Free objects in p-Banach lattices

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(joint work with P. Tradacete and N. Trejo Arroyo)

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- If we prefer, think of:
  - **Ban** = Banach spaces and (linear, continuous) operators.
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- There is a **forgetful** functor $\Box : \mathcal{D} \rightsquigarrow \mathcal{C}$.
- A **free functor** is (somewhat informally) a *left-adjoint* functor for $\Box$, that is, a functor $\mathcal{F} : \mathcal{C} \rightsquigarrow \mathcal{D}$ such that

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(C), D) \simeq \mathrm{Hom}_{\mathcal{C}}(C, \Box D)$$

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- For every morphism $f : C \to \square D$ (in $\mathcal{C}$) there is a unique morphism $\hat{f} : \mathcal{F}(C) \to D$ (in $\mathcal{D}$) such that $\hat{f}\delta = f$.

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3. The *Lipschitz free space*, as the adjoint of **Ban** $\rightsquigarrow$ $(\mathbf{Met}_{\{\bullet\}}, \mathrm{Lip})$ (Godefroy, Kalton).

Free Banach lattices

The *free Banach lattice* generated by a **Banach space** (Avilés, Rodríguez, Tradacete):

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Question: what about the *non-locally convex* setting?

Quasi-norms and p-norms

- A *quasi-norm* on a vector space E is a map $\|\cdot\| : E \to \mathbb{R}$ such that
  (1) $\|x\| = 0 \iff x = 0$.
  (2) $\|\lambda x\| = |\lambda| \cdot \|x\|$.
  (3) $\|x + y\| \leq C\,(\|x\| + \|y\|)$.

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- Given $p \in (0, 1]$, a p-norm $\|\cdot\| : E \to \mathbb{R}$ satisfies (1), (2) and
  (3') $\|x + y\|^p \leq \|x\|^p + \|y\|^p$.

- (Aoki-Rolewicz) Every quasi-norm is equivalent to some p-norm.

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- Every operator $T : E \to X$ uniquely extends to a lattice homomorphism $\widehat{T} : \mathrm{FpBL}[E] \to X$ with $\|\widehat{T}\| = \|T\|$.

Existence of $\mathrm{FpBL}[E]$

Theorem

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  - $\mathrm{FVL}[E]$ can be realized as the vector sublattice of $\{\delta_e : e \in E\}$ inside $\mathbb{R}^{E^\sharp}$.
  - Every *linear map* $T : E \to X$ has a unique extension to a *lattice-linear map* $\widehat{T} : \mathrm{FVL}[E] \to X$.

Existence of $\mathrm{FpBL}[E]$

- We put a "maximal" lattice p-seminorm on $\mathrm{FVL}[E]$ given by

$$\|f\|_{\mathrm{FpBL}} = \sup\{\|\widehat{T}f\|_X : T \in B_{\mathcal{L}(E,X)}\}$$

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We define $\mathrm{FpBL}[E]$ as the completion of such space.

Application to projectivity in pBan

A p-Banach lattice P is *projective* if given any lattice quotient $\pi : Z \to X$, any homomorphism $\tau : P \to X$ admits a lifting homomorphism $T : P \to Z$.

$$\begin{tikzcd}
X \arrow[r, two heads, "\pi"] & Z \\
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- Since $\ell_p(\Gamma)$ is a projective p-Banach *space*, $\mathrm{FpBL}[\ell_p(\Gamma)]$ is a projective p-Banach *lattice*.

$$\begin{tikzcd} X \arrow[r, two heads, "\pi"] & Z \\ & \ell_p(\Gamma) \arrow[u, "\tau"] \arrow[ul, bend left, "T"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, two heads, "\pi"] & Z \\ & \mathrm{FpBL}[\ell_p(\Gamma)] \arrow[u, "\widehat{\tau}"] \arrow[ul, bend left, "\widehat{T}"] \end{tikzcd}$$

Just let $\tau = \widehat{\tau}\delta$.

An application to projectivity in pBan

- Every complemented sublattice of a projective p-Banach lattice is a projective p-Banach lattice.
- As a consequence, one can show that ℓ_p is a projective p-Banach **lattice.**
- What about $\ell_p(\Gamma)$ for Γ uncountable?

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Definition

Let $p \in (0,1)$. A quasi-Banach lattice X is *p-convex* if there is $C > 0$ such that

$$\left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \leq C \left(\sum_{k=1}^{n} \|x_k\|^p\right)^{\frac{1}{p}}$$

We write $M^{(p)}(X)$ for the least of those constants C.

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We write $M^{(p)}(X)$ for the least of those constants C.

- A p-convex quasi-Banach lattice is p-normed.
- We say a quasi-Banach space is *natural* if it is embeddable in a p-convex quasi-Banach lattice for some $p \in (0,1)$.

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- $\delta : E \to \mathrm{FpBL}^{(p)}[E]$ is an into isometry.
- Every operator $T : E \to X$ to a **p-convex** quasi-Banach lattice uniquely extends to a lattice homomorphism $\widehat{T} : \mathrm{FpBL}^{(p)}[E] \to X$ with $\|\widehat{T}\| \leq M^{(p)}(X)\|T\|$.

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Construction of $\mathrm{FpBL}^{(p)}[E]$

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- For simplicity, suppose E is separable.
- Since E is p-natural, E embeds in some $\ell_\infty(\mathbb{N}, L_p(\mu_n))$, where $L_p(\mu_n) = \ell_p \oplus_p L_p[0,1]$ for every $n \in \mathbb{N}$.

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- An appropriate ambient space is

$$\widehat{E} = \{ f : S_{\mathcal{L}(E, \ell_p \oplus_p L_p)} \to \ell_p \oplus L_p \text{ bounded } \}$$

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— There is $\delta : E \to \widehat{E}$ given by $\delta(e)(f) = f(e)$.
— Furthermore, for every $T : E \to \ell_p \oplus_p L_p$ with $\|T\| = 1$, there is an extension

$$\widehat{T} : \widehat{E} \to \ell_p \oplus_p L_p, \quad \widehat{T}(f) = f(T).$$

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- Now, place a suitable "maximal p-norm" on $\widehat{E}$:

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- Define $\mathrm{FpBL}^{(p)}[E]$ as the closed lattice-linear span of $\{\delta_e : e \in E\}$ inside $\widehat{E}$.
- Show that, for any p-convex lattice X with $M^{(p)}(X)$ and $T : E \to X$ with $\|T\| = 1$, there is a unique norm-preserving extension $\hat{T} : \mathrm{FpBL}^{(p)}[E] \to X$.

Present and future plans

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(1) Functional representation of FpBL.

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(3) Properties of E vs. lattice properties of $\mathrm{FpBL}[E]$ and $\mathrm{FpBL}^{(p)}[E]$.

THANK YOU VERY MUCH!