Dense-range operators with the Kato property and Quasicomplemented subspaces

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Kato property for operators

Definition (Kato property for an operator)

Let $T : E \to F$ be a operator between Banach spaces. We say that T has the **Kato property** if for each finite-codimensional subspace $W \subset E$, the restriction operator $T|_W$ is not an isomorphism: for each $\varepsilon > 0$ there exists $x \in W$ such that

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Theorem (Fonf and Shevchik, 1994)

Let E and F be Banach spaces, with E separable, and let $T : E \to F$ be a operator with the Kato property. Then, for every $\varepsilon > 0$ there exists a closed subspace $E_1 \subset E$ with a basis such that $\|T_{|E_1}\| \leq \varepsilon$, $T_{|E_1}$ is compact and

$$\overline{T(E_1)} = \overline{T(E)}.$$

Operator ranges

Definition (Operator range)

Let E be a Banach space. We say that a subspace $R \subset E$ is an **operator range** if there exists a Banach space F and an operator $T : F \to E$ such that $R = T(F)$.

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$\mathcal{R}(E)$: the set of infinite-dimensional and infinite-codimensional operator ranges in E.
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Theorem (Saxon and Wilansky, 1977)

If E is a Banach space, TFAE:

(a) There exists a closed subspace $X \subset E$ such that E/X is separable.
(b) $\mathcal{R}_d(E) \neq \varnothing$.

A result about disjointness properties

Theorem (Fonf, Lajara, Troyanski and Zanco, 2019)

Let E be a separable Banach space, let $T : E \to F$ be an operator from E into another Banach space F, and let $L \subset E$ be a closed subspace of E and $R \in \mathcal{R}(E)$ such that

$$\operatorname{codim}(R + L) = \infty \quad \text{and} \quad R \cap L = \{0\}.$$

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Assume that, for every closed infinite-codimensional subspace $W \subset E$ containing L there is a vector $x \in W \setminus L$ such that

$$\|T(x)\| \leq \varepsilon \|Q_L(x)\|.$$

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$$\|T(x)\| \leq \varepsilon \|Q_L(x)\|.$$

Then, there exists a closed subspace $E_1 \subset E$ with the following properties:

(a) $L \subset E_1$ and $\dim(E_1/L) = \infty$,

(b) $E_1 \cap R = \{0\}$, and

(c) $\overline{T(E_1)} = \overline{T(E)}$.

Quasicomplemented subspaces

Definition (Quasicomplementary subspaces)

Let E be a Banach space and let $X, Y \subset E$ be two closed subspaces. We say that X and Y are **quasicomplementary** if

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We say that two closed subspaces $X, Y \subset E$ are **proper quasicomplements** if they are quasicomplements and

$$X + Y \neq E.$$

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The Kato property for operators and subspaces

Definition

Let $T : E \to F$ be an operator between Banach spaces and let $L \subset E$ be a closed subspace. We say that T has the **Kato property for** L if for every $\varepsilon > 0$ and for every finite-codimensional subspace $W \subset E$ such that $L \subset W$ there exists $x \in W \setminus L$ such that

$$\|Tx\| \leq \varepsilon \|Q_L(x)\|.$$

The Kato property for operators and subspaces

Let $T : E \to F$ be an operator and $L \subset E$ be a closed subspace. We define $\widetilde{T}_L : E/L \to F/\overline{T(L)}$ as the operator given by the formula

$$\widetilde{T}_L \circ Q_L(x) = Q_{\overline{T(L)}} \circ T(x), \quad x \in E.$$

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Proposition

Given an operator $T : E \to F$ between Banach spaces and a closed subspace $L \subset E$, TFAE:

(1) T has the Kato property for L.

(2) $\widetilde{T}_L$ has the Kato property.

(3) There exists a $L^{\perp}$-minimal sequence $\{x_n\}_n \subset E$ such that

$$\|Tx_n\| \leq n^{-1}\|Q_L(x_n)\| \quad \text{for every} \quad n \geq 1.$$

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Main Result

Theorem (Jiménez Sevilla, Lajara and Ruiz Risueño, 2025)

Let E be a Banach space with a separable quotient, let $T : E \to F$ be a operator from E into another Banach space F, and let $L \subset E$ be a closed subspace of E and $R \in \mathcal{R}_d(E)$ such that

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Assume that T has the Kato property for L.Then, there exists a closed subspace $E_1 \subset E$ with the following properties:

(a) $L \subset E_1$ and $\dim(E_1/L) = \infty$,

(b') $\operatorname{codim}_E(E_1 + R) = \infty$, and

(c) $\overline{T(E_1)} = \overline{T(E)}$.

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(b') $\operatorname{codim}_E(E_1 + R) = \infty$, and

(c) $\overline{T(E_1)} = \overline{T(E)}$.

If, in addition, E^* is weak*-separable, then E_1 can be constructed so that

(b) $E_1 \cap R = \{0\}$.

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Diminution of quasicomplements

Theorem (James, 1972; Johnson, 1973)

Let X and Y be a couple of proper quasicomplementary subspaces in a Banach space E. If X is a weakly compactly generated (WCG) space, then there exists a closed subspace $X_1 \subset X$ such that

$$X_1 + Y \quad \text{is} \ \text{dense} \ \text{in} \quad E.$$

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Corollary 1 (Jiménez Sevilla, Lajara and Ruiz Risueño, 2025)

Let X and Y be a couple of proper quasicomplementary subspaces in a Banach space E. If X has a separable quotient, then there exists a closed subspace $X_1 \subset X$ such that

$$X_1 + Y \quad \text{is dense in} \quad E.$$

Spaceability in the dual

Corollary 2 (Plichko, 1981; Jiménez Sevilla, Lajara and Ruiz Risueño, 2025)

Let $T : E \to F$ be a one-to-one and dense-range operator between Banach spaces, where E has a separable quotient, and let $R \in \mathcal{R}_d(E)$. If $T(E)$ is not closed then there exists a closed subspace $X \subset E$ such that E/X is separable and

$$\mathrm{codim}_E(R + X) = \infty \quad \text{and} \quad T^*(F^*) \cap X^\perp = \{0\}.$$

A characterization of weak*-separability of the dual

Corollary 3 (Jiménez Sevilla, Lajara and Ruiz Risueño, 2025)

Let E be a Banach space with a separable quotient. TFAE:

(1) E^* is weak*-separable.

(2) For every closed subspace $X \subset E$ such that E/X has a separable quotient and for every $R \in \mathcal{R}_d(E)$ there exists a quasicomplement Y of X such that
$$R \cap Y = \{0\}.$$

Theorem (Jiménez Sevilla and Lajara, 2024)

Let E be a Banach space with a separable quotient. TFAE:

(1) E^* is weak*-separable.

(2) For every $R \in \mathcal{R}_d(E)$ there exists a closed subspace $Y \subset E$ such that E/Y is separable and
$$R \cap Y = \{0\}.$$

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