphi_k(\{n\}))_k \in \ell_\infty$$

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We have then that

$$\|\sum_{n\in F} x_n\|_\infty = \sup_{k\in\mathbb{N}} (\sum_{n\in F} x_n)(k) = \sup_{k\in\mathbb{N}} \sum_{n\in F} x_n(k) =$$
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Similarly one sees that for non-pathological analytic P-ideals $\mathrm{Exh}(\varphi) = \mathcal{C}((x_n)_n)$. However, representations of the ideals $\mathcal{B}((x_n))$ and $\mathcal{C}((x_n))$ in the universal space ℓ_∞ are not optimal for a descriptive set-theoretic analysis. To overcome this limitation, we present effective representations of these ideals within the universal Polish spaces $C([0,1])$ and $C(2^{\mathbb{N}})$.

Tall Ideals and c_0-Saturation

The following is a classical characterization that uses the uniform boundedness principle.

Proposition

A series $\sum_{n\in\mathbb{N}} x_n$ in a Banach space X is weakly unconditionally convergent exactly when the numerical series $\sum_n x^*(x_n)$ converge unconditionally for every $x^* \in X^*$. Consequently, if the ideal $\mathcal{B}(\mathbf{x})$ is tall, then the sequence $\mathbf{x}$ is weakly null. □

Recall that an ideal of the form $\mathcal{C}(\mathbf{x})$ is tall if and only if $\mathbf{x}$ is norm-null (i.e. norm-converging to zero). Moreover, using a classical Bessaga-Pełczyński's theorem

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Theorem (Drewnowski and Labuda)

If c_0 does not embed in X, then $\mathcal{B}(\mathbf{x}) = \mathcal{C}(\mathbf{x})$ for every sequence, hence we have the same characterization of tallness.

However, using the well-known fact that weakly unconditional bases are precisely the sequences equivalent to the unit basis of c_0, we can show a general characterization which explicitly takes c_0 into account.

Theorem

Let $\mathbf{x} = (x_n)_n$ be a sequence in a Banach space X. The following are equivalent.

1. *$\mathcal{B}(\mathbf{x})$ is tall.*
2. *Every subsequence of $\mathbf{x}$ has a further subsequence that is either norm-null or equivalent to the unit basis of c_0.*

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Consequently,

3. *if $(x_n)_n$ does not have subsequences equivalent to the unit basis of c_0, then $\mathcal{B}(\mathbf{x})$ is tall exactly when $\mathbf{x}$ is a norm-null sequence, and*
4. *if X is isomorphic to a subspace of c_0, then $\mathcal{B}(\mathbf{x})$ is tall exactly when $\mathbf{x}$ is weakly-null.*

Summability

R. Filipów and J. Tryba recently proved in [4, Corollary 11.3] that every non-trivial $\mathcal{B}$-ideal can be extended to a non-trivial summable ideal.

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for every sequence $(y_k)_{k=1}^n$ in a Hilbert space we have that

$$\mathbb{E}_{(\theta_k)_k\in\{-1,1\}^n}\left(\|\sum_{k=1}^n \theta_k y_k\|^2\right) := \frac{1}{2^n}\sum_{(\theta_k)_k\in\{-1,1\}^n}\|\sum_{k=1}^n \theta_k y_k\|^2 = \sum_{k=1}^n \|y_k\|^2.$$

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Proposition

An ideal can be $\mathcal{B}$-represented in a Hilbert space by an unconditional basic sequence exactly when it is summable.

Non-trivial cotype

It is natural to ask for properties similar the generalized parallelogram identity in a given Banach space. Recall that a Banach space X has *cotype* $1 \leq q \leq \infty$ if there is a constant C such that for every finite sequence $(x_k)_{k=1}^n$ one has that

$$\sum_{k=1}^{n} \|x_k\|^q \leq C\mathbb{E}_\theta \left(\| \sum_{k=1}^{n} \theta_k x_k \|^q \right), \tag{1}$$

where when for $q = \infty$ the previous inequality has to be interpreted as $\max_{k=1}^n \|x_k\| \leq C\mathbb{E}_\theta \| \sum_{k=1}^n \theta_k x_k \|$. This means, by a simple use of the triangle inequality, that every space has cotype ∞, and by using constant sequences, that cotypes q are always at least 2. A Banach space X has *non trivial cotype* when it has cotype q for some $q < \infty$. We have the following.

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Proposition

For every sequence $\mathbf{x} = (x_n)_n$ in a space X with non-trivial cotype q one has that

$$\mathrm{Sum}((\|x_n\|)_n) \subseteq \mathcal{B}(\mathbf{x}) \subseteq \mathrm{Sum}((\|x_n\|^q)_n).$$

Cotype

A well-known result by B. Maurey and G. Pisier states that a space X has non-trivial cotype exactly when c_0 is not *finitely representable*, that, in the case of c_0, means that there is no constant C such that every ℓ_∞^n has C-isomorphic subspace of X. Combining this with the Drewnowski-Labuda Theorem we obtain the following.

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Corollary

If c_0 is not finitely representable in X, then there is some $2 \leq q < \infty$ such that

$$\mathrm{Sum}((\|x_n\|)_n) \subseteq \mathcal{B}(\mathbf{x}) = \mathcal{C}(\mathbf{x}) \subseteq \mathrm{Sum}((\|x_n\|^q)_n)$$

for every sequence $\mathbf{x} = (x_n)_n$ in X.

Representing on $C(2^{\mathbb{N}})$

We see how to effectively $\mathcal{B}$-represent an ideal $\mathcal{B}(\varphi)$, $\varphi = \sup_k \varphi_k$ on $C(2^{\mathbb{N}})$.

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$$T := \{\langle \varphi_k(0), \dots, \varphi_k(n) \rangle : k, n \in \mathbb{N}\}.$$

Since each L_n is finite, T is clearly finitely branching. Notice that each measure φ_k is a branch of T. Let $\rho : T \to 2^{<\omega}$ be an embedding, that is, $\rho(s) \prec \rho(s')$ iff $s \prec s'$, for all $s, s' \in T$.

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$$g_n := \sum_{s \in T, |s| = n+1} s(n) \mathbb{1}_{[\rho(s)]}. \tag{2}$$

Representing on $C(\alpha)$

We have observed the close relationship between tall $\mathcal{B}$-ideals and the space c_0. Thus, it is natural to expect that ideals represented in separable, c_0-saturated $C(K)$ spaces exhibit distinctive structural properties.

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i.e., $C(K)$ with K countable

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there is a borel $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{I}$ such that $\varphi(A) \subseteq A$ for every $A \in \mathcal{P}(\mathbb{N})$ (K)

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From where does the Borel choice function arise? Precisely from *c-coloring ideals*.

Coloring ideals

Given a Borel coloring $c : [\mathbb{N}]^\infty \to r$, the ideal $\mathrm{Hom}(c)$ generated by homogeneous sets is always tall.

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Galvin–Prikry theorem

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Proposition

Every tall F_σ ideal is an open coloring ideal.

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homogeneous sets is alway

a 2-coloring where one of the two colors is open

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Every tall coloring ideal $\mathcal{I}$ is of this form $\mathrm{Hom}(c)$, $c := \mathbb{1}_{\mathcal{I}}$.

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c-coloring ideals

Definition

An ideal is a *continuous coloring ideal (c-coloring ideal in short)* when it is of the form $\langle \mathrm{Hom}(c) \rangle$ for some continuous coloring $c : [\mathbb{N}]^\omega \to r \in \mathbb{N}$ or, equivalently, of the form $\langle \hom(\mathcal{c}) \rangle$ for some $\mathcal{c} : \mathcal{F} \to r$ defined on a front $\mathcal{F}$.

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Corollary

An ideal that admits a $\mathcal{B}$-representation in c_0 is tall if and only if it contains a c-coloring ideal induced by a coloring $c : [\mathbb{N}]^2 \to 2$.

Are c-coloring ideals non-pathological?

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Every tall ideal $\mathcal{I}$ that admits a $\mathcal{B}$-representation in a space $C(K)$, with K countable contains a non-pathological *c-coloring ideal.*

Are c-coloring ideals non-pathological?

A coloring $c : \mathcal{F} \to r$ on a family $\mathcal{F}$ of subsets of X induces r hypergraphs $\mathcal{H}_i(c)$ over X, where each $\mathcal{H}_i$ consists of the sets in $\mathcal{F}$ colored with i. When $\mathcal{F} \subseteq [X]^d$, these hypergraphs are d-uniform. If $\mathcal{F} = [\mathbb{N}]^d$ and $r = 2$, then $\mathcal{H}_0$ and $\mathcal{H}_1$ are complements.

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Question

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The c-coloring ideal $\langle \text{hom}(c) \rangle$ for a 2-coloring coincides with the ideal generated by cliques and anticliques of the hyperedges. More generally, for an r-coloring, it is the ideal of complete sub-hypergraphs of each $\mathcal{H}_i$.

non-pathological c-colorings

We do not know the full answer to this question, but we have some partial ones, for some colorings $c : [\mathbb{N}]^2 \to 2$ so that $c(\{m, n\}_<) = i$ induces a partial ordering on $\mathbb{N}$. To analyze these partial orders we will use *Dilworth's theorem.*

non-pathological c-colorings

For a partially ordered set $\mathcal{P}$, the supremum of the cardinalities of its antichains, when finite, equals the minimum number of chains needed to cover $\mathcal{P}$. The dual of Dilworth theorem, called *Mirsky's theorem* is also true.

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Theorem

Fix a coloring $c : [\mathbb{N}]^2 \to 2$ such that $m < n$ and $c(m,n) = i$ defines a partial ordering on $\mathbb{N}$. The corresponding c-coloring ideal $\hom(c)$ is non-pathological when

$$\hom_i(c) \subseteq \mathit{FIN} \text{ or when } \hom_{1-i}(c) \subseteq \mathit{FIN}.$$

Of particular interest are c-colorings $c : [\mathbb{N}]^2 \to r$ where both colors define a partial ordering. We call them *Sierpinski* colorings.

universal 2-colorings

A $\mathbb{Q}$-*coloring* is a Sierpinski coloring $\widehat{\theta} : [\mathbb{N}]^2 \to 2$ associated to some enumeration $\theta : \mathbb{N} \to \mathbb{Q}$, that is, $\widehat{\theta}(\{m, n\}_<) := 1$ exactly when $\theta(m) < \theta(n)$. The Sierpinski ideal associated to θ, $\langle \hom(\widehat{\theta}) \rangle$, will be called a $\mathbb{Q}$-*coloring ideal*.

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Theorem (Universality)

Suppose that $\theta : \mathbb{N} \to \mathbb{Q}$ is an enumeration. Then for every coloring $c : [\mathbb{N}]^2 \to l$ there is some $M \subseteq \mathbb{N}$ such that $\theta(M)$ is order-isomorphic to $\mathbb{Q}$ and such that $\hom(\widehat{\theta}) \restriction M \subseteq \hom(c)$.

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The proof uses crucially the following Galvin's Theorem

Theorem

For every coloring $c : [\mathbb{Q}]^2 \longrightarrow l$, there is $X \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ such that $|c[X]^2| \leq 2$.

universal d-colorings

This can be extended to any dimension d using the concept of *Devlin types*.

Theorem

For every $d \in \mathbb{N}$ there is a number $t_d \in \mathbb{N}$ satisfying:

1. *If $c : [\mathbb{Q}]^d \to r$ is an arbitrary (finite) coloring, then there is some order-isomorphic copy X of $\mathbb{Q}$ such that c uses at most t_d colors on $[X]^d$.*
2. *There exists a coloring $c_d : [\mathbb{Q}]^d \to t_d$ such that if $X \subseteq \mathbb{Q}$ is order-isomorphic to $\mathbb{Q}$, then c_d uses all colors on $[X]^d$.*

The number t_d represents the count of canonical patterns of arbitrary d points on the binary tree $2^{<\mathbb{N}}$. It is interesting to note that the sequence $(t_d)_d$ is a well known sequence of numbers known as the *odd tangent numbers*, because $t_d = T_{2d-1}$, where $\tan(z) = \sum_{n=0}^{\infty}(T_n/n!)z^n$.

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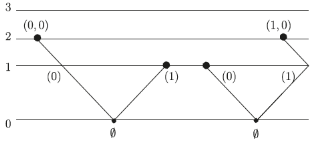

Figure 6.4 Devlin embedding types for $k = 2$.

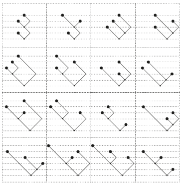

Figure 6.6 Devlin embedding types for $k = 3$.

1. *If* $c : [\mathbb{Q}]$ $\to r$ *is an arbitrary (finite) coloring, the* … *there is some order-isomorphic copy* X *of* $\mathbb{Q}$ *such that* c *uses at most* t_d *colors on* $[X]^d$.
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A $\mathbb{Q}_d$-*coloring* is some $\widehat{\theta}^d := c_d \circ \theta^n : [\mathbb{N}]^d \to t_d, \{n_j\}_{j<d} \mapsto c_d(\{\theta(n_j)\}_{j<d})$ for a bijection $\theta : \mathbb{N} \to \mathbb{Q}$.

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Pathological c-colorings

Recall that, in this context, a submeasure on X is called *non-pathological* when

$$\varphi = \sup\{\mu \ : \ \mu \text{ is a measure on } X \text{ such that } \mu \leq \varphi\}.$$

Pathological c-colorings

When φ is lower semicontinuous, these measures can be chosen to be σ-additive.

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The first known example of a pathological ideal is *Mazur's ideal*, which we now proceed to describe.

Mazur-like ideals

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1. One can define the *submeasure covering* $\psi_{\mathcal{S}}$ associated to $\mathcal{S}$ for $A \subseteq X$ by

$$\psi_{\mathcal{S}}(A) = \min\{\#\mathcal{T} : A \subseteq \bigcup \mathcal{T}, \mathcal{T} \subseteq \mathcal{S}\}.$$

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$$\delta(X, \mathcal{S}) := \frac{1}{\#\mathcal{S}} \min_{x \in X} \#\{S \in \mathcal{S} : x \in S\}.$$

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3. Certain submeasure coverings exhibit extreme violations of (3).

Mazur-like ideals

4. By amalgamating these covering submeasures, Mazur constructs a submeasure ψ on a countable set E, yielding the F_σ-ideal $\mathrm{Fin}(\psi)$. This ideal is *super-pathological*, meaning that it is not contained in any non-pathological F_σ-ideal.

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5. The index set X is a collection of subsets of a fixed finite set E not containing E. The family

$$\mathcal{S} := \{\widehat{e}_X \ : \ e \in E\}$$

is a covering of X, where

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6. We fix some $0 \leq \beta < 1$, a disjoint sequence $(E_n)_n$ of arbitrarily large finite sets and $X_n := [E_n]^{\leq \beta \# E_n}$. Then each covering submeasure ψ_n associated to the covering $\mathcal{S}_n$ guarantees that Kelley's fact fails badly, so that one can amalgamate the ψ_n's to obtain ψ on $\bigsqcup_n X_n$, defined by $\psi(A) := \sup_n \psi_n(A \cap X_n)$, to obtain a super pathological F_σ ideal.

Mazur-like ideals $\rightsquigarrow$ Mazur-coloring ideals

Observe that Mazur's ideal is generated by sets of the form $\bigsqcup_n (\widehat{e_n})_{X_n}$ for some sequence $(e_n)_n \in \prod_n E_n$ because each $\psi(\widehat{e}_X) = 1$, so $\psi(\bigsqcup_n (\widehat{e_n})_{X_n}) = 1$.

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Mazur-like ideals ⇝ Mazur-coloring ideals

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For example, fix $d \in \mathbb{N}$, recall that $X_n = [E_n]^{\leq \beta \# E_n}$. We define $\mathscr{c}_n : [X_n]^d \to 2$ for $\{A_j\}_{j=1}^d, A_j \in [E_n]^{\leq \beta \# E_n}$, by

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For example, fix $d \in \mathbb{N}$, recall that $X_n = [E_n]^{\leq \beta \# E_n}$. We define $c_n : [X_n]^d \to 2$ for $\{A_j\}_{j=1}^d, A_j \in [E_n]^{\leq \beta \# E_n}$, by

$$c_X(\{A_j\}_{j=1}^d) := \begin{cases} 1 & \text{if } \psi_n(\{A_j\}_{j=1}^d) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $c_n(\{A_j\}_j) = 1$ exactly when $\{A_j\}_j$ is not a covering of E_n.

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We freely amalgamate them: We define for the family $X := \bigsqcup_n X_n$ of subsets of $\bigsqcup_n E_n$ the *Mazur coloring* $c_X : [X]^d \to 2$ for $\{A_j\}_{j=1}^d \subseteq \mathcal{A}$ by

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Chromatic numbers

Observe that homogeneous of color zero must be finite because if $\{A_j\}_{j\in\mathbb{N}}$ is an infinite subset of X there must be $A_{j_1}, \cdots A_{j_d}$ belonging each of them to different X_n's so $c(\{A_{j_k}\}_{k=1}^{n}) = 1$.

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For each n, we have a d-uniform hypergraph $\mathcal{H}_n := (E_n, c_n^{-1}(0))$, and if the previous question has a positive answer then the chromatic number of $\mathcal{H}_n$ is at most r.

Theorem

For every $d \geq 3$, $0 < \beta < 1$, and $r \in \mathbb{N}$, there exists some $N = N(d, \beta, r) \in \mathbb{N}$ such that for every $n \geq N$,

$$\chi(\mathcal{H}_n) > r.$$

Chromatic numbers

The proof relies heavily on concentration of measure, applied to almost-centered families of finite sets.

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For every $d, r \geq 1$ *there exists a multiple* $\widetilde{n} = \widetilde{n}(d, r)$ *of* d *such that any* r*-coloring of* $[\widetilde{n}]^{\widetilde{n}/d}$ *contains a monochromatic set of cardinality* $d + r$ *that covers* $\widetilde{n}$*.*

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Is the ideal generated by the Random graph non-pathological?

Thanks!

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