Semadeni derivative of Banach spaces and functions on nonmetrizable rectangles

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Notation

- We write $X \simeq Y$ when Banach spaces X, Y are isomorphic and $X = Y$ when they are isometric.
- For Banach spaces X, Y we write $X \hookrightarrow Y$ for the isomorphic embedding, $X \twoheadrightarrow Y$ for a continuous linear surjection.
- The character of a point, denoted $\chi(x, F)$, in topological space F is the minimal cardinality of the base at the point x. $\chi(F) = \sup\{\chi(x, F) : x \in F\}$.

Motivations

Old problem (Banach, 1932)

Are Banach spaces $C([0,1])$ and $C([0,1]^2)$ isomorphic?

Answer (Miljutin, 1966): Yes.

Compact line is a linearly ordered space compact in the order topology.

Generalised problem

Consider compact lines $K_1, \ldots, K_n, L_1, \ldots, L_k$ for $n \neq k$. Is it possible that Banach spaces $C(\prod_{i=1}^{n} K_i)$ and $C(\prod_{j=1}^{k} L_j)$ are isomorphic?

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Known progress about the problem

Metrizable case

Characterisations of isomorphisms of Banach spaces by Bessaga, Pełczyński and Miljutin are solving the case for metrizable compact lines.

What happens for nonmetrizable compact lines?

Theorem (Michalak, 2020, [3])

Consider separable compact lines $K_1, \ldots, K_n, L_1, \ldots, L_k$ for $n \neq k$. Then Banach spaces $C(\prod_{i=1}^n K_i)$ and $C(\prod_{j=1}^k L_j)$ are not isomorphic.

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Semadeni-Pełczyński derivative

Definition

For a Banach space X and a cardinal number $\kappa \geq \omega$ put

$$\kappa X = \{x^{**} \in X^{**} : \forall A \subseteq X^* \ |A| \leq \kappa \ \exists x \in X \ x^{**}|A = x|A\}.$$

We define κ-Semadeni-Pełczyński derivative by $\mathcal{SP}_\kappa(X) = \kappa X / X$.

Fact

For any Banach spaces X, Y, if $X \hookrightarrow Y$, then $\mathcal{SP}_\kappa(X) \hookrightarrow \mathcal{SP}_\kappa(Y)$.

Theorem (Candido, 2022, [1])

For any family of Banach spaces $\{X_i : i \in I\}$ we have

$$\mathcal{SP}_\kappa(c_0(I, X_i)) = c_0(I, \mathcal{SP}_\kappa(X_i))$$

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The original result

Theorem (Semadeni, 1960, [4])

For cardinal numbers $\kappa \neq \lambda$ we have

$$C([0,\omega_1 \cdot \kappa]) \not\simeq C([0,\omega_1 \cdot \lambda])$$

In particular, for $\kappa = 1, \lambda = 2$, it follows that

$$C([0,\omega_1]) \not\simeq C([0,\omega_1] \times 2).$$

This theorem follows from the fact that for any κ we have

$$\mathcal{SP}_\omega(C([0,\omega_1 \cdot \kappa])) \simeq c_0(\kappa).$$

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Semadeni-Pełczyński dimension

Denote $\mathcal{SP}_\kappa^{(1)}(X) = \mathcal{SP}_\kappa(X)$ and $\mathcal{SP}_\kappa^{(n+1)}(X) = \mathcal{SP}_\kappa(\mathcal{SP}_\kappa^{(n)}(X))$.

Definition

Let X be any Banach space. We define κ-Semadeni-Pełczyński dimension of X by the following conditions

- $sp_\kappa(X) = -1$ if $X = \{0\}$,
- $sp_\kappa(X) = n$ if $\mathcal{SP}_\kappa^{(n+1)}(X) = \{0\}$ and $\mathcal{SP}_\kappa^{(n)}(X) \neq \{0\}$,
- $sp_\kappa(X) = \infty$ if for all $n \in \omega$ we have $\mathcal{SP}_\kappa^{(n)}(X) \neq \{0\}$.

Fact

If X, Y are Banach spaces and $X \hookrightarrow Y$ or $Y \twoheadrightarrow X$, then

$$sp_\kappa(X) \leq sp_\kappa(Y).$$

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If X, Y are Banach spaces and $X \hookrightarrow Y$ or $Y \twoheadrightarrow X$, then

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Inaccessible points

Definition

Let K be a compact line and κ a cardinal number. Say that a point is κ-inaccessible from the left [right] if

$$\chi(x,(\leftarrow,x]) > \kappa\big[\chi(x,[x,\rightarrow)) > \kappa\big].$$

$$K_L = \{x \in K : x \text{ is } \kappa\text{-inaccessible from the left}\},$$
$$K_R = \{x \in K : x \text{ is } \kappa\text{-inaccessible from the right}\}.$$

Definition

For a compact line K define its κ-continuous completion

$$\mathbb{K} = K_L \times \{-1\} \cup K \times \{0\} \cup K_R \times \{1\},$$

considered with the lexicographic order and order topology.
$\mathbb{K}$ is a compact line of character $\chi(\mathbb{K}) = \chi(K)$.

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Reproving Galego

In 2009, Galego [2] consistently proved a full isomorphic characterisation of spaces of the form $C(2^\kappa \times [0,\xi])$, where κ is an infinite cardinal and λ is an uncountable ordinal. We can prove a similar result using the framework of Semadeni-Pełczyński derivative.

Theorem

For any infinite cardinal numbers $\kappa, \kappa', \lambda, \lambda'$, if λ and λ' are below the first measurable cardinal, then

$$C(2^\lambda \times [0,\kappa^+]) \simeq C(2^{\lambda'} \times [0,\kappa'^+]) \implies \kappa = \kappa' \wedge \lambda = \lambda'.$$

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Buildup

Lemma

Consider compact lines $K_1, \ldots, K_n$. Then

$$\kappa C(\prod_{i=1}^{n} K_i) = C(\prod_{i=1}^{n} \mathbb{K}_i)$$

Lemma

Consider compact lines $K_1, \ldots, K_n$. Then

$$\mathcal{SP}_\kappa(C(\prod_{i=1}^{n} K_i)) \simeq \prod_{i=1}^{n} c_0\Big(K_{iL} \sqcup K_{iR}, C(\prod_{j=1, j\neq i}^{n} \mathbb{K}_j)\Big).$$

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Dimension of $C(\prod_{i=1}^n K_i)$

Corollary

Character of a compact line K is an isomorphic invariant i.e. if K_1, K_2 are compact lines such that $C(K_1) \simeq C(K_2)$, then $\chi(K_1) = \chi(K_2)$.

Theorem

If $K_1, \ldots, K_n$ are compact lines satisfying $\chi(K_i) > \kappa$ and L is a product of compact lines of character at most κ, then

$$sp_\kappa(C(L \times \prod_{i=1}^n K_i)) = n.$$

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Final results

Corollary

Consider compact lines $K_1, \ldots, K_n, L_1, \ldots, L_k$ of uncountable character. Then for $n > k$ we have

$$C(\prod_{i=1}^{n} K_i) \not\hookrightarrow C(\prod_{j=1}^{k} L_j)$$

and

$$C(\prod_{j=1}^{k} L_j) \not\twoheadrightarrow C(\prod_{i=1}^{n} K_i).$$

Remaining question

Question

For $n \neq k$ consider nonseparable compact lines $K_1, \dots, K_n, L_1, \dots, L_k$ of countable character. Is it possible that Banach spaces $C(\prod_{i=1}^n K_i)$ and $C(\prod_{j=1}^k L_j)$ are isomorphic?

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