Von Neumann-Maharam problem for vector lattices

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Note that the last condition can be replaced with disjoint subadditivity, or with $\rho(a \triangle b) \leq \rho(a) + \rho(b)$, for any $a, b \in A$.

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F is weakly (σ, ∞)-distr. iff $\forall e$ meager sets are nowhere dense in K_e.

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Theorem 3 (Kelley, 1959 + Kalton + Roberts, 1983)

Let A be a BA. A locally solid topology on A is generated by additive measures iff it is uniformly exhaustive, i.e. for every neighborhood U of 0_A there is $n \in \mathbb{N}$ such that there are no disjoint n-tuples in $A \backslash U$.

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If F is an AVL, then $F_{csp} := \{e \in F,\ F_e \text{ has the CSP}\}$ is the largest ideal in F with the CSP. Note that F_e has the CSP iff K_e has the CCC.

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In particular, there is a complete CCC BA such that τ_A exists, but is not uniformly exhaustive.

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In this case τ_F is complete iff $F = L_0(A)$, for some A, and τ_F is metrizable iff F has the CCC.

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Theorem 9 (Deng + de Jeu, 2024 + B., 2025)

If τ_F exists and F has the CSP, then $\tau_F = \mathrm{tuo}$. In fact, $(f_n)_{n\in\mathbb{N}}$ is τ_F-null iff every subsequences has a uo-null sub-subsequence. Conversely, this implies CSP, if F is atomless, under CH (false under MA + $\neg$CH).

Note that if τ_F exists, then F_{csp} is order dense in F, i.e. for every $f > 0_F$ there is $e \in (0_F, f] \cap F_{csp}$.

Question 2

- *Is it always true that $\tau_F = \mathrm{tuo}$?*
- *Find a version of Theorem 6 for AVL's.*

Theorem 10 (Preliminary)

If τ_F exists, TFAE:

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- *For all $e \in F$ there is a non-zero order continuous functional on F_e.*

THANK YOU!