On extremal nonexpansive mappings

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joint work with
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Research supported by the Austrian Science Fund (FWF): P 32523

Universität Innsbruck

Structures in Banach Spaces
ESI Vienna, 17-21 March 2025

The setting

Let X be a (real) Banach space and $C \subset X$ be a closed and convex set.

A mapping $f\colon C \to C$ is called nonexpansive if

$$\|f(x) - f(y)\| \leq \|x - y\|$$

for all $x, y \in X$.
We consider

$$\mathcal{M} = \mathcal{M}(C) = \{f\colon C \to C\colon f \text{ nonexpansive}\}.$$

If C is in addition bounded we equip $\mathcal{M}$ with the metric

$$d_\infty(f, g) = \sup_{x \in C} \|f(x) - g(x)\|$$

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Why nonexpansive mappings?

Theorem (Brouwer, 1911)

Let $C \subseteq \mathbb{R}^d$ be nonempty, bounded, closed and convex. Then every continuous mapping

$$f\colon C \to C$$

has a fixed point.

In infinite dimensions the situation is more complicated.

Theorem (Benyamini–Sternfeld, 1983)

Let X be a infinite dimensional Banach space. Then the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ is a Lipschitz retract of the unit ball $B_X = \{x \in C : \|x\| \leq 1\}$. In particular there is always a Lipschitz mapping $f\colon B_X \to B_X$ without a fixed point.

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Why nonexpansive mappings?

Theorem (Browder, Goehde, Kirk, 1965)

Let X be a uniformly convex Banach space and $C \subset X$ a bounded, closed and convex subset. Then every nonexpansive mapping $f\colon C \to C$ has a fixed point.

This is not true for every Banach space. For example let

$$C := \{g \in \mathcal{C}[0,1] \colon 0 = g(0) \leq g(t) \leq g(1) = 1 \text{ for } t \in [0,1]\}$$

and

$$T\colon C \to C, \quad (Tg)(t) := tg(t)$$

Then T is nonexpansive but has no fixed point.

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Let X be a Banach space and $C \subset X$ be a bounded, closed and convex set. Then $\mathcal{M}$ is a complete metric space, so Baire's theorem holds for $\mathcal{M}$.

We call a property (P) of nonexpansive mappings typical if the set

$$\mathcal{A} = \{f \in \mathcal{M} \colon f \text{ enjoys } (P)\}$$

contains a dense G_δ-set.

Theorem (de Blasi–Myjak, 1976)

The typical element of $\mathcal{M}$ has a unique fixed point.

Question

Is the typical element of $\mathcal{M}$ a strict contraction, i.e. does it satisfy $\|f(x) - f(y)\| \leq L\|x - y\|$ for some $L \in (0,1)$?

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We denote by

$$\operatorname{Lip} f = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}$$

the Lipschitz constant of f.

The typical nonexpansive mapping f on C satisfies $\operatorname{Lip} f = 1$ if

- X is a Hilbert space (de Blasi–Myjak, 1976)
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Extremal Mappings

We call a mapping $f \in \mathcal{M}$ extremal if

$$\lambda g + (1-\lambda)h \neq f$$

for every $\lambda \in (0,1)$ and every $g, h \in \mathcal{M} \setminus \{f\}$.

Given $\mathcal{F} \subset \mathcal{M}$ we say that $f \in \mathcal{F}$ is extremal among mappings from $\mathcal{F}$ if

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Observation

If f is not extremal, then there are $g, h \in \mathcal{M}$ and $\lambda \in (0,1)$ with

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Two questions

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The linear case: extreme contractions

Let X be a Banach space and $L(X)$ be the space of bounded linear operators on X.

$T \in L(X)$ is called a extreme contraction if T is an extreme point of the unit ball of $B_{L(X)}$, i.e. $\|T\| \leq 1$ and

$$T \neq \lambda S + (1-\lambda)R$$

for all $S, R \in L(X)$ with $\|S\|, \|R\| \leq 1$ and all $\lambda \in (0,1)$.

Theorem (Blumenthal–Lindenstrauss–Phelps, 1965)

Let K_1, K_2 compact Hausdorff spaces, K_1 metrisable. $T \colon C(K_1) \to C(K_2)$ with $\|T\| = 1$ is an extreme contraction iff there are continuous functions $\varphi \colon K_2 \to K_1$ and $\psi \colon K_2 \to \mathbb{T}$ with

$$(Tf)(x) = \psi(x) f(\varphi(x))$$

for $x \in K_1$.

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The linear case: extreme contractions

This problem is well studied, e.g.

- Characterisation of extreme contractions on $L_1(\Omega, \Sigma, \mu)$ for σ-finite μ: Iwanik, 1978
- Characterisation of extreme contractions on ℓ_∞: Kim, 1976
- Characterisation of extreme contractions between L_p-spaces: Kan, 1986
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Some observations

Let X be a Banach space and $C = B_X$ its unit ball. For $T \in L(X)$ with $\|T\| \leq 1$ the mapping $T|_{B_X}$ defines an element of $\mathcal{M}$.

- If T is not an extreme contraction, $T|_{B_X}$ cannot be extremal in $\mathcal{M}$.
- The mapping

$$T\colon c_0 \to c_0, \qquad (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$$

  is an extreme contraction but

$$T|_{B_{c_0}} = \frac{1}{2}f + \frac{1}{2}g$$

  with $f(x) = (1, x_1, x_2, \dots)$ and $g(x) = (-1, x_1, x_2, \dots)$, so $T|_{B_{c_0}}$ is not extremal in $\mathcal{M}$.

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Let X be a Banach space. $C \subset X$ is called a convex body if it is a closed convex set with nonempty interior.

We look at surjective isometries on a convex body C, i.e. surjective $f : C \to C$ with $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in C$.

Theorem (Mankiewicz, 1972)

Let X and Y be a Banach spaces and $C \subset X$ and $D \subset Y$ be convex bodies and $f : C \to D$ a surjective isometry, then there is an affine isometry $T : X \to Y$ with $T|_C = f$.

If $C = D = B_X$, then the extension T is even a linear isometry.

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An important observation

Let X be a Banach space and $C \subset X$ a convex body. Then every surjective isometry $C \to C$ is extremal in $\mathcal{M}$ if and only if the identity $\mathrm{id}\colon C \to C$ is extremal in $\mathcal{M}$.

Proof.

Suppose that $\mathrm{id} \in \mathcal{M}$ is extremal, let $f \in \mathcal{M}$ be a surjective isometry. Assume there are $\lambda \in (0,1)$ and $g, h \in \mathcal{M}$ such that

$$f = (1-\lambda)g + \lambda h.$$

By Mankiewicz's theorem f and f^{-1} are the restriction of an affine isometry $X \to X$. Hence,

$$\mathrm{id} = (1-\lambda)f^{-1} \circ g + \lambda f^{-1} \circ h$$

and $f^{-1} \circ g, f^{-1} \circ h \in \mathcal{M}$. id is extremal in $\mathcal{M}$, so $f^{-1} \circ g = \mathrm{id}$ and $g = f$. □

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and $f^{-1} \circ g, f^{-1} \circ h \in \mathcal{M}$. id is extremal in $\mathcal{M}$, so $f^{-1} \circ g = \mathrm{id}$ and $g = f$. $\square$

An important observation

Let X be a Banach space and $C \subset X$ a convex body. Then every surjective isometry $C \to C$ is extremal in $\mathcal{M}$ if and only if the identity $\mathrm{id}\colon C \to C$ is extremal in $\mathcal{M}$.

Proof.

Suppose that $\mathrm{id} \in \mathcal{M}$ is extremal, let $f \in \mathcal{M}$ be a surjective isometry. Assume there are $\lambda \in (0,1)$ and $g, h \in \mathcal{M}$ such that

$$f = (1-\lambda)g + \lambda h.$$

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Almost exposed points

Let X be a Banach space and $C \subset X$ a closed and convex set. A point $x \in C$ is called exposed if there is a hyperplane H supporting C in x with $C \cap H = \{x\}$.

A point $x \in C$ is called almost exposed if the intersection of all hyperplanes supporting C in x is the singleton x.

- Exposed points of C are almost exposed.
- If C is smooth almost exposed points are exposed.

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A general observations

Theorem

Let X be a Banach space with the property that its unit ball B_X is the closed convex hull of its almost exposed points. Then surjective isometries are extremal in the space of nonexpansive self-mappings of the unit ball.

Sketch of the proof.

Enough to show that id is extremal in $\mathcal{M}$. Let $\mathrm{id} = (1-\lambda)g + \lambda h$. Let E be the set of almost exposed points of B_X. Show that $g(x + te) = g(x) + te$ for all $t \in (0, 1 - \|x\|)$, $x \in B_X$ and $e \in E$ (using that e is almost exposed). By induction $g(z) = z + g(0)$ for $z \in \overline{\mathrm{conv}}(E) = B_X$. Hence $g = \mathrm{id}$. $\square$

If X has the Radon-Nikodym property, it satisfies the assumptions of the above theorem.

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The case $C = B_{C(K)}$

Theorem

Let K be a compact Hausdorff topological space. Surjective isometries on $B_{C(K)}$ are extremal in the space $\mathcal{M}$ of nonexpansive mappings $B_{C(K)} \to B_{C(K)}$.

Proof sketch.

Again, it is enough to show that the identity is extremal.

1. We show $\mathrm{id} = \lambda F + (1-\lambda)G$ implies that $F(f)(x) = G(f)(x) = \pm 1$ whenever $f(x) = \pm 1$.
2. We show that a nonexpansive mapping with the above property is already the identity.

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“Quantifying” (non-)extremality

Let X be a Banach space and $C \subset X$ be bounded, closed and convex. For $f \in \mathcal{M}$ we set

$$P_f = \{g \in \mathcal{M} \colon \exists \lambda \in [0,1] \exists h \in \mathcal{M} \text{ s.t. } f = (1-\lambda)g + \lambda h\}.$$

Then f is extremal iff $P_f = \{f\}$.

Properties of P_f:

1. P_f is F_σ: $P_f = \{f\} \cup \bigcup_{q\in(0,1/2)} P_{f,q}$

   $$P_{f,q} = \{g \in \mathcal{M} \colon \exists \lambda \in [q, 1-q] \exists h \in \mathcal{M} \text{ s.t. } f = (1-\lambda)g + \lambda h\}$$

2. $\mathcal{M} \setminus P_f$ is convex.
3. There is an affine subspace A_f of the space of continuous mappings $C \to X$ with $P_f = A_f \cap \mathcal{M}$.

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Small subsets of metric spaces

Let (M, ρ) be a complete metric space without isolated points. We consider a set $A \subset M$ to be large if it contains a dense G_δ-set.

So a set is small if it is a meagre set, i.e. a countable union of nowhere dense set.

Recall that $A \subset M$ is nowhere dense if for every $q \in A$ and every $\varepsilon > 0$ there is a $q' \in B_M(q, \varepsilon)$ and a $\delta > 0$ with $B_M(q', \delta) \cap A = \emptyset$.

A set $A \subset M$ is called upper porous at $q \in A$ if there is an $\alpha > 0$ such that for every $\varepsilon > 0$ there is a $q' \in B_M(q, \varepsilon)$ with $B_M(q', \alpha d(q, q')) \cap A = \emptyset$.

A set $A \subset M$ is called upper porous if it is upper porous at all its points.

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Almost extremal mappings

Theorem

Let $f \in \mathcal{M}$ with $\operatorname{Lip} f = 1$. Then the set P_f is σ-upper porous. More precisely, it is a countable union of closed upper porous subsets of $\mathcal{M}$.

Since the typical element $f \in \mathcal{M}$ satisfies $\operatorname{Lip} f = 1$ we obtain the following.

Corollary

For the typical $f \in \mathcal{M}$ the set P_f is σ-upper porous.

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The End

Thank you for your attention!

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