Somehow off-limits: The 3-space problem for isomorphic polyhedrality

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3-space problems

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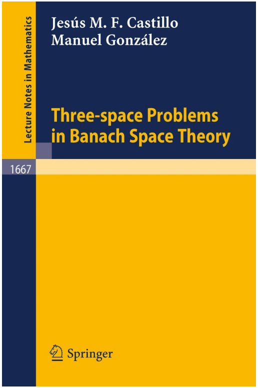
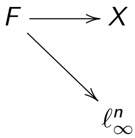
A property $\mathcal{P}$ is a 3-space property if whenever the spaces Y and X in an exact sequence of Banach spaces have property $\mathcal{P}$ then also Z has property $\mathcal{P}$. Equivalently, every twisted sum of two spaces having $\mathcal{P}$ has $\mathcal{P}$.



Polyhedrality

A Banach space X is *polyhedral* if every finite dimensional subspace $F \subset X$ is isometric to a subspace of some finite-dimensional ℓ_∞^n.



Banach spaces admitting a polyhedral renorming are called *isomorphically polyhedral*. Examples

- c_0 (polyhedral) $\quad$ c (isomorphically polyhedral)
- $C_0(\alpha)$ for countable α (polyhedral) $\quad$ $C(\alpha)$ (i.p.)
- All their subspaces (Schreier space; most combinatorial Banach spaces).
- There are others (Kunen's space, Gasparis constructed a polyhedral space having a Hibert quotient).

Exact sequences

An *exact sequence* of Banach spaces is a diagram of Banach spaces and (linear, continuous) operators

$$0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0 \tag{1}$$

in which the kernel of every operator agrees with the image of the preceding. This amounts to saying that j is an into isomorphism, q is a quotient map and X is isomorphic to $Z/j(Y)$. The middle space Z is called a *twisted sum* of Y and X.

The 3-space problem for isomorphic polyhedrality

Reasonable candidates to counterexamples:

- Some diabolic sequence

$$0 \longrightarrow c_0 \longrightarrow Z \longrightarrow c_0(\aleph) \longrightarrow 0$$

  The standard ones, generated by an uncountable almost disjoint family of subsets of $\mathbb{N}$, are called Nakamura-Kakutani sequences, and there is an extensive literature about them.

1. F. Cabello, J.M.F. Castillo, W. Marciscewski, G. Plebanek, A. Salguero, *Sailing over three problems of Koszmider*, J. Funct. Anal. 279 (2020) 108571
2. P. Koszmider, *On decomposition of Banach spaces of continuous functions on Mrówka's spaces*, Proc. Amer. Math. Soc. 133 (2005), 2137–2146.
3. P. Koszmider, N.J. Laustsen, *A Banach space induced by an almost disjoint family, admitting only few operators and decompositions*, arXiv:2003.03832.

The 3-space problem for isomorphic polyhedrality

Another reasonable candidate would be an exotic sequence of the type

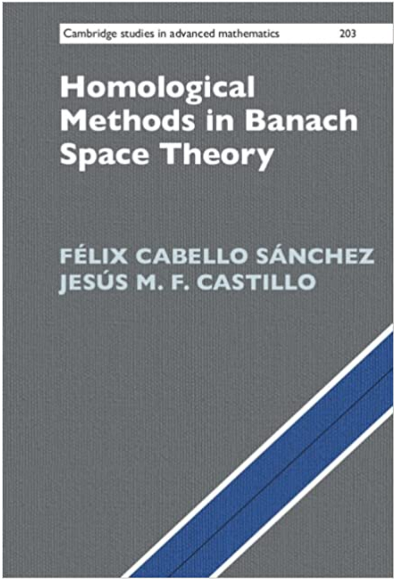
$$0 \longrightarrow C(\omega^\omega) \longrightarrow Z \longrightarrow c_0 \longrightarrow 0$$

The most exotic were constructed in

F. Cabello Sánchez, J.M.F. Castillo, N.J. Kalton, D.T. Yost, *Twisted sums with $C(K)$ spaces*, Trans. Amer. Math. Soc. 355 (2003) 4523–4541.

And they are so strange that have a strictly singular quotient map. The reader can find most of what is known about them in:

The book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Homological Methods in Banach Space Theory
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JESÚS M. F. CASTILLO

No

Claim, maybe Theorem.

None of those examples can work as a counterexample to the 3-space problem for isomorphic polyhedality.

Boundaries

Some of the most useful criteria for recognizing isomorphically polyhedral spaces involve the behaviour of *boundaries*.

- Given X a Banach space, a subset B of S_{X^*} is a *boundary* if for every $x \in X$ there is $x^* \in B$ so that $x^* x = \|x\|$.
- A boundary $\mathfrak{B}$ is said to have *property (*)* if whenever f is a weak*-accumulation point of $\mathfrak{B}$ and x_0 is a normalized element then $f(x_0) < 1$.
- Banach spaces with boundaries enjoying property $(*)$ form the most general class known nowadays of isomorphically polyhedral spaces. In fact, a separable Banach space is isomorphically polyhedral if and only if it admits a boundary having property $(*)$.

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Polyhedral norms on non-separable Banach spaces

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Abstract

We prove the existence of equivalent polyhedral norms on a number of classes of non-separable spaces, the majority of which being of the form $\mathcal{C}(K)$. In particular, we obtain a complete characterization of those trees $\mathcal{T}$, such that $\mathcal{C}_0(\mathcal{T})$ admits an equivalent polyhedral norm.

New results

Proposition

If X is a Banach space having a boundary $\mathfrak{B}$ with property $(*)$ then every twisted sum of $c_0(\kappa)$ and X has boundary with property $(*)$, hence it is isomorphically polyhedral

What makes this interesting is that it actually provides a polyhedral renorming of Z in

$$0 \longrightarrow c_0(\kappa) \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$$

And it does in two steps:

The homological part

One: there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c_0(\kappa) & \longrightarrow & C(L) & \xrightarrow{r} & C(B_{X^*}) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow \delta & & \\
0 & \longrightarrow & c_0(\kappa) & \xrightarrow{j} & Z & \xrightarrow{q} & X & \longrightarrow & 0
\end{array} \qquad (2)$$

where L is a compact space containing an homeomorphic copy of (B_{X^*}, w^*) such that $L \setminus B_{X^*}$ is a discrete space of size κ. Thus, after renorming, Z can be identified with the subspace

$$\{f \in C(B_{X^*} \cup \kappa) : \exists x \in X : \; f|_{B_{X^*}} = \delta(x)\}.$$

The topological part

Then, we show that

$$W = \{\delta_\alpha|_Z : \alpha < \kappa\} \cup \{2\delta_b|_Z : b \in \mathfrak{B}\}$$

is a boundary with property $(*)$ for anew norm on Z:

$$\nparallel z \nparallel = \sup\{|\langle z, v\rangle| : \; v \in W\}$$

Hence, it seems that counterexamples for the 3-space problem of the form

$$0 \longrightarrow c_0(\aleph) \longrightarrow Z \longrightarrow X \longrightarrow 0$$

likely do not exist.

Worse things happen at sea

The previous approach likely does not work for twistings with $C(\omega^\alpha)$. We however have:

Theorem

Let $\alpha < \omega_1$ be a countable ordinal. If X is an isomorphically polyhedral separable space with the BAP then every twisted sum of $C(\alpha)$ and X is isomorphically polyhedral.

With a purely drawable proof:

The proof

Set $C(\omega^\omega) \simeq c_0(\mathbb{N}, C(\omega^N)$ with projections P_N and draw:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c_0(\mathbb{N}, C(\omega^N)) & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle P_N} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & C(\omega^N) & \longrightarrow & Z_N & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

And draw:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c_0(\mathbb{N}, C(\omega^N)) & \xrightarrow{(\imath_N)} & c_0(\mathbb{N}, Z_N) & \longrightarrow & c_0(\mathbb{N}, X) & \longrightarrow & 0 \\
 & & \| & & & & & & \\
0 & \longrightarrow & c_0(\mathbb{N}, C(\omega^N)) & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

and keep drawing

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0(\mathbb{N}, C(\omega^N)) & \xrightarrow{(\imath_N)} & c_0(\mathbb{N}, Z_N) & \xrightarrow{(Q_N)} & c_0(\mathbb{N}, X) & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle v} & & \uparrow{\scriptstyle u} & & \\
0 & \longrightarrow & c_0(\mathbb{N}, C(\omega^N)) & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0
\end{array}
$$

Therefore, Z is isomorphic to a subspace of the product

$$c_0(\mathbb{N}, Z_N) \oplus_\infty X$$

which is isomorphically polyhedral.

Hence, it seems that counterexamples for the 3-space problem of the form

$$0 \longrightarrow C(\alpha) \longrightarrow Z \longrightarrow X \longrightarrow 0$$

likely do not exist.

Where finding a counterexample or a proof?

- At this moment there is no reasonable candidate to counterexample in sight.
- Worse than that: even if a counterexample were right in front of our eyes... because

A twisted sum spaces $0 \longrightarrow c_0 \longrightarrow Z \longrightarrow c_0(I) \longrightarrow 0$

1. Is c_0-saturated and c_0-uppersaturated.
2. Is an Asplund space. Consequently, it has weak*-sequentially compact dual ball, it is weak*-extensible and has the Gelfand-Phillips property.
3. Is isomorphic to a Lindenstrauss space; consequently, it has Pełczyński's property (V).
4. Is WCG if and only if is isomorphic to $c_0(I)$.
5. $Z \simeq Z \oplus c_0$.
6. Is separably injective and not universally separably injective.

This talk contains (I hope) ideas from

[1] J.M.F. Castillo, P.L. Papini, *On isomorphically polyhedral L_∞-spaces*, J. Funct. Anal. 270 (2016) 2336–2342.

[2] J.M.F. Castillo, A. Salguero Alarcón, *Twisted sums of $c_0(I)$*, Quaestiones Math. (2022) https://doi.org/10.2989/16073606.20.

[3] J.M.F. Castillo, A. Salguero Alarcón, *Polyhedrality for twisted sums with $C(\omega^\alpha)$*, Studia Mathematica (to appear)