# Diameter 2 properties – some glimpses

## - as seen by me -

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# The Banach group at UiA

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# Main collaborating institutions - not ranked!

- Murcia and Granada
- Tartu
- Berlin
- Paris VI and Beçanson
- Prague

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# Notation and conventions

**Let us agree about the following:**

Throughout

- $X$ is a Banach space and $B_X$ its unit ball.
- $S$ means a slice of $B_X$, i.e.,

$$S = \{x \in B_X \,:\, x^*(x) > \sup_{z \in B_X} \mathfrak{Re}\{x^*(z)\} - \varepsilon\}$$

  for some $x^* \in X$ and $\varepsilon > 0$.
- $S^n$ means a convex combination of $n$ slices of $B_X$.
- $U$ means a non-empty relatively open subset of $B_X$ in the weak topology, i.e., $U = V \bigcap B_X$ for some weakly open subset of $X$.

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# Bourgain's lemma – the way we need it

**Lemma**

Any $U$ contains some $S^n$.

# The first three diameter 2 properties

## Definitions (Abrahamsen, Lima, N 2013)

- $X$ has the local diameter 2 property [LD2P] if any $S$ has diameter 2.
- $X$ has the diameter 2 property [D2P] if any $U$ has diameter 2.
- $X$ has the strong diameter 2 property [SD2P] if any $S^n$ has diameter 2.

## Observation

Clearly D2P implies LD2P and, by Bourgain's lemma, SD2P implies D2P.

## Unknown to us when formulating the definitions

Are they really different?

# What we knew and what we proved in 2013

## Main examples

- Spaces with the Daugavet property have SD2P (Kadets-Shvidkoy-Siritkin-Werner ca. 2000)
- Uniform algebras have D2P (N-Werner 2001)
- $X^*$ being a proper L-summand in $X^{***}$ implies $X$ has D2P. In particular, proper M-embedded spaces have D2P (G. Lopez-Perez and M. Martin 2007)
- SD2P is stable with respect to $\oplus_1$ and $\oplus_\infty$ (Implicit in G. Lopez-Perez and M. Martin 2007)
- In all the above cases we obtain SD2P
- Perhaps most interesting: LD2P and D2P are stable with respect to $\oplus_p$ for any $1 \leq p \leq \infty$.

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# What happened VERY quickly after

Added in proof

Some few days before receiving the referee's report, the authors have been informed from University of Tartu, Estonia, that the answer to the above question (c) is negative and that, in particular, $c_0 \oplus_2 c_0$ does not have the strong diameter 2 property. A paper containing this result is in preparation.

# What happened quite quickly after

## Results

- In fact, taking $\oplus_p$ *always* destroys the SD2P if $1 < p < \infty$ (Independently discovered by Tartu and Granada).
- Any space containing an isomorphic copy of $c_0$ can be equivalently renormed to have LD2P, but not D2P. In fact, the renormed space has $U$'s of arbitrarily small diameter (Becerra-Guerrero, Lopez-Perez and Rueda Zoca 2015)
- All diameter 2 properties as well as the Daugavet property pass down to locally 1-complemented subspaces where the local retractions are almost isometric (ai-ideals) (Abrahamsen-Lima-N 2015).

# What now 1?

So, it is clear that LD2P, D2P and SD2P are different. Also, a 1989-result of Deville and Godefroy came into our knowledge:

**Theorem**

$X$ has SD2P if and only if $X^*$ is octahedral.

**Definition**

$X^*$ is octahedral if, for every finite-dimensional subspace $E$ of $X^*$ and every $\varepsilon > 0$, there is a $y^* \in X^*$ with $\|y^*\| = 1$ such that

$$\|x^* + y^*\| \geq (1 - \varepsilon)(\|x^*\| + \|y^*\|)$$

for all $x^* \in E$.

# What now 2?

## Main questions 2013 – 2015

- Are there octahedralities also corresponding to LD2P and D2P? (Yes, Tartu)
- Find properties that imply diameter 2 properties. (Almost squareness, stronger versions of SD2P, ... ) (Some 5 papers have been written)
- What about spaces that are inheritably (L)[S]D2P? (Only known examples to me are M-embedded spaces.)
- How rotund can diameter 2 spaces be? (Some 3–4 papers are written, I'll come back to this later)
- What about free spaces $\mathcal{F}(M)$ and their duals, $Lip_0(M)$ for various metric spaces $M$? (Many papers, I'll say a little more later)

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# Diameter 2 and convexity

When all slices have diameter 2, they are somehow parallel to a part of $S_X$. Thus, it feels like LD2P is a kind of "squareness".

## Observations

- However, there are proper M-embedded spaces that are strictly convex (and recall that such spaces are hereditarily SD2P)!
- Note that there are no points of continuity on $S_X$ when $X$ has LD2P. Thus such spaces are far from able to carry a LUR-norm.

# MLUR-renorming

**Midpoint Locally Uniform Rotundity**

$X$ is MLUR if every $x$ in $S_X$ is a strongly extreme point, i.e., for every sequence $(x_n)$ in $X$, we have that $x_n \to 0$ in norm whenever $\|x \pm x_n\| \to 1$.

**Theorem (Abrahamsen, Hajek, N, Talponen, Troyanski 2016)**

There is a (not very complicated) renorming of $C[0,1]$ which gives a simultaneous MLUR and D2P space.

**Remark**

The above renorming is not SD2P. I don't know whether MLUR and SD2P is possible at all.

# Renorming $L_1[0, 1]$

**weak Midpoint Locally Uniform Rotundity**

$X$ is weak MLUR if every $x$ in $S_X$ is a weak strongly extreme point, i.e., for every sequence $(x_n)$ in $X$, we have that $x_n \to 0$ weakly whenever $\|x \pm x_n\| \to 1$.

**Theorem (N, Põldvere, Troyanski, Viil 2024)**

There is a (more complicated) renorming of $L_1[0, 1]$ which gives a simultaneous w-MLUR and D2P space.

**Remark**

The above renorming is not SD2P and not MLUR.

# A question of Troyanski

Recall an old result that $X^{(4)}$ is never strictly convex...

**Troyanski's question appr. 2014**

Can $X^{**}$ be strictly convex if $X$ has any of the diameter 2 properties?

**Theorem (Abrahamsen, Lima, N, Troyanski, also 2016)**

If $X$ has SD2P, then $X^{**}$ is neither smooth nor strictly convex.

**Still open (to my knowledge)**

Can $X^{**}$ be strictly convex if $X$ has (L)D2P?

# Diameter 2 properties in $\mathcal{F}(M)$

Let $\mathcal{F}(M)$ denote the free space over a pointed metric space $M$ (=predual of $Lip_0(M)$.) Both $\mathcal{F}(M)$ and its dual have a very strong tendency of being diameter 2 spaces. In fact:

**Theorem (Langemets, Rueda Zoca)**

If $M$ is unbounded or not uniformly discrete, then $Lip_0(M)$ has (even a strong version of) the SD2P.

**The hunting plan**

Characterize $M$ such that $Lip_0(M)$ or $\mathcal{F}(M)$ gets (L)[S]D2P.

If one can do so, we get *constructed* spaces to illustrate the difference among diameter 2 properties, which I somehow like better than existence of renormings.

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# Hunting

## Example of successful hunting

- Tartu group has characterized $M$ for LD2P and SD2P in $Lip_0(M)$.
- I am sure more is already known and more is to come :-)

# End

Thank you for listening!