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Discrete subgroups of Banach spaces and lattice tilings

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Structures in Banach Spaces
March 17–21, 2025
Vienna, Austria

A net which is a subgroup?

Question (Medina, Vlachovice WS'24, problem session)

Does every Banach space admit a net closed under addition?

- $\mathcal{D}$ is r-**separated** if $\|d-h\| \geqslant r$ for $d \neq h \in \mathcal{D}$.
- $\mathcal{D}$ is R-**dense** if for all $x \in \mathcal{X}$ there is $d \in \mathcal{D}$ with $\|x-d\| \leqslant R$.
- $\mathcal{D}$ is a **net** if it is both (for some r, R).
- Motivation:
  - Does $\mathcal{F}(\mathcal{N})$ have a Schauder basis, for a net $\mathcal{N}$ in a separable $\mathcal{X}$?
  - A discretisation of $\mathcal{X}$, both in the metric and algebraic sense.

Answer (Doucha, *ibidem*)

- **Yes**, a 1-separated and $(1+\varepsilon)$-dense subgroup, if $\mathcal{X}$ separable.
  - **Dilworth, Odell, Schlumprecht, Zsák (2008).**
- (A later email) What about non-separable $\mathcal{X}$?

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And why should we care?

- **Rogers (1984).** A 1-separated and $(3/2+\varepsilon)$-dense subgroup.
- **Swanepoel (2009).** Can you get $(1+\varepsilon)$-dense?

Theorem (De Bernardi, R., Somaglia)

In every infinite-dimensional Banach space $\mathcal{X}$ there is a 1-separated and $(1+\varepsilon)$-dense subgroup.

- And... who cares, precisely?
- A simple constructive proof by induction, only using Riesz' lemma.
- If $\Gamma^\omega = \Gamma$, $\ell_2(\Gamma)$ contains a $(\sqrt{2}+)$-separated and 1-dense subgroup.
- **There exists a reflexive Banach space (isomorphic to $\ell_2(\Gamma)$) that is tiled by balls of radius 1.**
- **Fonf, Lindenstrauss (1998).** Can a reflexive space be tiled by translates of a convex body?
  - Repeated in **Guirao, Montesinos, Zizler (2016)** *Open problems...*

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Keep assuming that $\Gamma^\omega = \Gamma$

- $\ell_p(\Gamma)$ contains a $(2^{1/p}+)$-separated and 1-dense subgroup.

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