Some applications of Boolean algebras and the Stone representation theorem to fixed point theory

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Metric Fixed Point Theory: Standard definitions and brief background

- A metric space (M, d) has the FPP if every 1-Lipschitz map $T : M \to M$ (nonexpansive: $d(Tx, Ty) \le d(x, y)$), has a fixed point.

- A. Kirk (1965): If C is a convex weakly compact set of a Banach space with normal structure, then C has the FPP.

  Brodskii-Milman (1948): A c.c.b. set C of a Banach space X has normal structure if for all convex $H \subset C$, $\exists x_0 \in H$ with $H \subset B(x_0, r)$ and $0 < r < \text{diam}(H)$.

- F. Browder (1965): Every closed convex bounded subset of a UC Banach space has the FPP. In fact, if H is convex and bounded, $H \subset B(x_0, c \cdot diam(H))$ for $c = 1 - \delta_X(1) < 1$ for some $x_0 \in H$.

The failure of the FPP is, of course, possible

Let $(c_0, \|\cdot\|_\infty)$ be the Banach space of all null convergent sequences.
$T : B_{c_0} \to B_{c_0} \qquad T(x_1, x_2, x_3, \cdots) = (1, x_1, x_2, x_3, ...)$
is an isometry and fixed point free.

Let $(c, \|\cdot\|_\infty)$ be the Banach space of all convergent sequences.

$T : B_c \to B_c \qquad T(x_1, x_2, x_3, \cdots) = (1, -1, x_1, x_2, x_3, ...)$
is an isometry and fixed point free.

Note that: If $K = \mathbb{N} \cup \{\omega\}$, the one-point compactification of $\mathbb{N}$, then:

$$c = C(K) \qquad c_0 = C_0(K, \omega) = \{f \in C(K) : f(\omega) = 0\}.$$

The failure of the FPP is, of course, possible

K. Goebel, A. Kirk: **Topics in metric fixed point theory**. Cambridge, 1990.

$K = [-1, 1]$ and $T : B_{C(K)} \to B_{C(K)}$ given by:

$$T(f)(t) = \min\{1, \max\{-1, f(t) + t\}\}$$

T is nonexpansive and fixed point free.

If $T(f) = f$ then $f(t) = 1$ for $t \in (0, 1]$ and $f(t) = -1$ for $t \in [-1, 0)$.

Then T cannot have a fixed point in $C(K)$.

The closed unit ball of ℓ_∞ has the FPP

$(\ell_\infty, \|\cdot\|_\infty)$ contains an isometric copy of every separable Banach space. Thus, ℓ_∞ seems an unsuitable space to seek for the FPP. Nevertheless.....

P.M. Soardi, *Existence of fixed points of nonxpansive mappings in certain Banach lattices*. Proc. Amer. Math. Soc. (1979).

Theorem

Let X be the dual of an AL Banach lattice ($\|x+y\| = \|x\| + \|y\|$, $x, y \geq 0$). Let B_X be the closed unit ball of X.
Then, every nonexpansive mapping $T : B_X \to B_X$ has a fixed point.

Consequence: B_{ℓ_∞}, the closed unit ball of ℓ_∞, and
$B_{L_\infty(\mu)}$, the closed unit ball of $L_\infty(\mu)$, have the FPP.

The Ball Fixed Point Property

Definition

Let X be a Banach space. We say that X has the ball-FPP (BFPP) if its closed unit ball B_X has the FPP.

- ℓ_∞ and $L_\infty(\mu)$ enjoy the BFPP.
- c_0 fails the BFPP.
- c fails the BFPP

J. Borwein, B Sims *Nonexpansive mappings on Banach lattices and related topics*. Houston J. of Math, (1984):
Every convex weakly compact of either c_0 or c has the FPP (because c and c_0 are weakly orthogonal Banach lattices).

$C(K)$ and BFPP: Does K determine the BFPP for $C(K)$?

Let K be an infinite compact Hausdorff space.
$C(K) = \{f : K \to \mathbb{R} : \text{ continuous } \}$; $\|f\| = \sup\{|f(x)| : x \in K\}$.

$$K,\, L \text{ are homeomorphic} \iff C(K) \text{ and } C(L) \text{ are isometric}$$

$$K \text{ is metrizable} \iff C(K) \text{ is separable}$$

$$K \text{ is extremally disconnected} \iff C(K) \text{ is order-complete}$$

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$$K \text{ is extremally disconnected} \iff C(K) \text{ is order-complete}$$

$$K \quad ?? \qquad \iff C(K) \text{ has the BFPP}$$

Joint work with Antonio Avilés, Gonzálo Martínez (University of Murcia), and Chris Lennard and Adam Stawski (University of Pittsburgh).

A sufficient condition for the BFPP: ED compact spaces

Definition

A topological space is extremally disconnected if disjoint open sets have disjoint closures.

Example: $\beta\mathbb{N}$ is extremally disconnected and $\ell_\infty \equiv C(\beta\mathbb{N})$.

Theorem

If K is extremally disconnected, then $C(K)$ has the BFPP. Additionally if X is an injective Banach space, then X has the BFPP.

Proof: We will make use of the *uniform normal structure* and:
(Nachbin-Kelley): X is injective $\Leftrightarrow$ X isometric to $C(K)$ with K extremally disconnected $\Leftrightarrow$ every family of mutually intersection closed balls in $C(K)$ has a common point.

Uniform normal structure for admissible sets in a m.s.

- Let (M, d) be a metric space. A subset $A \subset M$ is said to be admissible if it is a nonempty intersection of closed balls.
- A metric space (M, d) has uniform normal structure (UNS) if there is some $0 < c < 1$ such that for all A admissible,

$$A \subset B(x_0, c \cdot diam(A)) \text{ for some } x_0 \in A$$

- A bounded metric space (M, d) with UNS has the FPP.

Example: If X is injective, then B_X has UNS for $c = \frac{1}{2}$.

Consequence: If K is extremally disconnected, $B_{C(K)}$ has the FPP (and there exist some non-duals $C(K)$ with K ED).

Hyperconvexity and FPP

N. Aronszajn, P. Panitchpakdi, *Extensions of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math., (1956).

Definition

A metric space (M, d) is hyperconvex if $\forall$ collection $(x_\alpha)_{\alpha\in I}$ and radius $\{r_\alpha\}_{\alpha\in I}$, $\bigcap_{\alpha\in I} B(x_\alpha, r_\alpha) \neq \emptyset$ whenever $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for $\alpha, \beta \in I$.

Given κ a cardinal, (M, d) is said to be κ-hyperconvex if the above holds for every collection $(x_\alpha)_{\alpha\in I} \subset M$ with $card(I) \leq \kappa$.

Corollary (Baillon, 1988)

Every bounded hyperconvex metric space has the FPP.

Question: If (M, d) is a bounded κ-hyperconvex metric space for some cardinal κ, does (M, d) have the FPP?

A necessary condition for the BFPP: F-spaces

Let $f : K \to \mathbb{R}$ continuos. $Cz(f) = \{x \in K : f(x) \neq 0\}$ (open F_σ).

Definition (L. Gillman, M. Jerison (Rings of Continuous Functions,1960))

K is an F-space if disjoint cozero sets have disjoint closures.

- Every extremally disconnected space is an F-space. Every closed subset of an extremally disconnected space is an F-space.
- $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ is an F-space, which is not extremally disconnected.
- TFAE:
  - K is an F-space
  - $C(K)$ is countable-order-complete: If $A, B \subset C(K)$ countable with $a \leq b$ $\forall a \in A, b \in B$, $\exists f \in C(K)$ with $a \leq f \leq b$, $\forall a \in A, b \in B$.
  - $C(K)$ is ω-hyperconvex.

A necessary condition for the BFPP

Theorem

If $C(K)$ verifies the BFPP, then K is an F-space.

Proof: U, V cozero sets with $U \cap V = \emptyset$, $\Rightarrow \overline{U} \cap \overline{V} = \emptyset$?

WLOG: $U = \{u > 0\}$, $V = \{v > 0\}$, $0 \leq u, v \leq 1$.

Define $T : B_{C(K)} \to B_{C(K)}$: $T(f) = [1 - (u + v)]\, f + (u - v)$.

$T(f)$ is continuous verifying: $T(f)(t) = (1 - u(t))f(t) + u(t)$ if $t \in U$;

$T(f)(t) = (1 - v(t))f(t) - v(t)$ if $t \in V$; and $T(f)(t) = f(t)$ otherwise.

T is nonexpansive. Hence $\exists f \in B_{C(K)}$ with $Tf = f$ (from the BFPP).

$Tf = f \Rightarrow (u + v)f = (u - v) \Rightarrow u(t)f(t) = u(t)\ t \in U$; $v(t)f(t) = -v(t)\ t \in V$.

$\Rightarrow f(t) = 1$ for $t \in U$; $f(t) = -1$ for all $t \in V$.

$\Rightarrow \overline{U} \subset f^{-1}(1)$, $\overline{V} \subset f^{-1}(-1)$. Consequently $\overline{U} \cap \overline{V} = \emptyset$. $\Box$

Examples of non-F spaces and the failure of the BFPP

- If K contains a nontrivial convergent sequence, then K is a non-F-space: Metrizable compact sets are non-F spaces.
- $K = [0,1]^{\mathbb{R}}$, $K = \beta\mathbb{Q}$, $K = \beta\mathbb{R}$ are non-F-spaces.
- Let K_1, K_2 be infinite and compact: $K_1 \times K_2$ is a non-F space.

Corollary

- *For every metric compact space, $C(K)$ fails the BFPP.*
- *$C(\beta\mathbb{R})$ or $C(\beta\mathbb{Q})$ fails the BFPP.*
- *$C(K_1 \times K_2)$ fails the BFPP: $C(\beta\mathbb{N} \times \beta\mathbb{N})$ fails the BFPP.*
- *Let $p, q \in \beta\mathbb{N} \setminus \mathbb{N}$, $X := \{f \in C(\beta\mathbb{N}) : f(p) = f(q)\}$.*

  Then X fails the BFPP since $X \equiv C(K)$ with K a non-F space.

A necessary condition for the BFPP in $C_0(K,p)$: P-points.

$p \in K$, $C_0(K,p) := \{f \in C(K) : f(p) = 0\}$. For instance:

- $c_0 = C_0(\mathbb{N} \cup \{\omega\}, \omega)$ fails the BFPP.
- Fix $n \in \mathbb{N}$. $C_0(\beta\mathbb{N}, n)$ has the BFPP (isometric to $C(\beta\mathbb{N})$).
- Fix some $p \in \beta\mathbb{N} \setminus \mathbb{N}$: Does $C_0(\beta\mathbb{N}, p)$ have the BFPP?

Definition

A point $p \in K$ is said to be a P-point if every G_δ set containing p is a neighbourhood of p. Otherwise, $p \in K$ is said to be a non-P-point.

p is a P-point $\Leftrightarrow$ every $f \in C(K)$ is constant on a neighbourhood of p.
p is a non-P-point $\Leftrightarrow$ $p \in \overline{Cz(f)}$ for some $f \in C_0(K,p)$.

- If $K = \mathbb{N} \cup \{\omega\}$, ω is a non-P-point (every sequence limit).
- Every $p \in \beta\mathbb{N} \setminus \mathbb{N}$ is a non-P-point.
- Every infinite compact set K contains some non-P-point.

Theorem

If $C_0(K, p)$ has the BFPP, then p is a P-point of K.

Proof: Let $u \in C(K)$. Is u constant on a neigh. of p?

WLOG: $u(p) = 0$, that is, $u \in C_0(K, p)$ and $0 \leq u \leq 1$.

Define $Tf = (1 - u)f + u$ for $f \in B_{C_0(K,p)}$. Since T is nonexpansive, $\exists f \in C_0(K, p)\ Tf = f \Rightarrow uf = u$. Then $u = 0$ in $f^{-1}(-1, 1)$, a neigh of p.

Some consequence:

- $c_0 = C(\mathbb{N} \cup \{\omega\}, w)$ fails the BFPP.
- For $p \in \beta\mathbb{N} \setminus \mathbb{N}$, $C_0(\beta\mathbb{N}, p)$ fails the BFPP.
- For every infinite compact space K there are points $p \in K$ such that $C_0(K, p)$ fails the BFPP.

The necessary conditions are not sufficient

Let $X_0 := \{f \in \ell_\infty([0,\omega_1)) : \text{ with countable support}\}$.

X_0 fails the BFPP

Define $T : B_{X_0} \to B_{X_0}$ in the following way.

- $Tf(0) = 1$
- For successor ordinals, $Tf(\alpha+1) = f(\alpha)$.
- If β is a limit ordinal $Tf(\beta) = \inf_{\alpha<\beta} \sup_{\gamma>\alpha} f(\gamma)$.

T is well defined and isometry. Assume that $Tf = f$:

Set $A := \{\alpha < \omega_1 : f(\alpha) < 1\} \neq \emptyset$ since $f \in X_0$.

Set $\mu := \min(A)$. This μ is limit ordinal. Indeed:
if $\mu = \alpha + 1$, $f(\alpha) = Tf(\alpha+1) = f(\alpha+1) = f(\mu) < 1 \Rightarrow \alpha \in A$.
Then $f(\gamma) = 1$ for all $\gamma < \mu$, so $f(\mu) = Tf(\mu) = \inf_{\alpha<\beta} \sup_{\gamma>\alpha} f(\gamma) = 1$,
which contradicts the fact that $\mu \in A$.

The necessary conditions are not sufficient

$X = \{f \in \ell_\infty([0,\omega_1)) : \ f \text{ is constant out a countable subset of } [0,\omega_1)\}.$

X fails the BFPP

Define $T : B_X \to B_X$ in the following way: $Tf(0) = 1$, $Tf(1) = -1$.

- For double successor ordinals, $Tf(\alpha + 2) = f(\alpha)$.
- If β is a limit ordinal: $$Tf(\beta) = \inf_{\alpha<\beta} \sup_{\gamma>\alpha} f(\gamma),$$ $$Tf(\beta+1) = \sup_{\alpha<\beta} \inf_{\gamma>\alpha} f(\gamma).$$

T is an isometry. Assume that $Tf = f$.
$A := \{\beta < \omega_1 \text{ limit ordinal} : (f(\beta), f(\beta+1)) \neq (1,-1)\}$. Set $\mu := \min(A)$.

For all limit ordinals $\beta < \mu$: $f(\beta) = 1$ and $f(\beta+1) = -1$.
By $f(\alpha+2) = Tf(\alpha+2) = f(\alpha)$, f takes value 1 on all even ordinals below μ and value -1 on all odd ordinals below μ. Then:
$f(\mu) = Tf(\mu) = \inf_{\alpha<\mu} \sup_{\gamma>\alpha} f(\gamma) = 1,$
$f(\mu+1) = Tf(\mu+1) = \sup_{\alpha<\mu} \inf_{\gamma>\alpha} f(\gamma) = -1$. This means that $\mu \notin A$.

Boolean algebras and the Stone theorem.

Let $(\mathcal{A}, +, \cdot, -, 0, 1)$ be a Boolean algebra ($x \leq y$ if $x + y = y$).

- A filter in $\mathcal{A}$ is a subset p of $\mathcal{A}$ verifying $0 \notin p$, $1 \in p$; if $x \in p$, $y \in \mathcal{A}$ and $x \leq y$ then $y \in p$; if $x, y \in p$ then $x \cdot y \in p$.
- A filter p in a Boolean algebra $\mathcal{A}$ is an ultrafilter if there is not filter in $\mathcal{A}$ strictly containing p.

Denote $Ult\mathcal{A} := \{p : p \text{ is an ultrafilter in } \mathcal{A}\}$. For every $A \in \mathcal{A}$, define

$$\widehat{A} := \{p : \ p \text{ is ultrafilter in } \mathcal{A} \text{ with } A \in p\}.$$

The sets $\{\widehat{A} : A \in \mathcal{A}\}$ form a base of neighbourhoods for a topology $\tau_{\mathcal{A}}$ on $Ult\mathcal{A}$. Moreover, for every $A \in \mathcal{A}$, $Ult\mathcal{A} \setminus \widehat{A} = \widehat{-A}$, which implies that every basic set $\widehat{A}$ is closed and open (clopen). Actually:

$(Ult\mathcal{A}, \tau_{\mathcal{A}})$ is Hausdorff, compact, has a basis of clopen sets.
$(Ult\mathcal{A}, \tau_{\mathcal{A}})$ is called the Stone topological space of $\mathcal{A}$.

The necessary conditions are not sufficient

$\Gamma := [0, \omega_1)$.
$\mathcal{A}_c := \{A \subset \Gamma : \text{either } A \text{ or its complementary } \Gamma \setminus A \text{ is countable}\}$.

$\mathcal{A}_c$ is a countable complete Boolean algebra.
Then $K := Ult\mathcal{A}_c$ is basically disconnected, that is, the closure of every F_σ open set is open. In particular K is an F-space.

Theorem

The following holds:

- *X, the space of all bounded functions defined on $[0, \omega_1)$ that are constant out of a countable set, is isometric to $C(Ult\mathcal{A}_c)$.*
- *Additionally, X_0 is isometric to $C_0(Ult\mathcal{A}_c, p)$, where*

$$p = \{A \in \mathcal{A}_c : A \textit{ is uncountable}\}$$

p is an ultrafilter in $\mathcal{A}_c$ which is P-point of $Ult\mathcal{A}_c$.

Some consequences and extensions

Corollary

- *K being a compact F-space (or even being basically disconnected) is not enough for the BFPP in $C(K)$.*
- *Likewise, the condition of being a P-point is not enough for BFPP for $C_0(K, p)$.*
- *Being ω-hyperconvex is not enough for the FPP.*

Theorem

For all infinite cardinal κ, $\exists$ a compact topological space K such that

- *$C(K)$ is κ-order complete (if $A \subset C(K)$ bounded from above with $card(A) \leq \kappa$, then $sup(A) \in C(K)$)*
- *$B_{C(K)}$ is κ-hyperconvex.*

and yet, $B_{C(K)}$ fails the FPP.

Summarizing:

Being an F-space is a necessary condition for the BFPP (but not sufficient), since we have shown a compact F-space for which the BFPP fails.

But what about $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$? Does $C(\mathbb{N}^*)$ have the BFPP?

- $\mathbb{N}^*$ is an F-space (closed subspace of $\beta\mathbb{N}$).
- $\mathbb{N}^*$ is not extremally disconnected: Take (A_n) infinite pair-disjoint sets in $\mathbb{N}$ and consider $A_n^* = \overline{A_n}^{\beta\mathbb{N}} \setminus \mathbb{N}$ is open (and closed) in $\mathbb{N}^*$, but $\overline{\bigcup_n A_n^*}^{\mathbb{N}^*}$ is not open in $\mathbb{N}^*$.
- $C(\mathbb{N}^*) \equiv (\ell_\infty / c_0, \| \cdot \|_\sim)$ isometrically.

  I.E. Leonard, J. Whitfield, *A classic Banach space ℓ_∞ / c_0*. Rocky Mountain J. Math, 1983.

Does $C(\mathbb{N}^*)$ have the BFPP?

Definition

Let K, L be compact spaces: L is a continuous retract of K is there exist $e : L \to K$ and $r : K \to L$ continuous such that $r \circ e = Id_L$.

Every continuous function $\phi : K_1 \to K_2$ induces a nonexpansive linear operator $\phi^0 : C(K_2) \to C(K_1)$ acting by composition $\phi^0(f) = f \circ \phi$.

Lemma

If L is a continuous retract of K and $C(K)$ has the BFPP, then so does $C(L)$.

Indeed, let $T : B_{C(L)} \to B_{C(L)}$ be nonexpansive.
Define $\tilde{T} = r^0 \circ T \circ e^0 : B_{C(K)} \to B_{C(K)}$ and let $\tilde{T}(g) = g$ with $g \in C(K)$.

Define $f := g \circ e$. Then $Tf = T(g \circ e) = (T \circ e^0)(g) = (T \circ e^0)(g) \circ r \circ e = (e^0 \circ r^0 \circ T \circ e^0)(g) = e^0(Tg) = e^0(g) = f$.

Under CH, $C(\mathbb{N}^*)$ fails the BFPP

J. V. Mill. *An introduction to $\beta\omega$.* Chapter 11. Handbook of set-theoretical topology. North Holland, 1984.

Theorem (Theorem 1.4.4. and Theorem 1.8.1)

Under (CH), every compact F-space K of weight ω_1 is a continuous retract of $\mathbb{N}^*$.

Remember the Boolean algebra:
$\mathcal{A}_c := \{A \subset [0, \omega_1) : \text{either } A \text{ or its complementary is countable}\}$,
$K = (Ult\mathcal{A}_c, \tau_{\mathcal{A}_c})$, compact F-space with a basis of open sets:
$\{\hat{A} : A \in \mathcal{A}_c\}$.

In particular, under (CH), the weight of $Ult\mathcal{A}_c$ is ω_1.

Under CH, $C(\mathbb{N}^*)$ fails the BFPP

Corollary

Under (CH), $Ult\mathcal{A}_c$ is a continuous retract of $\mathbb{N}^*$.

Corollary

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Under (CH), $C(\mathbb{N}^*)$ fails the BFPP.

Questions:

- Is it possible to construct an explicit example of a fixed point free nonexpansive mapping on the unit ball of $C(\mathbb{N}^*)$, or equivalently on the unit ball of ℓ_∞/c_0?
- Is there a non-trivial case where $C_0(K, p)$ has the BFPP?

The question:

$$K, L \text{ are homeomorphic} \iff C(K) \text{ and } C(L) \text{ are isometric}$$

$$K \text{ is metrizable} \iff C(K) \text{ is separable}$$

$$K \text{ is extremally disconnected} \iff C(K) \text{ is order-complete}$$

$$K \quad ?? \quad \iff C(K) \text{ has the BFPP}$$

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Conjecture:

$$K \text{ extremally disconnected} \iff C(K) \text{ has the BFPP ??}$$

Thank you very much for your attention

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