Uniqueness of unconditional basis in Banach spaces: overview and recent advances

José L. Ansorena. Joint work with F. Albiac

Structures in Banach Spaces
Erwin Schrödinger Institute, Vienna, March 16-22, 2025

Uniqueness of unconditional basis (UUB)

Let $(\boldsymbol{x}_n)_{n=1}^{\infty}$ be an unconditional basis of a Banach (or quasi-Banach) space. To properly state a uniqueness result we must take into account the following.

- If $(\lambda_n)_{n=1}^{\infty}$ are scalars, then $(\lambda_n \boldsymbol{x}_n)_{n=1}^{\infty}$ is un unconditional basis.
- If $(\boldsymbol{x}_n)_{n=1}^{\infty}$ is an unconditional basis and $(\boldsymbol{y}_n)_{n=1}^{\infty}$ is a small perturbation of $(\boldsymbol{x}_n)_{n=1}^{\infty}$, then $(\boldsymbol{y}_n)_{n=1}^{\infty}$ is an unconditional basis equivalent to $(\boldsymbol{x}_n)_{n=1}^{\infty}$.
- If π is a permutation of $\mathbb{N}$, then $(\boldsymbol{x}_{\pi(n)})_{n=1}^{\infty}$ is unconditional basis that could be nonequivalent to $(\boldsymbol{x}_n)_{n=1}^{\infty}$ unless $(\boldsymbol{x}_n)_{n=1}^{\infty}$ is symmetric.

So, we must restrict ourselves to normalized (or semi-normalized) bases, and we must consider uniqueness up to equivalence and permutation (UTAP).

Atomic lattices vs unconditional bases

- If the unit vectors of an atomic lattice $\boldsymbol{L}$ span the whole space, they are an unconditional basis of $\boldsymbol{L}$.
- Any unconditional basis of a Banach space $\mathbb{X}$ induces on $\mathbb{X}$ a lattice structure.

Proving that a Banach (or quasi-Banach) space has a unique unconditional basis (UUB for short) consists in proving that the unit vectors of an atomic lattice $\boldsymbol{L}$ are the unique unconditional basis of $\boldsymbol{L}$.

The pioneers

Having a unique unconditional bases is a rare property of Banach spaces.

Theorem (Pełczyński,1960)

Let $p \in (1\infty)$, $p \neq 2$. *Since*

$$\ell_p \simeq \left(\bigoplus_{n=1}^{\infty} \ell_2^n \right)_{\ell_p}$$

ℓ_p *does not have a UUB.*

Theorem (Köthe and Toeplitz, 1932)

ℓ_2 has a UUB.

Theorem (Lindenstrauss and Pełczyński, 1968)

ℓ_1 and c_0 have a UUB.

Three general problems

To search for new Banach spaces with a unique unconditional basis, we must pay attention to the following questions.

Question

Let $\mathbb{X}$ and $\mathbb{Y}$ be quasi-Banach spaces with a unique unconditional bases. Does $\mathbb{X} \oplus \mathbb{Y}$ has a unique unconditional basis?

Question

Let $\boldsymbol{L}$ be an atomic lattice and $\mathbb{X}$ be a quasi-Banach spaces. Suppose that $\boldsymbol{L}$ and $\mathbb{X}$ have a unique unconditional basis. Does

$$\boldsymbol{L}(\mathbb{X}) = \{(x_n)_{n=1}^\infty : x_n \in \mathbb{X}_n,\ (\|x_n\|)_{n=1}^\infty \in \boldsymbol{L}\}$$

have a unique unconditional basis.

Question

Let $\mathbb{Y}$ be a complemented subspace of a quasi-Banach $\mathbb{X}$ with a unique unconditional basis. If $\mathbb{Y}$ has an unconditional basis, it is unique?

Theorem (Edelstein and Wojtaszczyk, 1976)

$\ell_1 \oplus \ell_2$, $c_0 \oplus \ell_2$, $c_0 \oplus \ell_1$, *and* $c_0 \oplus \ell_1 \oplus \ell_2$ *have a UUB unconditional basis.*

Theorem (Bourgain, Casazza, Lindenstrauss, and Tzafriri, 1985)

$\ell_1(\ell_2)$, $c_0(\ell_2)$, $c_0(\ell_1)$ *and* $\ell_1(c_0)$ *have a UUB. Besides, their complemented subspaces inherit this property.*

Theorem (Bourgain, Casazza, Lindenstrauss, and Tzafriri, 1985)

Neither $\ell_2(\ell_1)$ nor $\ell_2(c_0)$ have a UUB.

Theorem (Bourgain, Casazza, Lindenstrauss, and Tzafriri, 1985)

The 2-convexified Tsirelson space $\mathcal{T}^{(2)}$ has a UUB. Besides, its complemented subspaces with an unconditional basis inherit this property.

It is hopeless to try to classify Banach spaces with a UUB!

Further advances

Theorem (Casazza and Kalton, 1998)

Tsirelson space $\mathcal{T}$, and the original Tsirelson space $\mathcal{T}^*$ has a UUB.

Theorem (Casazza and Kalton, 1998)

Let $\boldsymbol{P} = (p_n)_{n=1}^\infty$ be a nonincreasing sequence with

$$\lim_n p_n = 1.$$

Then, the variable exponent Lebesgue space $\ell_{\boldsymbol{P}}$ have a UUB as long as it is lattice isomorphic to its square.

Theorem

Let F be an Orlicz function such that

$$F(t) \sim \frac{t}{\log(t)}, \quad t \to 0^+.$$

There are complemented subspace $\mathbb{X}$ of ℓ_F and a complemented subspace $\mathbb{Y}$ of $\mathbb{X}$ such that $\mathbb{X}$ has UUB, and $\mathbb{Y}$ has an unconditional basis that is not unique.

Theorem (Casazza and Kalton, 1999)

Neither $c_0(\mathcal{T})$ not $c_0(\mathcal{T}^{(2)})$ have a UUB.

Recent advances

As far as Banach spaces are concerned, not much else is known.

Theorem (Albiac and A, 2021)

Any direct sum constructed from ℓ_1, c_0, $\mathcal{T}$, $\mathcal{T}^*$, ℓ_2 and $\mathcal{T}^{(2)}$,

$$\ell_2 \oplus \mathcal{T}^{(2)}$$

in particular, has a UUB.

As far as we know, the general question about finite direct sums is open.

Open questions

Question

Does $\ell_1(\mathcal{T})$, $\ell_1(\mathcal{T}^{(2)})$ or $\ell_2(\mathcal{T}^{(2)})$ have a UUB?

Question

Does $c_0(\ell_1(\ell_2))$ have a UUB?

Question

Let $\boldsymbol{P} = (p_n)_{n=1}^\infty$ be a sequence with $\lim_n p_n = 2$. Asume that the variable exponent Lebesgue space $\ell_{\boldsymbol{P}}$ is lattice isomorphic to its square. Does $\ell_{\boldsymbol{P}}$ have a UUB?

Quasi-Banach spaces

If we widen the scope and also consider non-locally convex spaces the list of quasi-Banach spaces with a UUB remarkably grows.

Theorem (Kalton,1977)

Let F be a concave Orlicz function. Then ℓ_F has UUB. In particular ℓ_p, $0 < p < 1$ has a UUB.

Theorem (Kalton, Leranoz, Wojtaszczyk, 1990)

$\ell_p(\ell_q)$, p, $q \in (0, 1)$.

Theorem (Wojtaszczyk, 1997)

Hardy spaces $H_p(\mathbb{D})$, $0 < p < 1$, have a UUB.

All the above non-locally convex lattices are *L-convex*, that is, are q-convex for some $q > 0$.
Also, they are strongly absolute. This means that are semi-normalized and that for every $\varepsilon > 0$ there is a constant $C(\varepsilon) \in (0, \infty)$ such that

$$\|f\|_1 \leq \max\{C(\varepsilon)\,\|f\|_\infty, \varepsilon\,\|f\|_{\boldsymbol{L}}\}, \quad f \in \boldsymbol{L}.$$

Quasi-Banach spaces with a Banach component

Theorem (Albiac, Kalton, Leranoz, Wojtaszczyk 1992–2004)

$\ell_p(\ell_2)$, $c_0(\ell_p)$, $\ell_p(c_0)$, $\ell_1(\ell_p)$ *and* $\ell_p(\ell_1)$, $0 < p < 1$, *have a UUB.*

Question

Does $\ell_2(\ell_p)$, $0 < p < 1$, has a UUB?

A tool for managing subbases

Aside that of $H_p(\mathbb{D})$, the subbases-structure of (the canonical basis) of all the above non-locally spaces lattices is simple. The following result allows us to manage spaces whose subbases-structure is not totally understood.

Theorem (Albiac, A, 2022)

Let $\mathcal{X}$ and $\mathcal{Y}$ be unconditional bases. Suppose that $\mathcal{X}^m$ is equivalent a subbasis of $\mathcal{Y}^m$ for some $m \in \mathbb{N}$. Then $\mathcal{X}$ is equivalent a subbasis of $\mathcal{Y}$.

Some applications

Theorem (Albiac, A, 2022)

Let $\mathbf{L}$ *be an L-convex strongly absolute lattice. Then* $\mathbf{L}(\mathcal{T})$, $\mathbf{L}(\mathcal{T}^*)$, $\mathbf{L}(\ell_1)$ *and* $\mathbf{L}(c_0)$ *have a UUB. In particular* $\ell_p(\mathcal{T}^*)$, $0 < p < 1$, *has a UUB.*

The Banach envelope of all the above non-locally convex spaces with a UUB has a UUB.
Note that the Banach envelope $\ell_p(\mathcal{T}^*)$ is $\ell_1(\mathcal{T}^*)$, which does not have a UUB.

Sufficiently Euclidean spaces

Theorem (Albiac, A, 2024)

$H_p(\ell_2)$, $H_p(\mathcal{T}^{(2)})$ and $\ell_p(\mathcal{T}^{(2)})$ have a UUB. More generally, if $\boldsymbol{L}$ is an L-convex strongly absolute lattice, then $\boldsymbol{L}(\ell_2)$ and $\boldsymbol{L}(\mathcal{T}^{(2)})$ have a UUB.

Aside from $\ell_p(\ell_2)$, all the above non-locally convex spaces $\mathbb{X}$ with a UUB are anti-Euclidean. The opposite condition, being sufficiently Euclidean, means that ℓ_2 is finitely complementably representable in $\mathbb{X}$.

The role of the squares

All the above spaces with a UUB are isomorphic to their squares.

Theorem (Albiac, A, 2025)

Let $\mathbb{G}$ be the Gowers space with an unconditional basis. Then $\mathbb{G}$ has a UUB.

The role the optimal convexity

Given an L-convex lattice $\boldsymbol{L}$ we set

$$\sigma(\boldsymbol{L}) = \sup \{p \in (0, \infty] \colon \boldsymbol{L} \text{ is lattice } p\text{-convex}\}.$$

All the above lattices $\boldsymbol{L}$ with a UUB satisfy $\sigma(\boldsymbol{L}) \in (0, 1] \cup \{2, \infty\}$.

Theorem (Albiac, A, 2025)

Let $\mathbb{G}$ be the Gowers space with an unconditional basis and let $p \geq 1$. Then the p-convexified Gowers space $\mathbb{G}^{(p)}$ has a UUB.

Acknowledgements

GOBIERNO DE ESPAÑA

UNIVERSIDAD DE LA RIOJA

Thank you for your attention.

Danke für Ihre Aufmerksamkeit.