Density of smooth functions without critical points

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In this talk I will present some results contained in the recent paper:

• D. Azagra, M. García-Bravo, M. Jiménez-Sevilla, *Approximate Morse-Sard type results for non-separable Banach spaces*, J. Funct. Anal. 287, no. 4 (2024).

The structure of the talk is as follows:

1. Introduction to the Morse-Sard theorem in infinite dimensions.
2. Approximate Morse-Sard results in the nonseparable case.
3. Main ideas and tools behind the proofs.

We begin with

1. Introduction to the Morse-Sard theorem in infinite dimensions.

Critical point

Definition (Critical point)

If we have a function $f : E \longrightarrow F$ between Banach spaces which is (Fréchet) differentiable at some point x we will say that x is a critical point if $Df(x) \in \mathcal{L}(E,F)$ is not a surjective operator.

- C_f = set of critical points.
- $f(C_f)$ = set of critical values.

Recall that the (Fréchet) derivative $Df(x)$ of f at x is defined as the unique linear continuous operator such that

$$\lim_{h\to 0} \frac{||f(x+h) - f(x) - Df(x)(h)||}{||h||} = 0.$$

Question: Can we make $f(C_f)$ *small* in some sense?

Motivation: Classical Morse-Sard theorem

The Morse-Sard theorem deals with the study of the set of critical points of differentiable functions $f : \mathbb{R}^n \to \mathbb{R}^m$.

Example (Striking Whitney example (1935))

Relying on his extension result from 1934, Whitney built a function $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^1 so that $\mathcal{L}(f(C_f)) > 0$, where

$$C_f = \{x \in \mathbb{R}^2 : \nabla f(x) = 0\}.$$

Indeed, the function f being nonconstant on a (nonrectifiable) curve $\Gamma \subset \mathbb{R}^2$ where $Df(x) = 0$ for all $x \in \Gamma$. Precisely $f(\Gamma) = [0, 1]$.

Observation: Dealing with a rectifible curve Γ, by the Fundamental Theorem of Calculus we would have f constant on Γ.

Why such a "weird" example is possible?

Classical Morse-Sard theorem

Theorem (Morse 1939, Sard 1942)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^k function with $k \geq \max\{n-m+1, 1\}$. Then the set of critical values, $f(C_f)$, is of Lebesgue measure zero in $\mathbb{R}^m$ ($\mathcal{L}^m(f(C_f)) = 0$).

Note that here

$$C_f = \{x \in \mathbb{R}^n : \, rank\, Df(x) < \min\{n, m\}\} \, .$$

• This result has been shown to be sharp in the class of functions C^j thanks to the famous counterexample of Whitney in 1935.

• This theorem has been generalized to other function spaces such as Hölder spaces $C^{k-1,1}$, Sobolev spaces $W^{k,p}$ with $p > n$ or to the space of functions of bounded variation BV_n.

Infinite dimension: $\dim(E) = \infty$, $\dim(F) = 1$

Even though there were some tries to find a "good" version of Morse-Sard in infinite dimensions (Smale, 1965)...

Kupka's counterexample (1965): There exist C^∞ functions $f : \ell_2 \to \mathbb{R}$ so that their sets of critical values $f(C_f)$ contain intervals.

Example (Bates and Moreira, 2001)

Let $f : \ell_2 \to \mathbb{R}$ be defined as

$$f\left(\sum_{n\geq 1} x_n e_n\right) = \sum_{n\geq 1} (3 \cdot 2^{-\frac{n}{3}} x_n^2 - 2x_n^3)$$

- f is C^∞ (a polynomial of degree three).
- $C_f = \left\{\sum_{n\geq 1} x_n e_n : x_n \in \{0, 2^{-\frac{n}{3}}\}\right\}$.
- $f(C_f) = [0, 1]$.

Conclusion: We cannot expect to have a good version of the Morse-Sard theorem for infinite dimensions !!

However around 20 years ago the following two results appeared, which can be considered as *approximate Morse-Sard results*.

Theorem

1. *For every continuous functions $f : \ell_2 \to \mathbb{R}$ and $\varepsilon : \ell_2 \to (0, \infty)$*
   - **(Eells, McAlpin 1968)** *, there exists a C^∞ function $g : \ell_2 \to \mathbb{R}$ for which $|f(x) - g(x)| \leq \varepsilon(x)$ for all $x \in E$ and $\mathcal{L}(g(C_g)) = 0$.*
   - **(Azagra, Cepedello 2004)** *, there exists a C^∞ function $g : \ell_2 \to \mathbb{R}$ for which $|f(x) - g(x)| \leq \varepsilon(x)$ for all $x \in E$ and $C_g = \emptyset$.*
2. **(Azagra, Jiménez-Sevilla 2007)** *Let E be an infinite dimensional Banach space with separable dual. Then for every continuous functions $f : E \to \mathbb{R}$ and $\varepsilon : E \to (0, \infty)$, there exists a C^1 function $g : E \to \mathbb{R}$ for which $|f(x) - g(x)| \leq \varepsilon(x)$ for all $x \in E$ and $C_g = \emptyset$.*

Approximate Morse-Sard for $\dim(E) = \dim(F) = \infty$

Theorem (Azagra, Dobrowolski, G-B (2019))

Let E be one of the classical Banach spaces c_0, ℓ_p or L^p, $1 < p < \infty$ and let F a (non zero) quotient of E (so there exists a bounded linear surjective operator from E onto F).
Then for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^k mapping $g : E \to F$ such that

1. $\|f(x) - g(x)\| \leq \varepsilon(x)$ *for every* $x \in E$, *and*
2. $Dg(x) : E \to F$ *is a surjective linear operator for every* $x \in E$.

Here k denotes the order of smoothness of the norm of the space E.

Warning: The previous results do not hold in finite dimensions.

Related topic: Failure of Rolle's theorem

Warning: The previous results do not hold in finite dimensions.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = |x|$ and let $\varepsilon = 1$. Clearly, any C^1 function $g : \mathbb{R} \to \mathbb{R}$ so that $|g(x) - f(x)| \leq \varepsilon$, according to the Rolle's Theorem, must satisfy that there exists $x_0 \in \mathbb{R}$ with $g'(x_0) = 0$.

¡Rolle's Theorem fails in infinite dimensions!

This was first noticed by Shkarin in 1992 for superreflexive spaces and non-reflexive spaces with smooth norms.

Theorem (Azagra, Jiménez-Sevilla (2001))

For every infinite-dimensional Banach space E with a C^1 smooth bump there exists another C^1 smooth bump $b : E \to \mathbb{R}$ so that $b'(x) \neq 0$ for every $x \in$ int(supp(b)).

Non separable approximate Morse-Sard

We continue now with

2. Approximate Morse-Sard results in the nonseparable case.

Question: Can the previous results be extended to the case of nonseparable Banach spaces?

Theorem (Azagra, G-B, Jiménez-Sevilla (2024))

Let Γ be an arbitrary infinite set, let $E = c_0(\Gamma), \ell_p(\Gamma)$, $1 < p < \infty$ and let F any (non zero) quotient of E. Then for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^k mapping $g : E \to F$ such that

1. $\|f(x) - g(x)\| \leq \varepsilon(x)$ *for every* $x \in E$, *and*
2. $Dg(x) : E \to F$ *is a surjective linear operator for every* $x \in E$.

Here k denotes the order of smoothness of the norm of the space E.

Definitions:

$$c_0(\Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \subset \mathbb{R}^\Gamma : \forall\, \varepsilon > 0 \text{ the set } \{\gamma \in \Gamma : |x_\gamma| \geq \varepsilon\} \text{ is finite}\}$$
$$\ell_p(\Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \subset \mathbb{R}^\Gamma : \sum_{\gamma\in\Gamma} |x_\gamma|^p < \infty\}$$

Other different versions:

Theorem (Azagra, G-B, Jiménez-Sevilla (2024))

Let E be a Banach space with C^1 partitions of unity. Assume moreover that E contains an infinite-dimensional separable complemented subspace Y. Then for every continuous mapping $f : E \to \mathbb{R}^m$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^1 mapping $g : E \to \mathbb{R}^m$ such that

1. $\|f(x) - g(x)\| \leq \varepsilon(x)$ *for every* $x \in E$, *and*
2. $Dg(x) : E \to \mathbb{R}^m$ *is a surjective linear operator for every* $x \in E$.

Remark: For this last theorem, in the case that Y cannot be taken to be reflexive, we need to perform an adequate renorming of the space Y.

Main tools of the proofs

We finally move to

3. Main ideas and tools behind the proofs.

We will distinguish between different cases:

1. Case of $c_0(\Gamma)$.
2. Case of $\ell_p(\Gamma)$, $1 < p < \infty$.

Case $c_0(\Gamma)$

The approximating function g has the form

$$g(x) = \sum_{\alpha \in A} \psi_\alpha(x)(f(x_\alpha) + T(x - x_\alpha))$$

1. $\{\psi_\alpha\}_{\alpha \in A}$ is a C^∞ smooth partition of unity subordinate to some open covering $\{U_\alpha\}_{\alpha \in A}$ with $x_\alpha \in U_\alpha$ that has locally the form

$$x \to \psi_\alpha(x) = \varphi_\alpha(e^*_{\gamma_1}(x), \ldots, e^*_{\gamma_n}(x)).$$

2. $T : c_0(\Gamma) \to F$ is a continuous surjective operator such that $T|_{c_0(\Gamma_n)}$ is surjective and

$$\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n, \quad \Gamma_i \cap \Gamma_j = \emptyset \ \forall i \neq j, \quad \#(\Gamma_n) = \#(\Gamma).$$

For all $x \in c_0(\Gamma)$ we have $\|f(x) - g(x)\| \leq \varepsilon(x)$ and

$$Dg(x) = T + L, \quad L \in \text{span}\{e^*_\gamma : \gamma \in \Gamma\} \quad \Rightarrow \quad Dg(x) \text{ is surjective.}$$

Case $\ell_p(\Gamma)$

We first define, in a similar way as before,

$$p(x) = \sum_{\alpha \in A} \psi_\alpha(x)(f(x_\alpha) + T(x - x_\alpha)).$$

Here $\{\psi_\alpha\}$ is a smooth parition of unity locally of the form

$$x \to \psi_\alpha(x) = \varphi_\alpha(\|x\|^p, e^*_{\gamma_1}(x), \dots, e^*_{\gamma_n}(x)).$$

The previous comes from the existence of certain homeomorphic embeddings from $\ell_p(\Gamma)$ into $c_0(\Gamma')$ (for some infinite set Γ') whose coordinate functions are C^k smooth, given by Toruńczyk in 1973.

In this case $C_p \neq \emptyset$, but we solve this situation by building a C^k diffeomorphism $h : E \to E \setminus C_p$ which "does not move too much the points". This way the final approximating function is

$$g = p \circ h$$

Observe that

$$Dg(x) = Dp(h(x)) \circ Dh(x) : E \to F \text{ is a surjective operator.}$$

Problem: Given a Banach space E and a closed subset $X \subset E$, we look for diffeomorphisms between E and $E \setminus X$?

Theorem (Negligibility theory)

Pioneering results:

1. *(Bessaga, 1966) There exists a C^∞ diffeomorphism $h : \ell_2 \to \ell_2 \setminus \{0\}$ so that $h = id$ outside the unit ball.*
2. *(West, 1969) For every compact set $K \subset \ell_2$ there exists a C^∞ diffeomorphism $h : \ell_2 \to \ell_2 \setminus K$ so that h is "as close to the identity as we want".*

For the case $\ell_p(\Gamma)$, in our work we need:

3. *For every close set $C \subset \ell_p(\Gamma)$ which is locally contained in a complemented subspace of infinite codimension subspace, there exists a C^k diffeomorphism $h : \ell_p(\Gamma) \to \ell_p(\Gamma) \setminus C$, so that h "is very close to the identity". (Again k denotes the order of smoothness of the $\ell_p(\Gamma)$ norm).*

Main final goal

The best possible result that one can aim for, but seems diifucult to handle (using our techniques) is:

Open question: Let E be a Banach space with C^1 smooth partitions of unity. Then for every conitnuous functions $f : E \to \mathbb{R}$ and $\varepsilon : E \to (0, \infty)$ there exists $g : E \to \mathbb{R}$ of class C^1 so that

$$|f(x) - g(x)| \leq \varepsilon(x) \quad \forall x \in E$$

and $g'(x) \neq 0$ for all $x \in E$. [Solved for separable case.]

THANK YOU FOR YOUR ATTENTION !!