A criterion for $2^{\mathfrak{c}}$ operator ideals on Banach spaces

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Notation: $[\mathbb{N}]$ and $[\mathbb{N}]^{<\infty}$ denote the families of infinite subsets of $\mathbb{N}$ and finite subsets of $\mathbb{N}$, respectably.

Banach spaces with completely described lattice of closed operator ideals

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- $C(K)$, for certain Koszmider-Mrowka compacta K [P.Koszmider 2005, G.Plebanek 2005]

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- Open problem: $L_1(0,1)$ and $C[0,1]$. So far a chain of continuum many closed operator ideals is known [W.B.Johnson, G.Pisier, G.Schechtman 2020]

A general criterion

A sequence $(F_n)_n$ of finite dimensional subspaces of a Banach space X forms a UFDD (an unconditional finite dimensional decomposition) of X, if any $x \in X$ is a sum of unconditionally convergent series $\sum_{n=1}^{\infty} x_n$, with $x_n \in F_n$ for each $n \in \mathbb{N}$.

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Theorem (Freeman, Schlumprecht and Zsák, after Johnson and Schechtman)

Let $\mathscr{C} \subset [\mathbb{N}]$ be an almost disjoint family, let X and Y be Banach spaces with UFDDs $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$, resp. and let $T \in \mathscr{L}(X, Y)$ satisfy the following.

- $TF_n \subset G_n$, $n \in \mathbb{N}$,
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Given any $A \subset \mathscr{C}$ let $J_A \subset \mathscr{L}(X, Y)$ be the smallest closed operator ideal containing all TQ_N, $N \in A$. Then the map $2^{\mathscr{C}} \ni A \mapsto J_A \subset \mathscr{L}(X, Y)$ is an embedding of $2^{\mathscr{C}}$ into the lattice of closed ideals in $\mathscr{L}(X, Y)$.

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The above tool is used explicitly or implicitly in all known results on $2^{\mathfrak{c}}$ closed operator ideals in $\mathscr{L}(X)$.

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no $N, M \in [\mathbb{N}]$ with $|N \setminus M| = \infty$ admit an operator $R \in \mathscr{L}(X)$ with $Rx_i \in \{x_j : j \in L_M\}$, $i \in L_N$.

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- Recall that the basis $(x_i)_i$ generates as a spreading model a basic sequence $(\tilde{x}_i)_i$, if for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ so that $(x_{i_1}, \ldots, x_{i_n})$ and $(\tilde{x}_1, \ldots, \tilde{x}_n)$ are $(1+\varepsilon)$-equivalent for any $m \leq i_1 < \cdots < i_n$.

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A sufficient condition for "inhomogeneity" of a basic sequence

Assume the basis $(x_i)_i$ with a spreading model $(\tilde{x}_i)_i$ admits no subsequence dominating $(\tilde{x}_i)_i$. Then some subsequence of $(x_i)_i$ is inhomogeneous.

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Proof: Gasparis dichotomy on compact families of finite sets of integers and Schreier families of countable order.

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We say that $(x_i)_i$ ∞-dominates $(e_i)_i$, if

(1) $(x_i)_i$ dominates $(e_i)_i$ (the map $Tx_i = e_i$, $i \in \mathbb{N}$, extends to $T \in \mathscr{L}(X, E)$).

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- The unit vector basis of ℓ_p ∞-dominates the unit vector basis of ℓ_q, provided $q > p \geq 1$.
- Assume $(x_i)_i$ admit no subsequence equivalent to the unit vector basis of c_0 and dominates $(e_i)_i$ via the operator T. Then the condition (2) implies strict singularity of T, but not vice versa.

A criterion

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Proof:

- Take $T \in \mathscr{L}(X, E)$ with $Tx_i = e_i$ for each i and the UFDD $(F_n)_n$ and $(G_n)_n$ defined by intervals witnessing by the inhomogeneity of the basis $(x_i)_i$ and test against the Johnson-Schechtman criterion:

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- Take $N, M \in [\mathbb{N}]$ with $|N \setminus M| = \infty$ and assume $dist(TQ_N, J_{TQ_M}) < 1$.

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- Take $T \in \mathscr{L}(X, E)$ with $Tx_i = e_i$ for each i and the UFDD $(F_n)_n$ and $(G_n)_n$ defined by intervals witnessing by the inhomogeneity of the basis $(x_i)_i$ and test against the Johnson-Schechtman criterion:
- Take $N, M \in [\mathbb{N}]$ with $|N \setminus M| = \infty$ and assume $dist(TQ_N, J_{TQ_M}) < 1$.
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A criterion

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- One of Gasparis-Leung tools provides an operator contradicting inhomogeneity of $(x_i)_i$.

Fix a family $\mathcal{F} \subset [\mathbb{N}]^{<\infty}$ which is hereditary (with respect to taking subsets), compact (in the product topology), covering $\mathbb{N}$ (contains all singletons) and large (any $L \in [\mathbb{N}]$ contains elements of $\mathcal{F}$ of arbitrary length).

Application - combinatorial spaces and their p-convex versions

Fix a family $\mathcal{F} \subset [\mathbb{N}]^{<\infty}$ which is hereditary (with respect to taking subsets), compact (in the product topology), covering $\mathbb{N}$ (contains all singletons) and large (any $L \in [\mathbb{N}]$ contains elements of $\mathcal{F}$ of arbitrary length). Define the p-convex combinatorial space $X_{\mathcal{F},p}$, $1 \leq p < \infty$, as the completion of c_{00} with

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The Schreier space [1937] is the prototype of combinatorial spaces, named by W.T.Gowers and studied by A.Antunes, K.Beanland and H.V.Chu, C.Brech, V.Ferenczi and A.Tcaciuc, P.Borodulin-Nadzieja and B.Farkas, S.Jachimek, APB. The p-convex versions were introduced by A.Bird, N.J.Laustsen and A.Zsák, and considered by M.Fakhoury.

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The criterion yields the following generalization of results in [A.Manoussakis, APB 2021, R.M.Causey, APB 2025, N.J.Laustsen, J.Smith 2025].

Corollary. There are $2^{\mathfrak{c}}$ many closed operator ideals in $\mathscr{L}(X_{\mathcal{F},p})$.

Fix a family $\mathcal{F} \subset [\mathbb{N}]^{<\infty}$ which is hereditary (with respect to taking subsets), compact (in the product topology), covering $\mathbb{N}$ (contains all singletons) and large (any $L \in [\mathbb{N}]$ contains elements of $\mathcal{F}$ of arbitrary length).

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Define the Baernstein space $B_{\mathcal{F},p}$, $1 < p < \infty$, as the completion of c_{00} with a norm

$$\|(a_i)_i\|_{B_{\mathcal{F},p}} = \sup_{F_1 < \cdots < F_m \in \mathcal{F}} \left\| \left(\|(a_i)_{i \in F_j}\|_1 \right)_{j=1}^{m} \right\|_p, \quad (a_i)_i \in c_{00}$$

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The criterion implies the following generalization of one of results in [N.J.Laustsen-J.Smith].

Corollary. There are $2^{\mathfrak{c}}$ many closed operator ideals in $\mathscr{L}(B_{\mathcal{F},p})$.

Fix $q < 2 < p$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $w = (w_i)_i \searrow 0$ with $\sum_i w_i^{2p/p-2} = \infty$.

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Spaces $X_{w,p}$ defined by different w's as above are isomorphic [H.Rosenthal].

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Application - Rosenthal spaces and their duals

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Corollary [W.B.Johnson, G.Schechtman]

There are $2^{\mathfrak{c}}$ many closed operator ideals in $\mathscr{L}(X_{w,p}^*)$ and $\mathscr{L}(X_{w,p})$.

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Application - a class of Lorentz sequence spaces

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- Fix $k \in \mathbb{N}$. Decompose $\mathbb{N} = \bigcup_i E_i^k$ into successive intervals of length k. For $v_k := \sum_{i \in E_1} w_i$ define $w_i^k = \frac{1}{v_k} \sum_{j \in E_i} w_j$ and $x_i^k = \frac{1}{v_k^{1/p}} \sum_{j \in E_i} e_j$. Then $(x_i^k)_i$ is 1-equivalent to $(e_i)_i \subset d(w^k, p)$ with $w^k = (w_i^k)_i$.

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Assume $(*)$ $\exists (k_n)_n \subset \mathbb{N}: \; w_i^{k_{n+1}} \geq w_i^{k_n}$ for any $n, i \in \mathbb{N}$ and $\sum_{i=1}^{s_n} w_i^{k_{n+1}} / \sum_{i=1}^{s_n} w_i^{k_n} \to \infty$ for some $(s_n)_n \subset \mathbb{N}$.

Fix $1 < p < \infty$ and $w = (w_i)_i \searrow 0$ with $\sum_i w_i = \infty$. Let Π be the set of all permutations of $\mathbb{N}$. Define the Lorentz sequence space $d_{w,p}$ as the completion of c_{00} with

$$\|(a_i)_i\|_{d_{w.p}} = \sup_{\pi\in\Pi} \|(w_i^{1/p} a_{\pi(i)})_i\|_p, \quad (a_i)_i \in c_{00}$$

- $d_{w,p}$ is reflexive, the unit vector basis $(e_i)_i \subset d_{w,p}$ is 1-unconditional and symmetric (i.e. equivalent to all of its subsequences).
- Any block sequence of averages of the basis spans a complemented subspace (general fact concerning symmetric bases).
- Fix $k \in \mathbb{N}$. Decompose $\mathbb{N} = \bigcup_i E_i^k$ into successive intervals of length k. For $v_k := \sum_{i\in E_1} w_i$ define $w_i^k = \frac{1}{v_k}\sum_{j\in E_i} w_j$ and $x_i^k = \frac{1}{v_k^{1/p}}\sum_{j\in E_i} e_j$. Then $(x_i^k)_i$ is 1-equivalent to $(e_i)_i \subset d(w^k, p)$ with $w^k = (w_i^k)_i$.

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Application - a class of Lorentz sequence spaces

Fix $1 < p < \infty$ and $w = (w_i)_i \searrow 0$ with $\sum_i w_i = \infty$. Let Π be the set of all permutations of $\mathbb{N}$. Define the Lorentz sequence space $d_{w,p}$ as the completion of c_{00} with

$$\|(a_i)_i\|_{d_{w.p}} = \sup_{\pi \in \Pi} \|(w_i^{1/p} a_{\pi(i)})_i\|_p, \quad (a_i)_i \in c_{00}$$

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Assume $(*)$ $\exists (k_n)_n \subset \mathbb{N}: \ w_i^{k_{n+1}} \geq w_i^{k_n}$ for any $n, i \in \mathbb{N}$ and $\sum_{i=1}^{s_n} w_i^{k_{n+1}} / \sum_{i=1}^{s_n} w_i^{k_n} \to \infty$ for some $(s_n)_n \subset \mathbb{N}$. Then there is a sequence $(x_i)_i \subset (x_i^k)_{i,k}$ that is inhomogeneous and ∞-dominates $(e_i)_i \subset d(w,p)$.

Corollary. There are $2^{\mathfrak{c}}$ many closed operator ideals in $\mathscr{L}(d_{w,p})$ for w with $(*)$.

Thank you !