AM-algebras

David Muñoz-Lahoz[1]
(Joint work with P. Tradacete)

[1]ICMAT-UAM (Madrid)

Structures in Banach Spaces
March 21, 2025

Outline

1. **Banach lattices and AM-spaces**

2. **AM-algebras with approximate unit (a characterization of the closed sublattice-algebras of $C(K)$)**

3. **Final remarks and applications**

Banach lattices (I)

Definition

A *Banach lattice* is a real Banach space X equipped with a partial order $\leq$ such that, for all $x, y, z \in X$:

1. $x \leq y$ implies $x + z \leq y + z$ and $\lambda x \leq \lambda y$ for all $\lambda > 0$;
2. $x \vee y$ (the least upper bound of $\{x, y\}$) exists;
3. setting $|x| = x \vee (-x)$, if $|x| \leq |y|$ then $\|x\| \leq \|y\|$.

Banach lattices (I)

Definition

A *Banach lattice* is a real Banach space X equipped with a partial order $\leq$ such that, for all $x, y, z \in X$:

1. $x \leq y$ implies $x + z \leq y + z$ and $\lambda x \leq \lambda y$ for all $\lambda > 0$;
2. $x \vee y$ (the least upper bound of $\{x, y\}$) exists;
3. setting $|x| = x \vee (-x)$, if $|x| \leq |y|$ then $\|x\| \leq \|y\|$.

Examples

1. $C(K) = C(K, \mathbb{R})$ with pointwise order,
2. $L_p(\mu)$, $1 \leq p \leq \infty$, with pointwise almost everywhere order.

Banach lattices (I)

Definition

A *Banach lattice* is a real Banach space X equipped with a partial order $\leq$ such that, for all $x, y, z \in X$:

1. $x \leq y$ implies $x + z \leq y + z$ and $\lambda x \leq \lambda y$ for all $\lambda > 0$;
2. $x \vee y$ (the least upper bound of $\{x, y\}$) exists;
3. setting $|x| = x \vee (-x)$, if $|x| \leq |y|$ then $\|x\| \leq \|y\|$.

Examples

1. $C(K) = C(K, \mathbb{R})$ with pointwise order,
2. $L_p(\mu)$, $1 \leq p \leq \infty$, with pointwise almost everywhere order.

From now on, X will denote a Banach lattice, and
$X_+ = \{\, x \in X : x \geq 0 \,\}$ its positive cone.

Banach lattices (II)

Definition

A vector subspace Y of X is a *sublattice* if $x \vee y \in Y$ for all $x, y \in Y$ (equivalently, if $|y| \in Y$ for all $y \in Y$).

Banach lattices (II)

Definition

A vector subspace Y of X is a *sublattice* if $x \vee y \in Y$ for all $x, y \in Y$ (equivalently, if $|y| \in Y$ for all $y \in Y$).

The dual of a Banach lattice

The dual of a Banach lattice is again a Banach lattice when equipped with the order: $x^* \leq y^*$ if and only if $x^*(x) \leq y^*(x)$ for all $x \in X_+$.

Banach lattices (II)

Definition

A vector subspace Y of X is a *sublattice* if $x \vee y \in Y$ for all $x, y \in Y$ (equivalently, if $|y| \in Y$ for all $y \in Y$).

The dual of a Banach lattice

The dual of a Banach lattice is again a Banach lattice when equipped with the order: $x^* \leq y^*$ if and only if $x^*(x) \leq y^*(x)$ for all $x \in X_+$.

Definition

Let X and Y be Banach lattices. An operator $T \colon X \to Y$ is said to be a *lattice homomorphism* if $T(x \vee y) = Tx \vee Ty$ for all $x, y \in X$.

Banach lattices (II)

Definition

A vector subspace Y of X is a *sublattice* if $x \vee y \in Y$ for all $x, y \in Y$ (equivalently, if $|y| \in Y$ for all $y \in Y$).

The dual of a Banach lattice

The dual of a Banach lattice is again a Banach lattice when equipped with the order: $x^* \leq y^*$ if and only if $x^*(x) \leq y^*(x)$ for all $x \in X_+$.

Definition

Let X and Y be Banach lattices. An operator $T\colon X \to Y$ is said to be a *lattice homomorphism* if $T(x \vee y) = Tx \vee Ty$ for all $x, y \in X$.

Example

Point evaluations $\delta_t\colon C(K) \to \mathbb{R}$, $\delta_t(f) = f(t)$, are lattice homomorphisms.

AM-spaces

Definition

We say that X is an *AM-space* if it satisfies

$$\|x \vee y\| = \max\{\|x\|, \|y\|\} \quad \text{for all } x, y \in X_+.$$

AM-spaces

Definition

We say that X is an *AM-space* if it satisfies

$$\|x \vee y\| = \max\{\|x\|, \|y\|\} \quad \text{for all } x, y \in X_+.$$

Let $e \in X_+$. We say that X is an *AM-space with unit e* if

$$\|x\| = \inf\{\, \lambda > 0 : |x| \leq \lambda e \,\} \quad \text{for all } x \in X.$$

In this case, e is also called a *(strong) order unit*.

AM-spaces

Definition

We say that X is an *AM-space* if it satisfies

$$\|x \vee y\| = \max\{\|x\|, \|y\|\} \quad \text{for all } x, y \in X_+.$$

Let $e \in X_+$. We say that X is an *AM-space with unit* e if

$$\|x\| = \inf\{\, \lambda > 0 : |x| \leq \lambda e \,\} \quad \text{for all } x \in X.$$

In this case, e is also called a *(strong) order unit*.

Certainly, every AM-space with unit is an AM-space.

AM-spaces

Definition

We say that X is an *AM-space* if it satisfies

$$\|x \vee y\| = \max\{\|x\|, \|y\|\} \quad \text{for all } x, y \in X_+.$$

Let $e \in X_+$. We say that X is an *AM-space with unit* e if

$$\|x\| = \inf\{\, \lambda > 0 : |x| \leq \lambda e \,\} \quad \text{for all } x \in X.$$

In this case, e is also called a *(strong) order unit*.

Certainly, every AM-space with unit is an AM-space.

Examples

1. $C(K)$ is an AM-space with unit 1_K.
2. Every closed sublattice of $C(K)$ is an AM-space.

Kakutani's theorem

Theorem (S. Kakutani, 1941)

Kakutani's theorem

Theorem (S. Kakutani, 1941)

1. *Every AM-space with unit is lattice isometric to $C(K)$ for some compact Hausdorff space K, with the unit corresponding to the constant one function 1_K.*

Kakutani's theorem

Theorem (S. Kakutani, 1941)

1. *Every AM-space with unit is lattice isometric to $C(K)$ for some compact Hausdorff space K, with the unit corresponding to the constant one function 1_K.*
2. *Every AM-space is lattice isometric to a closed sublattice of $C(K)$ for some compact Hausdorff space K.*

Kakutani's theorem

Theorem (S. Kakutani, 1941)

1. *Every AM-space with unit is lattice isometric to $C(K)$ for some compact Hausdorff space K, with the unit corresponding to the constant one function 1_K.*
2. *Every AM-space is lattice isometric to a closed sublattice of $C(K)$ for some compact Hausdorff space K. More precisely, there exists a family of pairs of points $\{(t_i, s_i)\}_{i \in I} \subseteq K \times K$ and scalars $\{\lambda_i\}_{i \in I} \subseteq [0, 1)$ such that the AM-space is lattice isometric to the closed sublattice of $C(K)$:*

$$\{ f \in C(K) : f(t_i) = \lambda_i f(s_i) \text{ for all } i \in I \}.$$

What about the algebraic structure?

- By Kakutani's theorem, we can interpret the AM-space condition as an intrinsic characterization of the closed sublattices of $C(K)$.

What about the algebraic structure?

- By Kakutani's theorem, we can interpret the AM-space condition as an intrinsic characterization of the closed sublattices of $C(K)$.
- But $C(K)$ also has a natural algebraic structure (the pointwise product) compatible with the lattice structure (for example, the product of positive functions is positive, i.e., it is a *Banach lattice algebra*).

What about the algebraic structure?

- By Kakutani's theorem, we can interpret the AM-space condition as an intrinsic characterization of the closed sublattices of $C(K)$.
- But $C(K)$ also has a natural algebraic structure (the pointwise product) compatible with the lattice structure (for example, the product of positive functions is positive, i.e., it is a *Banach lattice algebra*).

Question

Can we characterize intrinsically the AM-spaces that embed as a closed sublattice-algebra of $C(K)$?

AM-algebras with unit

We need to find an appropriate compatibility condition between the AM-space and algebraic structures. The following result points in this direction.

AM-algebras with unit

We need to find an appropriate compatibility condition between the AM-space and algebraic structures. The following result points in this direction.

Theorem (L. Martignon, 1980)

Let A be a Banach lattice algebra with algebraic identity e. If A is also an AM-space with unit e, then A is both lattice and algebra isometric to a $C(K)$, with e going to 1_K.

AM-algebras with unit

We need to find an appropriate compatibility condition between the AM-space and algebraic structures. The following result points in this direction.

Theorem (L. Martignon, 1980)

Let A be a Banach lattice algebra with algebraic identity e. If A is also an AM-space with unit e, then A is both lattice and algebra isometric to a $C(K)$, with e going to 1_K.

Definition

Let A be a Banach lattice algebra with identity e. We say that A is an *AM-algebra with unit e* if A is also an AM-space with unit e.

AM-algebras with unit

We need to find an appropriate compatibility condition between the AM-space and algebraic structures. The following result points in this direction.

Theorem (L. Martignon, 1980)

Let A be a Banach lattice algebra with algebraic identity e. If A is also an AM-space with unit e, then A is both lattice and algebra isometric to a $C(K)$, with e going to 1_K.

Definition

Let A be a Banach lattice algebra with identity e. We say that A is an *AM-algebra with unit e* if A is also an AM-space with unit e.

How can we extend this idea to general AM-spaces, where no unit is present?

Approximate order units

This characterization of order units...

Lemma

A Banach lattice X is an AM-space with unit $e \in X_+$ if and only if $x^*(e) = \|x^*\|$ for all $x^* \in (X^*)_+$.

...led us to the following notion.

Approximate order units

This characterization of order units...

Lemma

A Banach lattice X is an AM-space with unit $e \in X_+$ if and only if $x^*(e) = \|x^*\|$ for all $x^* \in (X^*)_+$.

...led us to the following notion.

Definition

Let X be a Banach lattice, and let $(e_\gamma) \subseteq X_+$ be a net. We say that X is an *AM-space with approximate unit* (e_γ) if

$$x^*(e_\gamma) \to \|x^*\| \quad \text{for every } x^* \in (X^*)_+.$$

We also say that (e_γ) is an *approximate order unit* of X.

Approximate order units

This characterization of order units...

Lemma

A Banach lattice X is an AM-space with unit $e \in X_+$ if and only if $x^*(e) = \|x^*\|$ for all $x^* \in (X^*)_+$.

...led us to the following notion.

Definition

Let X be a Banach lattice, and let $(e_\gamma) \subseteq X_+$ be a net. We say that X is an *AM-space with approximate unit* (e_γ) if

$$x^*(e_\gamma) \to \|x^*\| \quad \text{for every } x^* \in (X^*)_+.$$

We also say that (e_γ) is an *approximate order unit* of X.

Does having an approximate order unit imply that X is an AM-space?

AM-algebras with approximate unit

Yes! Moreover, the converse is also true.

Lemma

A Banach lattice X has an approximate order unit if and only if X is an AM-space.

AM-algebras with approximate unit

Yes! Moreover, the converse is also true.

Lemma

A Banach lattice X has an approximate order unit if and only if X is an AM-space.

Recall that, in a Banach algebra A, a net (e_γ) is an approximate (algebraic) identity if $e_\gamma a \to a$ and $ae_\gamma \to a$ for all $a \in A$.

Definition

Let A be a Banach lattice algebra and let $(e_\gamma) \subseteq A_+$. We say that A is an *AM-algebra with approximate unit* (e_γ) if (e_γ) is both an approximate order unit and an approximate algebraic identity.

AM-algebras with approximate unit

Yes! Moreover, the converse is also true.

Lemma

A Banach lattice X has an approximate order unit if and only if X is an AM-space.

Recall that, in a Banach algebra A, a net (e_γ) is an approximate (algebraic) identity if $e_\gamma a \to a$ and $ae_\gamma \to a$ for all $a \in A$.

Definition

Let A be a Banach lattice algebra and let $(e_\gamma) \subseteq A_+$. We say that A is an *AM-algebra with approximate unit* (e_γ) if (e_γ) is both an approximate order unit and an approximate algebraic identity.

This is the interaction between the AM-space and algebraic structures we were looking for!

Kakutani's theorem...

Theorem (S. Kakutani, 1941)

1. *Every AM-space with unit is lattice isometric to $C(K)$ for some compact Hausdorff space K, with the unit corresponding to the constant one function 1_K.*
2. *Every AM-space is lattice isometric to a closed sublattice of $C(K)$ for some compact Hausdorff space K. More precisely, there exists a family of pairs of points $\{(t_i, s_i)\}_{i\in I} \subseteq K \times K$ and scalars $\{\lambda_i\}_{i\in I} \subseteq [0,1)$ such that the AM-space is lattice isometric to the closed sublattice of $C(K)$:*

$$\{\, f \in C(K) : f(t_i) = \lambda_i f(s_i) \text{ for all } i \in I \}.$$

...for Banach lattice algebras

Theorem (M.-L., Tradacete)

...for Banach lattice algebras

Theorem (M.-L., Tradacete)

1. *Every AM-algebra with unit is lattice and algebra isometric to $C(K)$ for some compact Hausdorff space K, with the unit corresponding to the constant one function 1_K.*

...for Banach lattice algebras

Theorem (M.-L., Tradacete)

1. *Every AM-algebra with unit is lattice and algebra isometric to $C(K)$ for some compact Hausdorff space K, with the unit corresponding to the constant one function 1_K.*
2. *Every AM-algebra with approximate unit is lattice and algebra isometric to a closed sublattice-algebra of $C(K)$ for some compact Hausdorff space K.*

...for Banach lattice algebras

Theorem (M.-L., Tradacete)

1. *Every AM-algebra with unit is lattice and algebra isometric to $C(K)$ for some compact Hausdorff space K, with the unit corresponding to the constant one function 1_K.*
2. *Every AM-algebra with approximate unit is lattice and algebra isometric to a closed sublattice-algebra of $C(K)$ for some compact Hausdorff space K.More precisely, there exists a closed set $F \subseteq K$ such that the AM-algebra is lattice and algebra isometric to the closed sublattice-algebra of $C(K)$:*

$$\{ f \in C(K) : f(t) = 0 \text{ for all } t \in F \}.$$

In particular, it embeds as an order and algebraic ideal in $C(K)$.

Remarks on approximate order units

Remarks

Recall: $(e_\gamma) \subseteq X_+$ is an approximate order unit if $x^*(e_\gamma) \to \|x^*\|$ for all $(X^*)_+$.

Remarks on approximate order units

Remarks

Recall: $(e_\gamma) \subseteq X_+$ is an approximate order unit if $x^*(e_\gamma) \to \|x^*\|$ for all $(X^*)_+$.

1. Approximate order units need not be bounded or even have bounded tails.

Remarks on approximate order units

Remarks

Recall: $(e_\gamma) \subseteq X_+$ is an approximate order unit if $x^*(e_\gamma) \to \|x^*\|$ for all $(X^*)_+$.

1. Approximate order units need not be bounded or even have bounded tails.
2. Even if a Banach lattice algebra has approximate order units and approximate identities, it may not be an AM-algebra with approximate unit.

Remarks on approximate order units

Remarks

Recall: $(e_\gamma) \subseteq X_+$ is an approximate order unit if $x^*(e_\gamma) \to \|x^*\|$ for all $(X^*)_+$.

1. Approximate order units need not be bounded or even have bounded tails.
2. Even if a Banach lattice algebra has approximate order units and approximate identities, it may not be an AM-algebra with approximate unit.
3. If A is an AM-algebra with approximate unit, then $(B_A)_+$ is an approximate order unit and an approximate identity.

Remarks on approximate order units

Remarks

Recall: $(e_\gamma) \subseteq X_+$ is an approximate order unit if $x^*(e_\gamma) \to \|x^*\|$ for all $(X^*)_+$.

1. Approximate order units need not be bounded or even have bounded tails.
2. Even if a Banach lattice algebra has approximate order units and approximate identities, it may not be an AM-algebra with approximate unit.
3. If A is an AM-algebra with approximate unit, then $(B_A)_+$ is an approximate order unit and an approximate identity.
4. Let A be an AM-space and a Banach lattice algebra. Then A is an AM-algebra with approximate unit (i.e., lattice-algebra embeds in a $C(K)$) if and only if $(B_A)_+$ is an approximate identity.

Applications (I)

As an application, we can establish precisely when Banach lattice algebras have a C^*-algebra structure that preserves the positive cone. Since the natural scalar field for C^*-algebras is $\mathbb{C}$, we now introduce complex Banach lattice algebras.

Applications (I)

As an application, we can establish precisely when Banach lattice algebras have a C^*-algebra structure that preserves the positive cone. Since the natural scalar field for C^*-algebras is $\mathbb{C}$, we now introduce complex Banach lattice algebras.

Complex Banach lattices

Let X be a Banach lattice and let $X_{\mathbb{C}} = X + iX$. For every $z = x + iy \in X_{\mathbb{C}}$, the supremum

$$|z| = \sup\{\cos\theta x + \sin\theta y : \theta \in [0, 2\pi]\}$$

exists in X and is called the *modulus* of z. Define $\|z\|_{\mathbb{C}} = \||z|\|$. Then $(X_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$ is a complex Banach space. Any Banach space of this form is called a *complex Banach lattice*.

Applications (I)

As an application, we can establish precisely when Banach lattice algebras have a C^*-algebra structure that preserves the positive cone. Since the natural scalar field for C^*-algebras is $\mathbb{C}$, we now introduce complex Banach lattice algebras.

Complex Banach lattices

Let X be a Banach lattice and let $X_{\mathbb{C}} = X + iX$. For every $z = x + iy \in X_{\mathbb{C}}$, the supremum

$$|z| = \sup\{\cos\theta x + \sin\theta y : \theta \in [0, 2\pi]\}$$

exists in X and is called the *modulus* of z. Define $\|z\|_{\mathbb{C}} = \||z|\|$. Then $(X_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$ is a complex Banach space. Any Banach space of this form is called a *complex Banach lattice*.

Example

$C(K)_{\mathbb{C}} = C(K, \mathbb{C})$.

Applications (II)

Complex Banach lattice algebras

Let A be a Banach lattice algebra. The complex Banach lattice $(A_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$, equipped with the usual product

$$(x+iy)(z+it) = (xz-yt) + i(yz+xt) \quad \text{where } x, y, z, t \in A,$$

becomes a complex Banach algebra. This product is compatible with the complex Banach lattice structure: $|z_1 z_2| \leq |z_1||z_2|$. The space $A_{\mathbb{C}}$ is said to be a *complex Banach lattice algebra*.

Applications (II)

Complex Banach lattice algebras

Let A be a Banach lattice algebra. The complex Banach lattice $(A_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$, equipped with the usual product

$$(x+iy)(z+it) = (xz-yt) + i(yz+xt) \quad \text{where } x, y, z, t \in A,$$

becomes a complex Banach algebra. This product is compatible with the complex Banach lattice structure: $|z_1 z_2| \leq |z_1||z_2|$. The space $A_{\mathbb{C}}$ is said to be a *complex Banach lattice algebra*.

Question

When can a complex Banach lattice algebra $A_{\mathbb{C}}$ be equipped with a C^*-algebra structure so that the positive elements of the C^*-algebra (i.e., the self-adjoint elements with positive spectrum) are precisely A_+?

Applications (III)

Corollary

Let A be a Banach lattice algebra. Its complexification $A_{\mathbb{C}}$ can be endowed with a C^*-algebra structure in such a way that A_+ is the cone of self-adjoint elements with positive spectrum if and only if A is an AM-algebra with approximate unit.

Applications (IV)

Definition

A Banach lattice algebra A is said to be an *f-algebra* if $a \wedge b = 0$ implies

$$(ca) \wedge b = (ac) \wedge b = 0 \quad \text{for all } a, b, c \in A_+.$$

(That is, if the left and right multiplications are orthomorphisms).

Applications (IV)

Definition

A Banach lattice algebra A is said to be an *f-algebra* if $a \wedge b = 0$ implies

$$(ca) \wedge b = (ac) \wedge b = 0 \quad \text{for all } a, b, c \in A_+.$$

(That is, if the left and right multiplications are orthomorphisms).

Example

Every closed sublattice-algebra of $C(K)$ is an f-algebra.

Applications (IV)

Definition

A Banach lattice algebra A is said to be an *f-algebra* if $a \wedge b = 0$ implies

$$(ca) \wedge b = (ac) \wedge b = 0 \quad \text{for all } a, b, c \in A_+.$$

(That is, if the left and right multiplications are orthomorphisms).

Example

Every closed sublattice-algebra of $C(K)$ is an f-algebra.

Corollary

Every AM-algebra with approximate unit is an f-algebra.

Work in progress (I)

Work in progress (I)

Question

Does every AM-space admit a product making it into an AM-algebra with approximate unit?

Work in progress (I)

Question

Does every AM-space admit a product making it into an AM-algebra with approximate unit?

Work in progress

- No. But when it does, it is unique.

Work in progress (I)

Question

Does every AM-space admit a product making it into an AM-algebra with approximate unit?

Work in progress

- No. But when it does, it is unique.
- This AM-algebra product can be characterized in many “nice” ways. For instance, it is the unique product for which the norm-one lattice homomorphisms are also algebra homomorphisms.

Work in progress (I)

Question

Does every AM-space admit a product making it into an AM-algebra with approximate unit?

Work in progress

- No. But when it does, it is unique.
- This AM-algebra product can be characterized in many "nice" ways. For instance, it is the unique product for which the norm-one lattice homomorphisms are also algebra homomorphisms.
- Another way: lattice embed your AM-space X in a $C(K)$ space in such a way that it is also a subalgebra. Then the pointwise product of $C(K)$ coincides with the canonical product in X.

Work in progress (I)

Question

Does every AM-space admit a product making it into an AM-algebra with approximate unit?

Work in progress

- No. But when it does, it is unique.
- This AM-algebra product can be characterized in many "nice" ways. For instance, it is the unique product for which the norm-one lattice homomorphisms are also algebra homomorphisms.
- Another way: lattice embed your AM-space X in a $C(K)$ space in such a way that it is also a subalgebra. Then the pointwise product of $C(K)$ coincides with the canonical product in X.
- When the AM-space has unit (i.e., it is a $C(K)$), this AM-algebra product is, of course, the pointwise product.

Work in progress (II)

In other words...

"Many" AM-spaces come with a canonical product that is the natural generalization of the pointwise product in AM-spaces with unit (i.e., $C(K)$ spaces).

Work in progress (II)

In other words...

"Many" AM-spaces come with a canonical product that is the natural generalization of the pointwise product in AM-spaces with unit (i.e., $C(K)$ spaces).

Work in progress

- Most of the familiar examples of AM-spaces admit such a product.

Work in progress (II)

In other words...

"Many" AM-spaces come with a canonical product that is the natural generalization of the pointwise product in AM-spaces with unit (i.e., $C(K)$ spaces).

Work in progress

- Most of the familiar examples of AM-spaces admit such a product.
- We have several characterizations of the spaces that admit an AM-algebra product.

Work in progress (II)

In other words...

"Many" AM-spaces come with a canonical product that is the natural generalization of the pointwise product in AM-spaces with unit (i.e., $C(K)$ spaces).

Work in progress

- Most of the familiar examples of AM-spaces admit such a product.
- We have several characterizations of the spaces that admit an AM-algebra product.
- We believe that several results that relate the lattice and algebraic structures of $C(K)$ can be extended to general AM-spaces with this canonical product.

References

D. Muñoz-Lahoz and P. Tradacete
"Banach lattice AM-algebras"
To appear in Proc. Am. Math. Soc. (2025)

Shizuo Kakutani
"Concrete representation of abstract (M)-spaces (A characterization of the space of continuous functions)"
Annals of Mathematics 42 (4), pp. 994–1024 (1941)

Laura Martignon
"Banach f-algebras and Banach lattice algebras with unit"
Bol. Soc. Bras. Mat. 11 (1), pp. 11–18 (1980)