On w^*-binormality of the dual space $C_k(X)'$

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The presentation is based on the results from the recent joint paper

Jerzy Kąkol, Ondřej Kurka, Arkady Leiderman,
"On Asplund spaces $C_k(X)$ and w^*-binormality",
Results Math. **78:203** (2023).

What is binormality?

Definition 1.

Let X be a nonempty set and σ, τ be two topologies on X. We say that (X, σ, τ) is binormal, if for every disjoint pair S, T of subsets of X such that S is closed in σ and T is closed in τ, there is a disjoint pair of sets V and U such that $S \subset V$, $T \subset U$, V is open in τ, and U is open in σ.

This property is called pairwise normality in the paper J.C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. 13 (1963), 71–89.

Examples of binormality

(1) Luzin-Menchoff property of the pair of the Euclidean and the density topologies on the real line $\mathbb{R}$.

Luzin-Menchoff Theorem. If F is a perfect subset of a measurable subset $U \subset \mathbb{R}$ consisting exclusively of its points of density, then there exists a perfect set K with the set K^* consisting of density points of K, such that the following inclusions hold:

$$F \subset K^* \subset K \subset U.$$

(For the history see the paper ”Urysohn Lemma or Luzin-Menshov Theorem?” by Jerzy Mioduszewski).

Examples of binormality

(2) (P. Holický, 1997) Let (X, τ) be a locally convex space and let σ be its weak topology. Assume that (X, τ) is Lindelöf. Then X is binormal with respect to σ and τ.
Corollary: Every separable Banach space is binormal with respect to its norm and weak topologies.

(3) (P. Holický, 1997) The non-separable space Banach space ℓ^∞ is not binormal with respect to its norm and weak topologies.

Examples of binormality

Nevertheless, many non-separable Banach spaces are binormal with respect to norm and weak topologies (O. Kurka, 2010).

(4) The space $C([0, \mu])$ is binormal for every ordinal μ.

(5) Every weakly compactly generated, in particular, every reflexive Banach space is binormal.

(6) Every dual to an Asplund space is binormal.
(The reason: all those spaces have a projectional resolution of the identity (PRI). For the duals of Asplund spaces this is a result by M. Fabian and G. Godefroy.)

(7) There is a locally compact space T such that the function space $C_0(T)$ is Asplund and admits a locally uniformly rotund (LUR) norm but $C_0(T)$ is not binormal.

- For a Tychonoff space X by $C_k(X)$ we denote lcs of continuous real-valued functions on X with the compact-open topology.
- If E is a lcs with its dual E', by $\beta(E', E)$ we mean the strong topology of E', i.e. the topology of uniform convergence on bounded subsets of E.
- If E is a Banach space then the strong topology of E' is just the normed topology.

Specifically for the dual lcs we formulate the definition of w^*-binormality as follows:

Definition 2.

For a locally convex space E we will say that its dual E' is w^*-binormal if for every disjoint $\beta(E', E)$-closed $A \subset E'$ and w^*-closed $B \subset E'$ there exist disjoint $\beta(E', E)$-open $D \subset E'$ and w^*-open $C \subset E'$ such that $A \subset C$ and $B \subset D$.

Results on w^*-binormality

Theorem 0. (O. Kurka, 2010)

If the dual E' of a Banach space E is w^*-binormal, then E is Asplund.

Theorem 1.

Let X be a pseudocompact space and assume that the strong dual $C_k(X)'$ of $C_k(X)$ is w^*-binormal. Then X is a Δ-space.

In particular, for a compact space X, if the dual space $C(X)'$ is w^*-binormal, then X is a Δ-space.

Δ-spaces

Definition 3.

A topological space X is said to be a Δ-space if for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of X with empty intersection, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of X, also with empty intersection, and such that $D_n \subset V_n$ for every $n \in \omega$.

A systematic study of the class of Δ-spaces was originated in the paper

1) J. Kąkol, A. Leiderman, *A characterization of X for which spaces $C_p(X)$ are distinguished and its applications*, Proc. Amer. Math. Soc., series B, **8** (2021), 86–99.

and continued in

2) J. Kąkol, A. Leiderman, *Basic properties of X for which the space $C_p(X)$ is distinguished*, Proc. Amer. Math. Soc., series B, **8** (2021), 267–280.
3) A. Leiderman, P. Szeptycki, *On Δ-spaces*, published in Israel J. Math., 2025.

Some topological properties of compact Δ-spaces

- Every compact Δ-space X is scattered.
- Every compact Δ-space X has countable tightness, i.e. if $x \in cl(A)$ in X then there is a countable $M \subset A$ such that $x \in cl(M)$ in X.
- If X is a compact Δ-space and Y is its continuous image, then Y also is a compact Δ-space.
- If $X = \bigcup_{n\in\omega} X_n$, and every X_n is a compact Δ-space, then X also is a Δ-space.

Corollary.

Let X be a compact space X.
If the dual $C(X)'$ is w^*-binormal, then X is scattered, or, equivalently, the Banach space $C(X)$ is Asplund.

The compact space of ordinals $X = [0, \omega_1]$ is not a Δ-space.

Hence Theorem 1 yields that the dual of $C(X)$ is not w^*-binormal for the scattered compact space $X = [0, \omega_1]$, in other words, $C(X)$ is Asplund, but the dual of $C(X)'$ is not w^*-binormal.

We don't know whether the converse implication in Theorem 1 also holds.

Problem 1.

Assume that X is a compact Δ-space. Is the dual Banach space $C(X)'$ w^*-binormal?

We have however a positive answer to Problem 1 for Corson compact spaces.

Theorem 2.

Let K be a Corson compact space. The following assertions are equivalent:

(i) K is a Δ-space.

(ii) K is a scattered Eberlein compact space.

(iii) The Banach space $C(K)$ is Asplund.

(iv) The dual Banach space $C(K)'$ is w^*-binormal.

Effectively Δ-spaces

Now we define a property of a topological space which is stronger than the property of being a Δ-space.

Definition 4.

A topological space X is called an effectively Δ-space if there exists a system of open neighborhoods $\{P_x^n \ni x : x \in X, n \in \omega\}$ such that

$$\bigcap_{n \in \omega} P_{x_n}^n \neq \emptyset$$

implies that $\{x_n : n \in \omega\}$ is a finite set.

Effectively Δ-spaces

1. Every effectively Δ-space is a Δ-space.
2. Let $\varphi : X \to Y$ be a continuous closed map onto. If X is an effectively Δ-space then Y is also an effectively Δ-space.
3. Every scattered Eberlein compact space is an effectively Δ-space.
4. Moreover, if a compact space K can be represented as a countable union of scattered Eberlein compact spaces, then K is an effectively Δ-space.
5. In particular, for every Isbell-Mrówka Ψ-space its one-point compactification is an effectively Δ-space.

Now we define a property which formally is stronger than the property of being a w^*-binormal space.

Definition 5.

The dual E' of a Banach space E will be called effectively w^*-binormal if there exists in the dual of E a system of w^*-open neighborhoods $\{U_x^n \ni x : x \in E', n \in \omega\}$, such that

$$\bigcap_{n\in\omega}(U_{x_n}^n + \epsilon_n B_{E'}^*) \neq \emptyset$$

implies that the sequence $\{x_n : n \in \omega\}$ is norm-relatively compact, whenever $\epsilon_n \searrow 0$, and $B_{E'}^*$ is the closed unit ball in the dual of E.

Main result

Theorem 3.

Let K be a compact space. Then the dual space $C(K)'$ is effectively w^*-binormal if and only if K is an effectively Δ-space.

Corollary 4.

(1) $C(K)'$ is w^*-binormal for every compact K which is representable as a countable union of scattered Eberlein compact spaces.

(2) $C(K)'$ is w^*-binormal for every compact K which is an effectively Δ-space.

(3) Let X and Y be two compact spaces such that the Banach spaces $C(X)$ and $C(Y)$ are topologically isomorphic. If X is an effectively Δ-space then Y also is an effectively Δ-space.

(4) The one-point compactification of any Isbell-Mrówka Ψ-space provides an example of a compact space K such that the dual Banach space $C(K)'$ is w^*-binormal but K is not Eberlein.

Problem 2.

Let X be a compact space. Is it true that
X is a Δ-space iff X is an effectively Δ-space, i.e.
$C(K)'$ is a w^*-binormal space iff $C(K)'$ is an effectively
w^*-binormal space?

Problem 3.

Let X and Y be two compact spaces such that the Banach spaces $C(X)$ and $C(Y)$ are topologically isomorphic.
Assume that X can be represented as a countable union of scattered Eberlein compact spaces.
Is it true that Y also can be represented as a countable union of scattered Eberlein compact spaces?

Some new results

Proposition 5.

Let $X = \bigcup_{n \in \omega} X_n$, where each X_n is closed in X. If every X_n is an effectively Δ-space then X also is an effectively Δ-space.

Proposition 6.

Let $f : X \to Y$ be a continuous countable-to-one surjective map such that all its fibers are scattered. If Y is an effectively Δ-space then X also is an effectively Δ-space.

Theorem 7.

Let X be a compact space and let $f : X \to Y$ be a continuous countable-to-one surjective map. If Y is a countable union of scattered Eberlein compact spaces then so is X.

Thank you !