Solution to an old question of Joram Lindenstrauss

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Structures in Banach spaces
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The coauthors

- V. Montesinos and J. Orihuela. *Separable slicing and locally uniformly rotund renormings.* To appear in PAFA 2025 .
- V. Montesinos and J. Orihuela. *Weak compactness and separability of faces for convex renormings of Banach spaces.* Preprint 2025.
- A. Moltó, J. Orihuela, S. Troyanski, and M. Valdivia. *A nonlinear transfer technique for renorming*, Volume 1951 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2009.

LUR norms

Definition

If $(E, \|\cdot\|)$ is a normed space, the norm $\|\cdot\|$ is said to be **locally uniformly rotund** (**LUR**, for short) if

$$\left[\lim_n \left(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2\right) = 0\right] \Rightarrow \lim_n \|x - x_n\| = 0 \tag{1}$$

for any sequence $\{x_n\}_{n=1}^{\infty}$ and any x in E.

An equivalent, more geometrical, definition of the LUR property of the norm reads: If $\{x, x_1, x_2, \ldots\} \subset S_E$ and $\|x + x_n\| \to 2$, then $\|x - x_n\| \to 0$.

$$\boxed{Q_{\|\cdot\|}(x, y) := 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2.} \tag{2}$$

A main result

Theorem

A Banach space E with a norming subspace $F \subset E^*$ has an equivalent $\sigma(E,F)$-lower semicontinuous **LUR** *norm if, and only if,there is a sequence $\{A_n : \ n = 1, 2, \cdots\}$ of subsets of E such that, given any $x \in E$ and $\epsilon > 0$, there is a $\sigma(E,F)$-open half-space H and $p \in \mathbb{N}$ such that $x \in H \cap A_p$ and the slice $H \cap A_p$ can be covered with countable many sets of diameter less than ϵ.*

Therefore, this renorming can be achieved as soon as

$$x \in H \cap A_p \subset S_H^p + B(0, \epsilon),$$

where S_H^p is some separable subset of E, in particular any one with finite Hausdorff measure should be good here.

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Corollary

Let E be a Banach space, and let $F \subset E^*$ be a norming subspace of E^*. Let us assume that for every $\epsilon > 0$ and every $x \in S_E$ there is a $\sigma(E, F)$-open half-space H with $x \in H$ and $B_E \cap H$ is covered with countably many sets of diameter less or equal to ϵ. Then E admits an equivalent $\sigma(E, F)$-lower semicontinuous and locally uniformly rotund norm.

Corollary

A Banach space E with a norming subspace $F \subset E^*$, has an equivalent $\sigma(E, F)$-lower semicontinuous LUR norm if, and only if, it has anoher one with separable denting faces of its closed unit ball.

Solutions to open questions

- This result completely solves four problems asked by Moltó, Troyanski, Valdivia and myself. It generalizes Troyanski's fundamental results (see Chapter IV in Deville, Godefroy, Zizler), and Raja's theorems in LUR renormings, as well as García, Oncina, Troyanski and myself where finite covers were considered too .
- E has an equivalent $\sigma(E, F)$-lower semicontinuous **LUR** norm if, and only if, there is another one with separable denting faces. This add an answer to Lindenstrauss' question for LUR norms
- Banach spaces $C(K)$, *where K is a Rosenthal compact space $K \subset \mathbb{R}^\Gamma$ (i.e., a compact space of Baire one functions on a Polish space Γ,) with at most countably many discontinuity points for every $s \in K$,* This solves questions of Haydon, Moltó and myself.

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Theorem (Open Localization Theorem)

Let A be a bounded subset in E and $\mathcal{C} = \{\Theta_i : \; i \in I\}$ be $\sigma(E, F)$-closed convex subsets of E.
Then, there is an equivalent $\sigma(E, F)$-lower semicontinuous norm $\| \cdot \|_{\mathcal{C},A}$ such that:
If $x \in A \setminus \Theta_{i_0}$ for some $i_0 \in I$, and $\{x_n\}_{n=1}^{\infty}$ is a sequence in E such that $\lim_n Q_{\| \cdot \|_{\mathcal{C},A}}(x_n, x) = 0$, then there is a sequence $\{i_n\}_{n=1}^{\infty}$ in I such that:
There is $n_0 \in \mathbb{N}$ such that $x \in A \setminus \Theta_{i_n}$ for each $n \geq n_0$.
Moreover, if for some $n \geq n_0$ we have $x_n \in A$, then $x_n \in A \setminus \Theta_{i_n}$.

Theorem (Open localization plus aproximation theorem)

Let A be a bounded subset in E and $\mathcal{C} := \{\Theta_i : i \in I\}$ be a family of convex and $\sigma(E,F)$-closed subsets of E.
Then there is an equivalent $\sigma(E,F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{C},A}$ on E such that given $x \in A \setminus \Theta$ for some $\Theta \in \mathcal{C}$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in E with $\lim_n Q_{\|\cdot\|_{\mathcal{C},A}}(x_n, x) = 0$, then there is a sequence $\{i_n\}_{n=1}^{\infty}$ in I verifying the two following properties:

(i) *There is $n_0 \in \mathbb{N}$ with $x \in A \setminus \Theta_{i_n}$ for each $n \geq n_0$. Moreover, if $x_n \in A$ for some $n \geq n_0$, then $x_n \in A \setminus \Theta_{i_n}$.*

(ii) *Additionally we will still have the following approximation: For every $\delta > 0$ there is some $n_\delta \in \mathbb{N}$ such that*

$$x, x_n \in \overline{\operatorname{co}\left(A \setminus \Theta_{i_n}\right) + \delta B_E}^{\sigma(E,F)} \text{ for all } n \geq n_\delta. \tag{3}$$

Theorem (Δ-Convex Networking)

The following are equivalent:

(i) *E admits a $\sigma(E,F)$-lower semicontinuous equivalent LUR norm.*

(ii) *There are sequences $\{A_n\}_{n=1}^{\infty}$ of $\sigma(E,F)$-closed convex subsets of E together with families $\mathcal{C}_n$ of convex and $\sigma(E,F)$-closed subsets such that*

$$\bigcup\{A_n \setminus \Theta : \ \Theta \in \mathcal{C}_n, n \in \mathbb{N}\}$$

is a network for the norm topology in E.

(iii) *There is a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of E together with families $\mathcal{C}_n$ of convex and $\sigma(E,F)$-closed subsets such that*

$$\bigcup\{A_n \setminus \Theta : \ \Theta \in \mathcal{C}_n, n \in \mathbb{N}\}$$

is a network for the norm topology in E.

Question: Characterize those Banach spaces which have an equivalent strictly convex norm.

It is easily verified that every separable Banach space has an strictly convex norm. The same is true for a general WCG space. On the other hand, it was shown by Day that there exist Banach spaces which do not have an equivalent strictly convex norm.

Some conjectures concerning a possible answer to the question were shown to be false by Dashiell and Lindenstrauss.

This results shows that even for $C(K)$ spaces it seems to be a delicate and presumably difficult question to decide under which condition there exists an equivalent strictly convex norm.

Definition

We say that a topological space (X, τ) **is a $T_0(*)$-space** or that the topology τ **is** $T_0(*)$ if there is a system $\{\mathcal{W}_n : \; n \in \mathbb{N}\}$, where each $\mathcal{W}_n$ is a family of open sets, such that for $x \neq y$ there is some $p \in \mathbb{N}$ for which either we have $y \notin \mathrm{Star}(x, \mathcal{W}_p) \neq \emptyset$ or $x \notin \mathrm{Star}(y, \mathcal{W}_p) \neq \emptyset$.

For a family $\mathcal{F}$ of subsets of X, let us remind you:

$$\mathrm{Star}(x, \mathcal{F}) := \bigcup \{F : x \in F \in \mathcal{F}\}.$$

Systems $\{\mathcal{W}_n : n \in \mathbb{N}\}$ are said to $T_0(*)$**- separate points of** E. For a system $\{\mathcal{G}_n : n \in \mathbb{N}\}$, where each $\mathcal{G}_n$ consists of functions from E into $\mathbb{R}$, we say that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ $T_0(*)$**- separates points of** E whenever the system $\{\mathcal{O}_n : n \, in \mathbb{N}\}$ $T_0(*)$-separates points of E, where $\mathcal{O}_n := \{O_g : \; g \in \mathcal{G}_n\}$ for $n \in \mathbb{N}$, and

$$O_g := \{x \in E : \; g(x) > 0\}. \tag{4}$$

A solution

Theorem (Strictly Convex Renorming)

Let E be a normed space with a norming subspace $F \subset E^*$. Then E admits an equivalent $\sigma(E,F)$-lower semicontinuous and strictly convex norm if, and only if, there are families $\mathcal{G}_n$, $n \in \mathbb{N}$, of $\sigma(E,F)$-lower semicontinuous convex functions defined on E such that the system $\{\mathcal{G}_n : n \in \mathbb{N}\}$ $T_0(*)$-separates points of E.

Answer to Lindenstrauss

Theorem

Let E be a normed space with a norming subspace F in E^*. E admits an equivalent $\sigma(E, F)$-lower semicontinuous rotund norm if, and only if, it has another one with $\sigma(E, F)$-closed and norm separable faces.

Corollary

Every monolithic compact space K has a Banach $C(K)$ with an equivalent strictly convex norm.

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Let E be a normed space with a norming subspace F in E^*. Let B_E be the $\sigma(E,F)$-closed, convex unit ball of E and $\mathcal{H} = \{H_i : i \in I\}$ be a family of $\sigma(E,F)$-closed afine hyperplanes of E such that

$$S_E \cap H_i$$

is non void but separable for every $i \in I$. Then there is an equivalent $\sigma(E,F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{H},A}$ with the following property: For every $x \in A \cap H_{i_0}$ for some $i_0 \in I$ and every $y \in E$ such that

$$\left(\frac{1}{2}\|x\|^2_{\mathcal{H},A} + \frac{1}{2}\|y\|^2_{\mathcal{H},A} - \left\|\frac{x+y}{2}\right\|^2_{\mathcal{H},A}\right) = 0, \tag{5}$$

we should have $x = y$.

Answer to R. Smith

Theorem

Let E^* be a dual Banach space. Then E^* admits a dual rotund equivalent norm if, and only if, its unit ball B_{E^*} has $T_0(*)$ for the w^* topology

Corollary

Dual Banach spaces which are w**-homeomorphic preserve the property of being dual strictly convex renormable.*

The dual LUR case

Theorem (Bing-Nagata-Smirnov meeting renormings)

In a dual space E^* we have a dual LUR norm if, and only if, the norm topology admits a σ-discrete and w^*-relatively discrete basis.

Corollary

Asplund spaces with the same density character and w^*-homeomorphic dual spaces preserve the property of being dual LUR renormable.

Rigidity through δ-disjointness

Definition

Two disjoint faces

$$A := \{x \in B_E : f^*(x) = 1\} \text{ and } B := \{x \in B_E : g^*(x) = 1\}$$

for $f^*, g^* \in S_{E^*}$ in a normed space E are said to be δ-disjoint, for some $\delta > 0$, if, and only if

$$A - \delta := \{x \in B_E : E : f^*(x) \geq 1 - \delta\}$$

doest not meet

$$B - \delta := \{x \in B_E : E : g^*(x) \geq 1 - \delta\}$$

Press down LUR renorming

Theorem

Let E be a Banach space and F a norming subspace in E^*. Let us fix a a non void familly of δ- disjoint $\sigma(E,F)$-closed and separable faces $\mathcal{J}$ of the uni ball B_E for some fixed $\delta > 0$. Then E admits an equivalent $\sigma(E,F)$-lower semicontinuous norm which is going to be locally uniformly rotund at every point of

$$\overline{\bigcup\{G : G \in \mathcal{J}\}}^{\|\cdot\|}.$$

Even more, if there is a countable family of sets of faces $\{\mathcal{J}_n : n = 1, 2, \cdots\}$ where every one of the sets $\mathcal{J}_n$ is formed by δ_n- disjoint $\sigma(E,F)$-closed and separable faces for some $\delta_n > 0$, then the norm can be constructed to be LUR on

$$\bigcup\{\overline{G}^{\|\cdot\|} : G \in \mathcal{J}_n : n = 1, 2, \cdots\}$$

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Theorem

Let E be a Banach space and F a norming subspace in E^*. Then E admits an equivalent $\sigma(E, F)$-lower semicontinuous norm which is going to be locally uniformly rotund at every point of

$$\bigcup\{\mathcal{F}_n : n = 1, 2, \cdots\}$$

where every one of the families $\mathcal{F}_n$ is a weakly relatively discrete family of norm separable and $\sigma(E, F)$-compact faces.

Historical context of our research

LUR-renorm. $\rightarrow$ Kadec-renorm.$\rightarrow$ Descriptive space$\rightarrow$ weakly Cech-analytic $\rightarrow$ σ-fragmentable

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M. Jimenez, J.P. Moreno, L. Oncina, F. Garcia, S. Lajara, A. Avilés, S. Ferrari, L.C. García Girola, M. Raja, A. Moltó, M. López Pellicer, M. Valdivia, A.J. Guirao, V. Montesinos, $\cdots$ etc.

MAIN QUESTIONS:

- Does every Banach space E with the RNP admits an equivalent **LUR** norm?
- Does every Banach space with Fréchet differentiable norm admits an equivalent **LUR** norm?

We only know the following:

Theorem (Ferrari, Oncina, Raja and J.O.)

A descriptive Banach space, in particular every σ-fragmentable and hereditarely weakly-θ-refinable space, admits an equivalent **Kadec** *F-norm.*

Theorem

A Banach space with a Fréchet differentiable norm with separable faces admits an equivalent **LUR** *norm.*

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