Combinatorial Banach spaces

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Structures in Banach spaces

This represents joint works with
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Theorem If a space $Exh(\|\cdot\|)$ does not contain a copy of ℓ_1, then $Fin(\|\cdot\|)$ is isometric to the double dual of $Exh(\|\cdot\|)$.

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Convention: $X_{\mathcal{F}} = Exh(\mathcal{F})$.

The canonical example

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The family $\mathcal{S}$ is compact (in the Cantor topology): a hereditary family $\mathcal{F}$ is compact iff there is no infinite $N \subseteq \omega$ such that $[N]^\infty \subseteq \mathcal{F}$.

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So, if $\mathcal{F}$ is compact, then $X_\mathcal{F}$ is c_0-saturated.

Each countable compact space is scattered (i.e. it does not contain a Cantor set). One can analyze $X_\mathcal{F}$ for compact families in terms of Cantor-Bendixson rank of $\mathcal{F}$ (higher order Schreier spaces).

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Let $\mathcal{A} = \{\text{finite antichains}\}$ and $\mathcal{C} = \{\text{finite chains}\}$.

Farah space

Let $I_n = [2^n, 2^{n+1})$ for every n. Let

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In general:

Schur property $\implies$ ℓ_1-saturation $\implies$ no copies of c_0

Schur property vs ℓ_1-saturation

It was an open problem if ℓ_1-saturation implies the Schur property. Solved by Bourgain in negative, then another examples appeared (Azimi-Hagler, Popov).

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Then $X_{\mathcal{F}}$ does not have the Schur property but it is ℓ_1-saturated.

Back to dyadic trees

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So, combinatorial spaces may be quite reach in terms of subspaces.

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Theorem. $X_{\mathcal{F}}$ is universal for Banach spaces with unconditional basis. Moreover, each Banach space with unconditional basis has a complemented copy in $X_{\mathcal{F}}$.

By a theorem due to Pełczyński $X_{\mathcal{F}}$ is isomorphic to so called Pełczyński space.

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Theorem No, $X_{\mathcal{F}}$ is not. In $X_{\mathcal{F}}$ the base is not universal (contrary to the case of the Pełczyński space).

A structure problem

Theorem. TFAE

- $X_{\mathcal{F}}$ does not contain an isomorphic copy of ℓ_1,
- $X_{\mathcal{F}}$ is c_0-saturated,
- $\mathcal{F}$ is compact,
- $\mathcal{F}$ is scattered.

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Problem. How to characterize in a combinatorial way families $\mathcal{F}$ such that $X_{\mathcal{F}}$ does not contain an isomorphic copy of c_0?

Back to dyadic trees again.

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- If $\mathcal{F}$ and $\mathcal{G}$ are geometrically dual, then they are combinatorially dual.

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An infinite graph G is *perfect* if each vertex generated finite subgraph of G is perfect.

Perfect graph conjecture

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Corollary (from the proof) If G is perfect then $\mathcal{C}(G)$ and $\mathcal{A}(G)$ are geometrically dual, i.e.

$$\mathrm{Exh}(\mathcal{C}(G))^* = \mathrm{Fin}(\mathcal{A}(G)).$$

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Corollary A graph G is perfect if and only if $\mathcal{C}(G)$ and $\mathcal{A}(G)$ are geometrically dual.

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What about $X_{\mathcal{C}(G_f)}$?

Thanks.