Generic operators

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The setup

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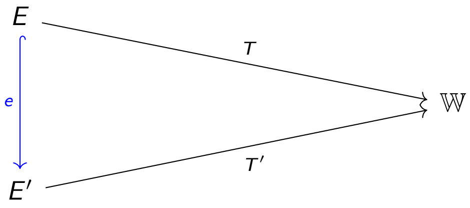

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We say that an operator $\Theta \colon V \to \mathbb{W}$ is generic over $\langle \mathfrak{K}, \mathbb{W} \rangle$ if the second player has a strategy such that, no matter how the first player plays, there is a linear isometry $h \colon V \to E_\infty$ satisfying $\Theta = T_\infty \circ h$.

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Claim

A generic operator (if exists) is unique, up to isometry.

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(E) Given a finite-dimensional space E and its subspace E_0, given an isometric embedding $e_0 \colon E_0 \to G$, for every $\varepsilon > 0$ there is an isometric embedding $e \colon E \to G$ such that

$$\|e \restriction E_0 - e_0\| < \varepsilon.$$

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Let $\mathbb{W}$ be a separable Banach space. Then there exists a non-expansive operator $\Omega_{\mathbb{W}} \colon G_{\mathbb{W}} \to \mathbb{W}$ with the following properties.

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Claim

The projection $\Omega_{\mathbb{W}}$ is generic over $\langle \mathfrak{B}, \mathbb{W} \rangle$.

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Fraïssé theory in slice categories. Inspired by [Pech & Pech 2014].

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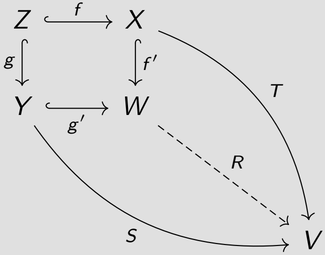
Pushouts

Given embeddings of Banach spaces $f\colon Z \to X$, $g\colon Z \to Y$, there exists a uniqely determined space $W = X \oplus_W Y$, together with embeddings $f'\colon X \to W$, $g'\colon Y \to W$ satisfying $f' \circ f = g' \circ g$ and such that, given any operators $T\colon X \to V$, $S\colon Y \to V$, there exists a unique operator $R\colon W \to V$ satisfying $R \circ f' = T$ and $R \circ g' = S$.



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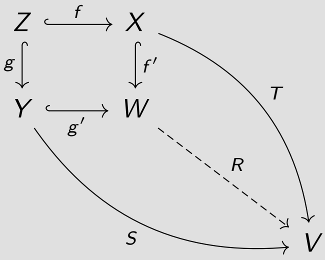

Furthermore, $\|R\| = \max\{\|T\|, \|S\|\}$.

A general result

Theorem

Let $\mathfrak{K}$ be a category of finite-dimensional spaces, containing the trivial space and closed under pushouts. Then for every separable space $\mathbb{W}$, there exists a $\mathfrak{K}$-generic operator with codomain $\mathbb{W}$.

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Furthermore, the space $\P$ does not depend on $\mathbb{W}$ and is isomorphic to the universal Kadec – Pełczyński – Wojtaszczyk space.

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Remark

In case $\mathbb{W} = \{0\}$, this is the result of J. Garbulińska-Węgrzyn from 2014.

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