The Banach-Saks rank of a separable weakly compact set

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joint work in progress with J. López-Abad

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Mazur's and Banach-Saks' Theorem

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Every weakly convergent sequence in a Banach space has a norm-convergent block subsequence of convex combinations.

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Theorem (Banach, Saks, 1930)

Every bounded sequence in $L^p(0,1)$, $1 < p < \infty$, has a Cesàro convergent subsequence.

Definition

A subset A of a Banach space X has the Banach-Saks property if every sequence in A has a Cesàro convergent subsequence. We say that X has the Banach-Saks property if its unit ball B_X has it.

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$$\Big\downarrow \; \textit{Banach, Saks, '30}$$

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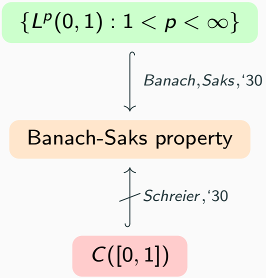

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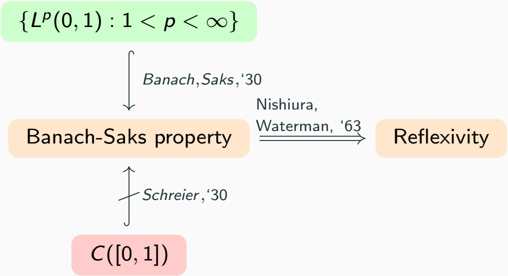

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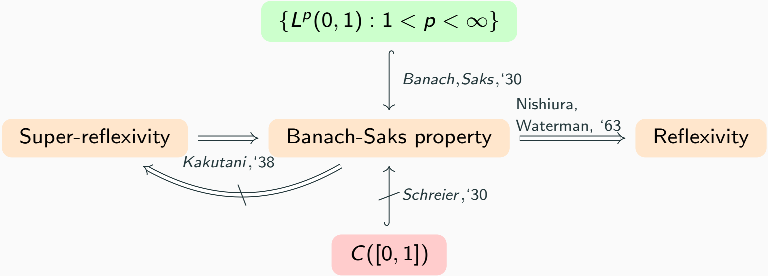

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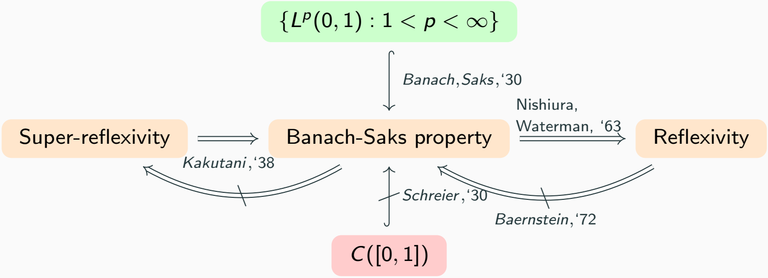

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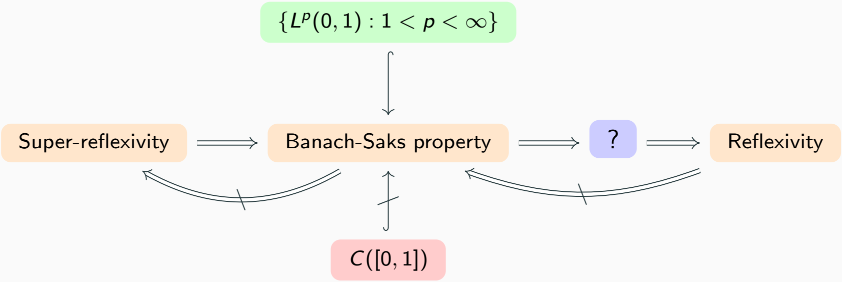

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For a sequence $(x_n)_n$ the following are equivalent:

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Once $\mathcal{S}_\xi$ is defined, we set

$$\mathcal{S}_{\xi+1} = \mathcal{S}_\xi \otimes \mathcal{S}_1 := \left\{ \bigcup_{j=1}^{n} s_j : (s_j)_{j=1}^{n} \text{ is block, } s_j \in \mathcal{S}_\xi,\ n \leq \min s_1 \right\}.$$

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If ξ is a limit ordinal we fix a sequence $(\xi_n)_{n\in\mathbb{N}} \nearrow \xi$ and set

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Definition

We say that a sequence $(x_n)_n$ in a Banach space is an ℓ_1-spreading model if there exists $\varepsilon > 0$ such that for every sequence of coefficients $(a_n)_n$ we have

$$\left\| \sum_{n \in s} a_n x_n \right\| \geq \varepsilon \sum_{n \in s} |a_n| \quad \text{for every } s \in \mathcal{S}.$$

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Definition (Alspach, Argyros, '92)

We say that a sequence $(x_n)_n$ in a Banach space is an ℓ_1^ξ-spreading model, $\xi < \omega_1$, if there exists $\varepsilon > 0$ such that for every sequence of coefficients $(a_n)_n$ we have

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Theorem (Alspach, Argyros, '92)

If a sequence contains ℓ_1^ξ-spreading models for every $\xi < \omega_1$, then it contains a subsequence equivalent to the unit vector basis of ℓ_1.

ξ-Repeated averages

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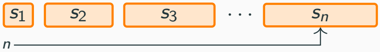

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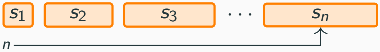

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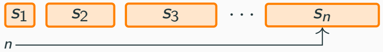

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Definition

A sequence $\mathbf{x} = (x_n)_n$ is ξ-summable if there is a block sequence $(\boldsymbol{a}^k)_k$ of ξ-repeated averages such that $\langle \boldsymbol{a}^n, \mathbf{x} \rangle$ converges in norm.

Theorem (Argyros, Mercourakis, Tsarpalias, '92)

For a weakly null sequence $(x_n)_n$ and $\xi < \omega_1$ exactly one of the following holds:

(i) *Every subsequence has a further subsequence which is ξ-summable.*

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Definition

Given a separable subset A of a Banach space, we define its Banach-Saks rank $\varrho_{BS}(A)$ as the minimum ordinal $\xi < \omega_1$ (if it exists) such that every sequence in A has a ξ-summable subsequence.

Some new results

- The coefficients of the convex combinations are not important:

Theorem

A separable subset A of a Banach space has Banach-Saks rank $\leq \xi$ if and only if every $(x_n)_n \subseteq A$ has a subsequence for which there is a block subsequence of convex combinations with supports in $\mathcal{S}_\xi$ converging in norm. One can replace $\mathcal{S}_\xi$ by any spreading and ω^ξ-uniform family.

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- We can obtain information about $\varrho_{BS}(\mathbf{x})$ from the sequence $\mathbf{x}$:

Proposition

For a sequence $\mathbf{x}$ weakly convergent to some $x \in X$ and $\varepsilon > 0$ define

$$\mathcal{C}(\mathbf{x}, \varepsilon) := \{s \in [\mathbb{N}]^{<\infty} : \exists x^* \in B_{X^*} \text{ such that } |x^*(x_n - x)| \geq \varepsilon \text{ for every } n \in s\}.$$

For a separable subset $A \subseteq X$ set

$$\beta(A) := \sup \left\{ \alpha < \omega_1 : \begin{array}{c} \exists (x_n)_n \subseteq A \text{ weakly convergent and } \varepsilon > 0 \\ \text{such that } \mathcal{C}(\mathbf{x}, \varepsilon) \text{ is } \alpha\text{-uniform} \end{array} \right\}.$$

Then

$$\beta(A) \leq \omega^{\varrho_{BS}(A)} \leq \beta(A) \cdot \omega.$$

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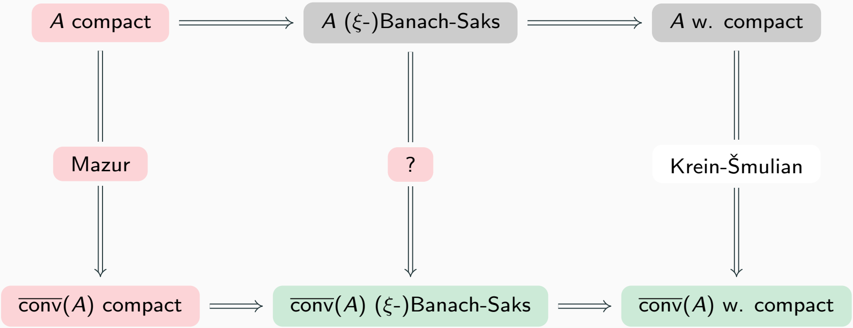

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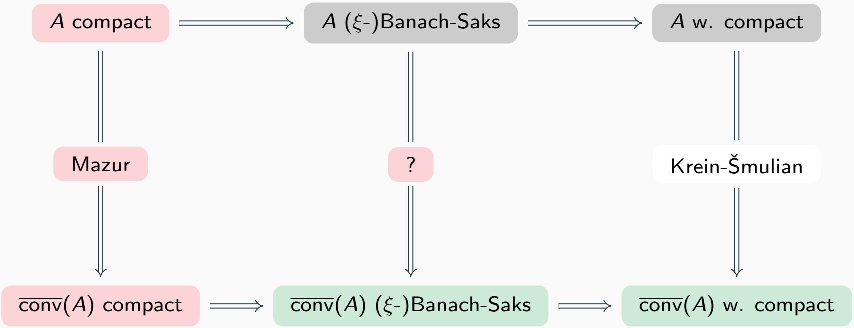

Theorem (López-Abad, Ruiz, Tradacete, 2013)

There exists a separable Banach-Saks set whose closed convex hull is not Banach-Saks.

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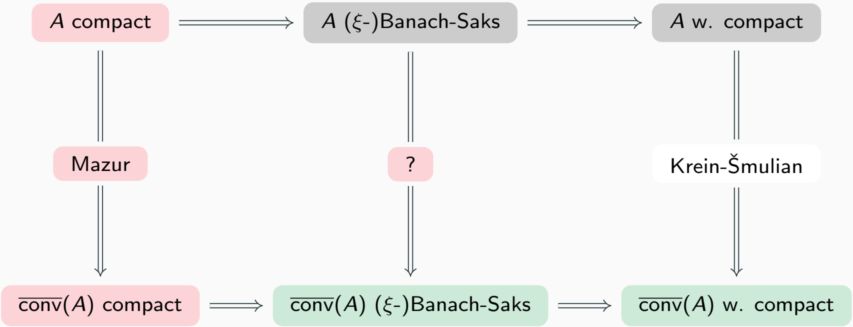

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Question 1

Do we have $\varrho_{BS}(\overline{\text{conv}}(A)) \leq \varrho_{BS}(A) + 1$ for all separable weakly compact sets A?

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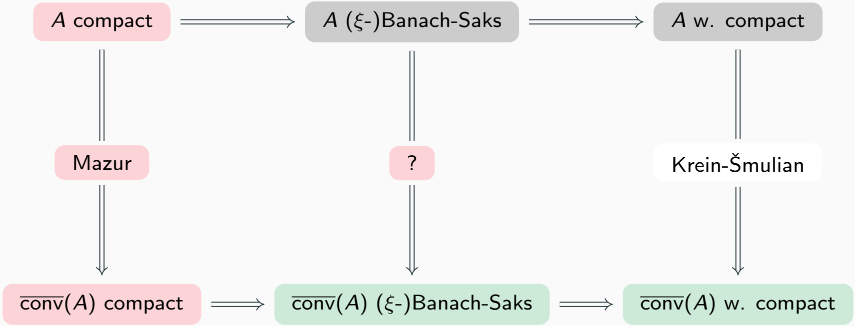

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Question 2

Is there an ordinal function $f : \omega_1 \to \omega_1$ such that $\varrho_{BS}(\overline{\text{conv}}(A)) \leq f(\varrho_{BS}(A))$ for all separable weakly compact sets A?

A first attempt: co-analytic ranks

Let $\mathfrak{F}$ be the family of all closed subsets of $C([0,1])$ and endow it with the Effros-Borel σ-algebra, i.e. the one generated by the sets

$$\{F \in \mathfrak{F} : F \cap U \neq \emptyset\} \quad \text{with } U \subseteq C([0,1]) \text{ open.}$$

This space is an standard Borel space, meaning that there is a Polish topology generating the σ-algebra.

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The subfamily $RWC \subseteq \mathfrak{F}$ of relatively weakly compact sets is co-analytic ($\mathbf{\Pi}^1_1$) by a result of James. We take the notion of $\mathbf{\Pi}^1_1$-ranks $\varrho : RWC \to \omega_1$ from Descriptive Set Theory, which satisfies:

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(i) *For every $\xi < \omega_1$ the set $\{x \in A : \varrho(x) \leq \xi\}$ is Borel.*

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$$\{F \in \mathfrak{F} : F \cap U \neq \emptyset\} \quad \text{with } U \subseteq C([0,1]) \text{ open.}$$

This space is an standard Borel space, meaning that there is a Polish topology generating the σ-algebra.

The subfamily $RWC \subseteq \mathfrak{F}$ of relatively weakly compact sets is co-analytic ($\mathbf{\Pi}^1_1$) by a result of James. We take the notion of $\mathbf{\Pi}^1_1$-ranks $\varrho : RWC \to \omega_1$ from Descriptive Set Theory, which satisfies:

Theorem (Boundedness Theorem for $\mathbf{\Pi}^1_1$-ranks)

Let $\varrho : A \to \omega_1$ be a co-analytic rank on a $\mathbf{\Pi}^1_1$ subset A of a standard Borel space.

(i) *For every $\xi < \omega_1$ the set $\{x \in A : \varrho(x) \leq \xi\}$ is Borel.*

(ii) *If $B \subseteq A$ is analytic then $\sup\{\varrho(x) : x \in B\} < \omega_1$.*

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(ii) *If $B \subseteq A$ is analytic then $\sup\{\varrho(x) : x \in B\} < \omega_1$.*

(iii) *If ϱ is another co-analytic rank on A, then there exists an increasing function $f : \omega_1 \to \omega_1$ such that $\varrho' \leq f(\varrho)$.*

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Question 3

Is $\varrho_{BS} : RWC \to \omega_1$ a co-analytic rank?

No!

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Proposition

The Banach-Saks rank $\varrho_{BS} : RWC \to \omega_1$ is not a co-analytic rank.

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Idea of the proof.

- We define the uniform rank of a compact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ as

$$\text{urk}(\mathcal{F}) \coloneqq \sup\{\alpha < \omega_1 : \exists M \in [\mathbb{N}] \text{ s.t. } \mathcal{F} \cap \mathcal{P}(M) \text{ is } \alpha\text{-uniform on } M\}.$$

ϱ_{BS} is related to the uniform rank of the families $\mathcal{C}(\mathbf{x}, \varepsilon)$ above. If ϱ_{BS} is co-analytic, so is urk.

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Question 2 (again)

Is there an ordinal function $\phi : \omega_1 \to \omega_1$ such that $\varrho_{BS}(\overline{\text{conv}}(A)) \leq \phi(\varrho_{BS}(A))$ for all $A \in RWC$?

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Thank you for your attention!