Continuous functions on Fedorchuk compacta

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Structures in Banach Spaces

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Constructing topological spaces: Resolutions (Fedorchuk ('68))

- Y, X_y for $y \in Y$ - topological spaces.
- $h_y : Y \setminus \{y\} \to X_y$ - continuous mappings.
- $R(Y, X_y, h_y) := \big\{\{y\} \times X_y : y \in Y\big\}$.
- $\{y\} \times V \cup \left(\bigcup \left\{y' \times Y_{x'} : x' \in (U \cap h_y^{-1}(V))\right\}\right)$ for $U \subset Y, V \subset X_y$ open - basic open sets.

Preliminaries

Definition (Fedorchuk ('68))

Let X and Y be Hausdorff compact spaces and $f : X \to Y$ a continuous onto mapping. f is called *fully closed* if for any two closed disjoint subsets F_1, F_2 of X, the set $f(F_1) \cap f(F_2)$ is finite.

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Proposition

The canonical mapping $\pi : R(Y, X_y, h_y) \to Y$ (the resolution mapping) is fully closed.

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Definition

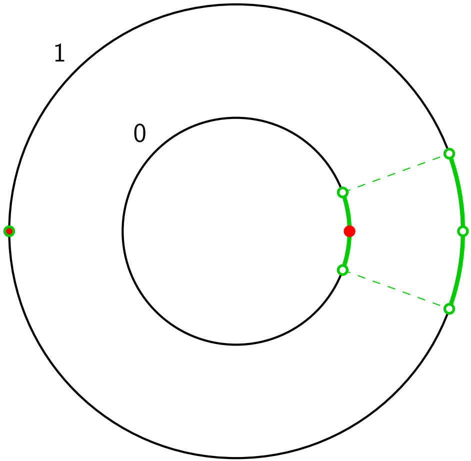
Let $S := \{X_\alpha, \pi_\alpha^\beta, \alpha, \beta \in \gamma\}$ be a well-ordered continuous inverse system of Hausdorff compacta with X_0 a singleton. We say that S is an *F-system* if the neighboring bonding mappings, $\pi_\alpha^{\alpha+1}$ for $\alpha + 1 \in \gamma$, are fully closed. We say that S is an *F-decomposable system* if all bonding mappings, π_β^α for $\alpha, \beta \in \gamma$, are fully closed.

Definition (Ivanov ('84))

Let $S := \{X_\alpha, \pi_\alpha^\beta, \alpha, \beta \in \gamma\}$ be an F-system for which the fibers of all neighboring bonding mappings are metrizable. Then the limit $X := \varprojlim S$ is called a *Fedorchuk compact* (*F-compact*) and the length of the system is called its spectral height (denoted $\mathrm{sh}(X)$). If S is moreover F-decomposable, we say that X is an F_d-*compact*.

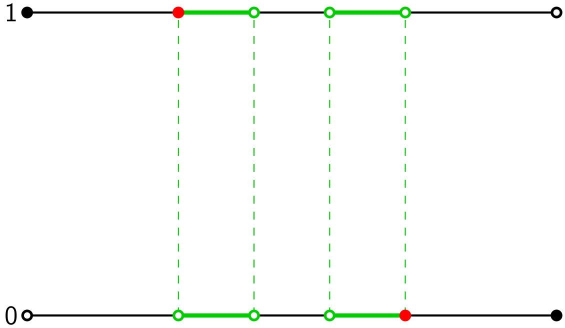
Double circle (Doubling of a compact space)

$Y = S^1$, $X_y = \{0, 1\}$, $h_y(y') = 0\ \forall y \neq y' \in Y$.



Double arrow (Any separable compact linearly ordered space (Ostraszewski ('74)))

$Y = [0,1]$, $X_y = \{0,1\}\ \forall y \in Y$, $h_y(y') = 0$ if $y' < y$ and $h_y(y') = 1$ if $y' > y$.



Lexicographic square (cube)

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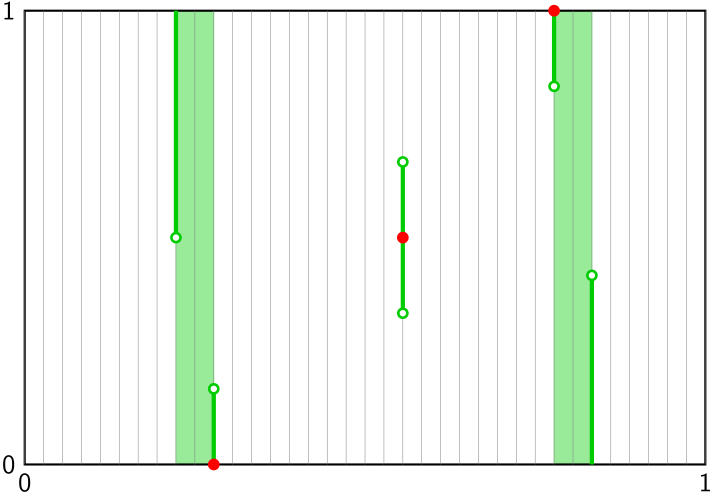

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Let E be a Banach space. The norm on E is called *locally uniformly rotund (LUR)* if for any point x in the unit sphere S_E and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset S_E$ we have that

$$\lim_{n\to\infty} \left\| \frac{x + x_n}{2} \right\| = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} \|x - x_n\| = 0$$

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Question

For which compact spaces K does $C(K)$ admit an equivalent LUR norm?

Some examples

Theorem (Haydon, Rogers ('90))

Let K be a scattered compact space with $K^{(\omega_1)} = \emptyset$. Then $C(K)$ admits an equivalent LUR norm.

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Any scattered compact space is homeomorphic to the limit of an F-decomposable system, with the fibers of neighboring bonding mappings homeomorphic to the one-point compactification of a discrete set.

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Theorem (Haydon, Jayne, Namioka, Rogers ('00))

The space $C([0,1]^{\gamma}_{lex})$ admits an equivalent LUR (strictly convex) norm if and only if γ is countable.

The problem for F-compacta

Theorem (Gul'ko, Ivanov, Shulikina, Troyanski ('20))

X admits an equivalent pointwise-lower semicontinuous LUR norm whenever X is a Fedorchuk compact of spectral height 3.

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Let X and Y be Hausdorff compacta and π a fully closed mapping from X onto Y. Then $C(X)$ admits an equivalent τ_p-lower semicontinuous LUR norm provided that the spaces $C(Y)$ and $C\left(\pi^{-1}(y)\right)$ for $y \in Y$ admit equivalent τ_p-lsc LUR norms.

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Theorem (M. (25'))

X admits an equivalent pointwise-lower semicontinuous LUR norm whenever X is an F_d-compact of countable spectral height.

The main tool

Theorem (Moltó, Orihuela, Troyanski ('97))

Let E be a Banach space and F a norming subspace of its dual. Then E admits an equivalent $\sigma(E,F)$-lower semicontinuous LUR norm if and only if for any $\epsilon > 0$ there exists a countable decomposition $X = \bigcup_{n\in\mathbb{N}} X_n$ such that for all $n \in \mathbb{N}$ and $x \in X_n$ there exists H an open half space containing x and satisfying:

$$\|\cdot\|\text{-diam}\,(X_n \cap H) < \epsilon.$$

A characterization (fully closed mapping)

Proposition (Gul'ko, Ivanov, Shulikina, Troyanski ('20))

Let $\pi : X \to Y$ be a continuous surjective mapping between Hausdorff compacta. Then π is fully closed if and only if for all $f \in C(X)$ we have

$$\left(\operatorname{osc}_{\pi^{-1}(y)} f : y \in Y\right) \in c_0(Y)$$

.

The tree of an inverse system

Let $S := \{X_\alpha, \pi^\beta_\alpha, \alpha, \beta \in \gamma\}$ be a continuous inverse system of Hausdorff compacta.

Define $\Upsilon(S) := \bigcup\{X_\alpha, \alpha \in \gamma\}$ and a partial order $\preceq$ on $\Upsilon(S)$ as follows. If $y \in X_\alpha, x \in X_\beta$, then

$$y \preceq x \quad \Longleftrightarrow \quad \alpha \leq \beta \text{ and } \pi^\beta_\alpha(x) = y.$$

We shall refer to $\Upsilon(S)$ as the *tree of the system S*.

A characterization (F-decomposable system)

Lemma

Let S and $\Upsilon(S)$ be as above and $X := \varprojlim S$. Consider the mapping

$$\Phi : C(X) \to l_\infty(\Upsilon(S))$$

$$\Phi(f)(x) := \operatorname{osc}_{\pi_\alpha^{-1}(x)} f.$$

Then Φ maps $C(X)$ into $C_0(\Upsilon(S))$ if and only if S is an F-decomposable system.

Another characterization (fully closed mapping)

Notation

Let X and Y be topological spaces and π a continuous mapping from X onto Y. If $M \subset Y$, by Y^M we will denote the quotient space corresponding to the following equivalence classes:

$$[x] \quad = \quad \begin{cases} x, & x \in \pi^{-1}(M); \\ \pi^{-1}\left(\pi(x)\right), & \pi(x) \in Y \setminus M. \end{cases}$$

We will denote the corresponding quotient mapping from X to Y^M by p^M and by $\pi^M : Y^M \to Y$ the unique mapping such that $\pi = \pi^M \circ p^M$.

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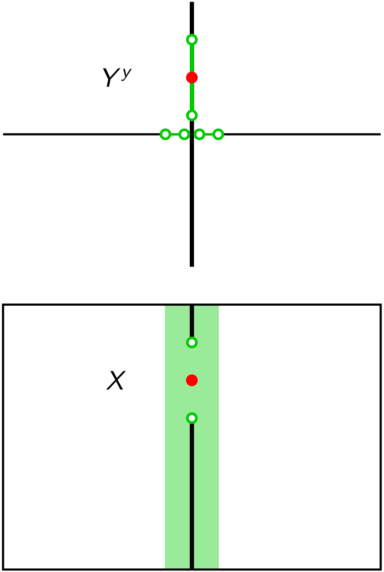
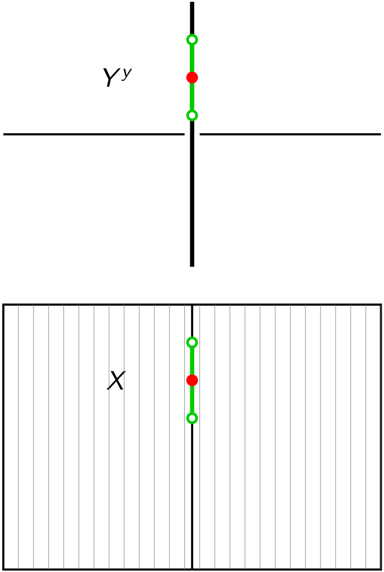
Proposition (Fedorchuk ('06))

Let X and Y be Hausdorff compacta and $f : X \to Y$ a continuous mapping. Then f is fully closed if and only if for any $M \subset Y$, the space Y^M is Hausdorff.

$\mathbb{R}^2$ square



Lexicographic square



Idea of the proof

$$
\begin{tikzcd}
X \arrow[rr, "f_2"] \arrow[drrr, bend right=15, "\pi^{M_1} \circ f_2"] \arrow[ddr, "q"] & & Y \arrow[rr, "f_1"] \arrow[dr, "\pi^{M_1}"] & & Z \\
& & & Z^{M_1} \arrow[ur, "p^{M_1}"] & \\
& \left(Z^{M_1}\right)^{f_1(M_2)} \arrow[urr, "p^{f(M_2)}"] & & &
\end{tikzcd}
$$

Idea of the proof

$$
\begin{tikzcd}
X \arrow[rr, "q"] \arrow[d, dotted, "\pi_{n+1}"] \arrow[rrd, "q^{\omega}_{n+1}"] & & K \arrow[d, dotted, "\lambda_{n+1}"] \\
X_{n+1} \arrow[rr, "q_{n+1}"] \arrow[d, "\pi^{n+1}_{n}"] \arrow[rrd, "q^{n+1}_{n}"] & & K_{n+1} \arrow[d, "\lambda^{n+1}_{n}"] \\
X_n \arrow[rr, "q_n"] \arrow[dr, dotted, "\pi^{n}_{1}", "q^{n}_{1}"'] & & K_n \arrow[dl, "\lambda^{n}_{1}"'] \\
& X_1 \cong K_1 &
\end{tikzcd}
$$

Thank you for the attention!