The Grothendieck property for spaces $Lip_0(M)$ of Lipschitz functions

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Based on a joint work with Ch. Bargetz and D. Sobota

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5. (CH) There is compact X without $\beta\mathbb{N}$ and such that $C(X)$ is Grothendieck but without quotients isomorphic to ℓ_∞ (Talagrand).

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5. (CH) There is compact X without $\beta\mathbb{N}$ and such that $C(X)$ is Grothendieck but without quotients isomorphic to ℓ_∞ (Talagrand).
6. $((MA) \wedge \sim (CH))$ Every nonreflexive Grothendieck space has a quotient isomorphic to ℓ_∞ (Haydon-Levy-Odell).

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Theorem 1 (Cembranos)

A Banach space $C(X)$ is Grothendieck iff $C(X)$ does not contain a complemented copy of c_0.

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Theorem 2 (Cembranos-Freniche)

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4. We will study the same problem but for Banach spaces $Lip_0(M)$ of Lipschitz functions on a metric space M.

1. $Lip(M)$ - **Banach space of all bounded real-valued Lipschitz functions on** M with the norm $\|f\|_{Lip(M)} = \|f\|_\infty + lip(f)$, where $\|f\|_\infty = \sup_{x\in M} |f(x)|$, and $lip(f)$ *Lipschitz constant* of f.

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Theorem 3 (Hájek–Novotný)

M an infinite metric space. Then $Lip_0(M)$ contains an isomorphic copy of $\ell_\infty(d(M))$. Hence $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(d(M))$.

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3. Indeed, the space $\mathcal{F}(M)$ contains a complemented copy of the space $\ell_1(d(M))$. Hence $\mathcal{F}(M)$ contains a complemented copy of ℓ_1.
4. Since complemented subspaces of a Banach space with the Grothendieck property are Grothendieck, and ℓ_1 fails the Grothendieck property, the claim holds.

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2. Recall that a Banach space E is a Grothendieck space iff E^* is weakly sequentially complete and E has no quotient isomorphic to c_0 (Räbiger).

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2. Recall that a Banach space E is a Grothendieck space iff E^* is weakly sequentially complete and E has no quotient isomorphic to c_0 (Räbiger).
3. To prove that $Lip_0(M)$ is not Grothendieck (using Räbiger's criterion), we need either to look for a quotient of $Lip_0(M)$ isomorphic to c_0 or to show that the dual space $Lip_0(M)^*$ is not weakly sequentially complete.

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4. Grothendieck property is preserved by continuous (open) linear surjections, so $Lip_0(E)$ **lacks the Grothendieck property if E^* fails to have it** (since $Lip_0(E)$ contains a complemented copy of E^*). This provides examples of non-Grothendieck spaces $Lip_0(E)$ over Banach spaces E.

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Theorem 4 (Bargetz-Kąkol-Sobota)

Let E be a Banach space satisfying any of the following conditions: (1) There is a continuous linear surjection $T\colon E^* \to \ell_1$. (2) E is separable and contains an isomorphic copy of a predual of ℓ_1. (3) E contains a complemented copy of ℓ_1. (4) E has property (V) and E is not Grothendieck. Then $Lip_0(E)$ is not a Grothendieck space.

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Corollary 5

If $E = c_0$ or $E = \ell_1$, then $Lip_0(E)$ is not a Grothendieck space.

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Let E be a separable Banach space not weakly sequentially complete. Then $Lip_0(E)$ is not a Grothendieck space.

2. Indeed, $\mathcal{F}(E)$ contains an isometric copy of E. If E is separable, (by the lifting property) the space $\mathcal{F}(E)$ contains a linear isometric copy of E (Godefrey-Kalton).

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3. Hence $Lip_0(E)^* \simeq \mathcal{F}(E)^{**}$ is not weakly sequentially complete, so Räbiger's theorem applies.

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1. Summing up the above results, we have the following selected cases of metric spaces M for which the space $Lip_0(M)$ does not have the Grothendieck property:

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4. **M is a net in c_0 or ℓ_1.**

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4. M **is a net in** c_0 **or** ℓ_1.
5. M **is a** $C(K)$**-space,** $L_1(\mu)$**-space,** $Lip_0(M)$**-space, or** $\mathcal{F}(M)$**-space.**

1. A subspace $\mathcal{N}$ of a metric space M is a *net* if there are $\varepsilon, \delta > 0$ such that $\rho(x, y) \geq \varepsilon$ for every $x \neq y \in \mathcal{N}$ and for every $x \in M$ there is $y \in \mathcal{N}$ with $\rho(x, y) < \delta$. $Lip_0(\mathcal{N})$, where $\mathcal{N}$ is a net in either c_0 or ℓ_1, admits a continuous operator onto ℓ_1 (Candido, Cúth, Doucha)

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3. Banach spaces c_0 and $C(K)$ for K metric compact are absolute Lipschitz retracts (Lindenstrauss).

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Theorem 8 (Bargetz-Kąkol-Sobota)

Let E be a separable Banach space which is an absolute Lipschitz retract and contains c_0. If M contains a bilipschitz copy of S_E of E, then $Lip_0(M)$ is not Grothendieck (since it admits a continuous operator onto ℓ_1.)

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Does there exist a Banach space E of dimension at least 2 such that $Lip_0(E)$ admits no continuous linear surjection onto ℓ_1? Can such $Lip_0(E)$ still admit a continuous linear surjection onto c_0?

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Problem 11

Is there an infinite-dimensional Banach space E for which the space $Lip_0(E)$ is a Grothendieck space?

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3. Note that $Lip_0(\mathbb{R}^n)^*$ $(n \geq 1)$ is not Grothendieck, since it contains $\mathcal{F}(\mathbb{R}^n)$ complemented (Cuth-Kalenda-Kaplicky).

1. **If $Lip_0(\ell_2)$ is Grothendieck**, then for any $d \in \mathbb{N}$ the space $Lip_0(\mathbb{R}^d)$ is Grothendieck, since $\mathbb{R}^d$ is a Lipschitz retract of ℓ_2.

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2. **If $Lip_0(\ell_2)$ is not Grothendieck** and one can find a separable infinite-dimensional Banach space E such that $Lip_0(E)$ is Grothendieck, then this would answer in negative a question (Candido, Cúth, Doucha) whether $Lip_0(\ell_2)$ is complemented in $Lip_0(F)$ for every separable infinite-dimensional Banach space F.