On the spaces dual to combinatorial Banach spaces

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Combinatorial Banach space

Definition

- Let $\mathcal{F} \subseteq [\omega]^{<\omega}$ be
  (i) *hereditary* i.e $\forall A \in \mathcal{F} (B \subseteq A \Rightarrow B \in \mathcal{F})$
  (ii) covering ω.

A *combinatorial Banach space* is a space, denoted by $X_{\mathcal{F}}$, being a completion of c_{00} with respect to the norm

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- Equivalently

$$X_{\mathcal{F}} = \{x \in \mathbb{R}^{\omega} : \|x|_{[n,\infty)}\|_{\mathcal{F}} \to 0\} \; (= \mathrm{EXH}(\| \cdot \|_{\mathcal{F}})$$

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- If $\mathcal{F} = [\omega]^{<\omega}$, then $X_{\mathcal{F}} = \ell_1$.
- Let $\mathcal{P} = \{[2^n, 2^{n+1}) : n \in \omega\}$, where $[2^n, 2^{n+1}) = \{2^n, 2^n + 1, ..., 2^{n+1} - 1\}$ and let $\mathcal{F} = h(\mathcal{P})$ - hereditary closure of $\mathcal{P}$. Then $X_{\mathcal{F}} = \bigoplus_{c_0} \ell_1^{2^n}$, where

$$\bigoplus_{c_0} \ell_1^{2^n} = \{x = (x_n) \in \prod_n \ell_1^{2^n} : \|x_n\|_1 \to 0\}$$

is a Banach space equipped with a norm

$$\|x\| = \sup_n \|x_n\|_1$$

Important example

- Let $\mathcal{S} = \{A \in [\omega]^{<\omega} : |A| \leq \min(A)\}$. It is called the *Schreier family* and $X_{\mathcal{S}}$ is the *Schreier space*.

Goal

Describe the dual to $X_{\mathcal{F}}$.

Candidate for the dual norm

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- Then we define

$$X^{\mathcal{F}} = \mathrm{EXH}(\|\cdot\|^{\mathcal{F}}) := \{x \in \mathbb{R}^{\omega} : \|x|_{[n,\infty)}\|^{\mathcal{F}} \to 0\}$$

$$= \{x \in \mathbb{R}^{\omega} : \|x\|^{\mathcal{F}} < \infty\} =: \mathrm{FIN}(\|\cdot\|^{\mathcal{F}})$$

Example of calculation

- Consider finitely supported sequence $y = (4, 3, 2, 1, 3, 2, 1, 2, 1, 1, 0, 0, 0, \ldots)$ and let $\mathcal{F}$ be the Schreier family.

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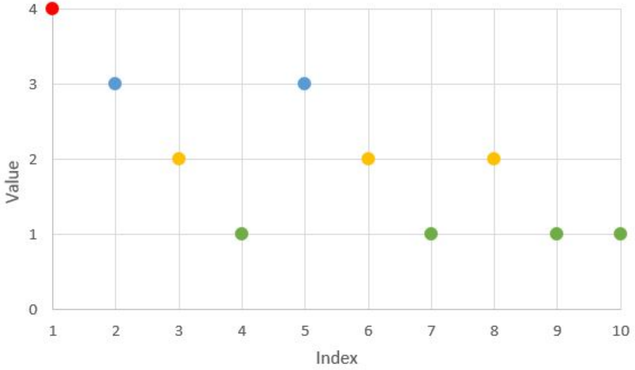
- To find $\|y\|^{\mathcal{F}}$ we need to find the partition for which the infimum is attained.

Example of calculation



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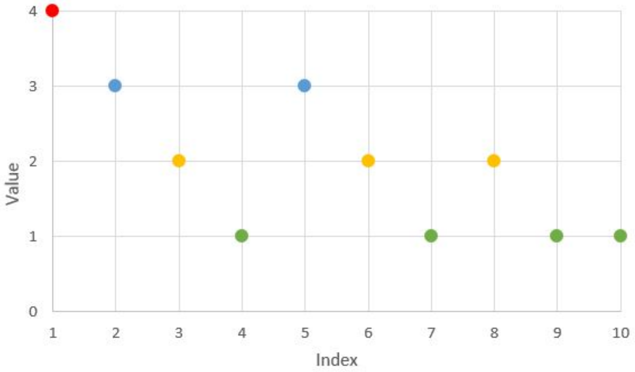

- Good partition must contain $\{1\}, \{2,5\}, \{3,6,8\}, \{4,7,9,10\}$
- $\|y\|^{\mathcal{F}} = 4 + 3 + 2 + 1 = 10.$

Question

Fix regular family $\mathcal{F}$ (hereditary, covering ω and compact). Is $X^{\mathcal{F}}$ isomorphic to $X^*_{\mathcal{F}}$? In particular, does it hold for the Schreier family $\mathcal{S}$?

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It turns out that $\|\cdot\|^{\mathcal{F}}$ is a quasi-norm.
For all $x, y \in X^{\mathcal{F}}$

$$\|x + y\|^{\mathcal{F}} \leq 2(\|x\|^{\mathcal{F}} + \|y\|^{\mathcal{F}})$$

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Then $X^{\mathcal{F}}$ is a quasi-Banach space.

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- Let $\mathcal{P} = \{[2^n, 2^{n+1}) : n \in \omega\}$ and $\mathcal{F} = h(\mathcal{P})$. Then

$$X^{\mathcal{F}} = \bigoplus_{\ell_1} c_0^{2^n} = \Big(\bigoplus_{c_0} \ell_1^{2^n}\Big)^* = X_{\mathcal{F}}^*$$

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In particular, the answer is negative for the Schreier space.

Banach envelopes

For every $x \in X^{\mathcal{F}}$ let

$$|||x|||^{\mathcal{F}} = \inf \{ \sum_{i=1}^{n} \|x_i\|^{\mathcal{F}} : n \in \omega, x_1, ..., x_n \in X^{\mathcal{F}}, x = \sum_{i=1}^{n} x_i \}$$

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If $\mathcal{F}$ is a regular family then $\widehat{X^{\mathcal{F}}}$ is isometrically isomorphic to $X^*_{\mathcal{F}}$.

Corollary

$(X^{\mathcal{F}})^*$ is isometrically isomorphic to $X^{**}_{\mathcal{F}}$.

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- *CSRP* (**C**onvex **S**eries **R**epresentation **P**roperty): for every x from the unit ball

$$x = \sum_{n=1}^{\infty} \lambda_n e_n$$

where $\lambda_n > 0$, $\sum_{n=1}^{\infty} \lambda_n = 1$ and e_n is an extreme point of the unit ball.

Problem

Let X be a quasi-Banach space and let Y be its Banach envelope. Which properties of X are shared by Y?

THANK YOU!