Q-measures, Q-points, and L-orthogonality.

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Workshop Structure in Banach Spaces
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Joint work with Antonio Avilés, Gonzalo Martínez-Cervantes, and Alejandro Poveda.

The plan

Banach Space questions

Special Ultrafilters and similar objects

Comments and Open questions

Some definitions

Definition

Let $(X, \|\cdot\|)$ be a Banach space

1. $\|\cdot\|$ *is* **octahedral** *if for any* $x_0, \ldots, x_{n-1} \in X$, $\epsilon > 0$, *there is* $y \in S_X$ *such that* $\|x_i + y\| > \|x_i\| + \|y\| - \epsilon$.

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Renorming results

Theorem (Godefroy (1989))

For a Banach space X, TFAE:

- *X contains a isomorphic copy of ℓ_1.*
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Proposition (Avilés, Martínez-Cervantez, Rueda Zoca (2022))

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Spoiler

The answer to both questions is independent of the usual set theory axioms (ZFC).

Ultrafilters

Definition (Filters)

A family $\mathscr{F} \subseteq \mathcal{P}(\mathbb{N})$ *is a* **filter over** $\mathbb{N}$ *if*

1. $\varnothing \notin \mathscr{F}$
2. *If* $F \in \mathscr{F}$, $F \subseteq E$ *then* $E \in \mathscr{F}$
3. *If* $F, F' \in \mathscr{F}$, *then* $F \cap F' \in \mathscr{F}$

- *A filter over* $\mathbb{N}$ *is* **free** *if it extends the Fréchet filter.*
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Definition

A free ultrafilter $\mathscr{U} \subseteq \mathcal{P}(\mathbb{N})$ is a Q-point if for any partition of $\mathbb{N}$ into finite sets $(I_n)_{n\in\mathbb{N}}$, there is a selector $S \in \mathscr{U}$ such that for any $n \in \omega$, $|S \cap I_n| = 1$.

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The existence of Q-points is independent of ZFC.

Answering the second question.

Definition

*Given a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X, and filter $\mathscr{F}$ over $\mathbb{N}$, the $\mathscr{F}$-**limit** with respect to the sequence is $x \in X$ ($x = \mathscr{F}\text{-}\lim x_n$) iff for every neighborhood V of x,*

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Theorem (Hrušák, S. (2024))

For any Banach Space X, $(x_n)_{n\in\mathbb{N}}$ an L-orthogonal sequence, and $\mathscr{U}$ a Q-point, $\mathscr{U}\text{-}\lim x_n \in X^{**}$ (taken in the w^* topology of X^{**}) is L-orthogonal.

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If there are no Q-points, then there is a Banach space X with an L-orthogonal sequence such that no $x^{**} \in \overline{\{x_n : n \in \mathbb{N}\}}^{w^*}$ is L-orthogonal.

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Theorem

Fix an ultrafilter $\mathscr{U}$. $\mathscr{U}$ is a Q-point iff for any Banach space X, and $(x_n)_{n\in\mathbb{N}}$ an L-orthogonal sequence, $\mathscr{U}\text{-}\lim x_n \in X^{**}$ is L-orthogonal.

Special Measures

Definition

Let $\mu : \mathcal{P}(\mathbb{N}) \longrightarrow [0, +\infty)$ be a finitely additive measure defined on the subsets of $\mathbb{N}$ and vanishing on finite sets.

1. μ is a ***Q*-measure** if every partition of $\mathbb{N}$ into finite sets has a selector of positive measure.
2. μ is a **fit *Q*-measure** if for every partition of $\mathbb{N}$ into finite sets and every $\delta < \frac{1}{2}$ there is a finite union of selectors of measure greater than $\delta \cdot \mu(\mathbb{N})$.
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Definition

Given a sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X, and a measure of bounded variation $\mu \colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}$. Let $T \colon \ell_1 \to X$ be defined as $T(e_n) := x_n$ for $n \in \mathbb{N}$. Define the μ-limit of $(x_n)_{n\in\mathbb{N}}$ as

$$\mu\text{-}\lim x_n := T^{**}(\mu).$$

Here $T^{**} \colon \ell_1^{**} \to X^{**}$ and remember that $\ell_1^{**} = \ell_\infty^*$ is naturally identified with the set of finitely additive signed finite measures on $\mathcal{P}(\mathbb{N})$. In the case where μ stems from an ultrafilter $\mathcal{U}$ this coincides with the classical $\mathcal{U}$-limit.

Answering the second question

Theorem

Let X be a Banach space, $(x_n)_{n\in\mathbb{N}}$ an L-orthogonal sequence, and $\mu\colon \mathcal{P}(\mathbb{N}) \to [0,1]$ a strong Q-measure with $\mu(\mathbb{N}) = 1$. Then $\mu\text{-}\lim x_n$ is an L-orthogonal element.

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Can you provide a characterization of when, for any measure μ, the $\mu\text{-}\lim x_n$ of an L-orthogonal sequence is L-orthogonal?

Comments on the proofs

Theorem (Mathias)

Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, TFAE.

1. *$\mathscr{U}$ is selective.*
2. *For any tall analytic ideal $\mathcal{I}$, $\mathcal{I} \cap \mathscr{U} \neq \varnothing$.*

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Theorem (Hrušák, Meza, Minami (2010))

Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, TFAE.

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Theorem

Let $\mu\colon \mathcal{P}(\mathbb{N}) \to [0,1]$ be a measure and $\varepsilon > 0$. TFAE.

1. *μ is an ε-strong Q-measure.*
2. *For every analytic hereditary family $\mathcal{H}$ that is countably hitting, there is $A \in \mathcal{H}$, such that $\mu(A) \geq \varepsilon$.*

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Theorem

TFAE

1. $\mathfrak{d} = cov(\mathcal{M})$.
2. *Any filter $\mathscr{F}$ of character less that $\mathfrak{d}$ can be extended to a Q-point.*
3. *For any measure $\mu : \mathbb{B} \to [0, 1]$, such that $[\mathbb{N}]^{<\mathbb{N}} \subseteq \mathbb{B} \subseteq \mathcal{P}(\mathbb{N})$ is Boolean Algebra, μ vanishes on finite sets, and has density less than $\mathfrak{d}$, there exists an atomless strong Q-measure $\nu : \mathcal{P}(\mathbb{N}) \to [0, 1]$ extending μ.*

Thanks!

References

[1] Antonio Avilés, Gonzalo Martínez-Cervantes, Alejandro Poveda, and Luis Sáenz. *A banach space with l-orthogonal sequences but without l-orthogonal elements*. Preprint, 2024.

[2] Antonio Avilés, Gonzalo Martínez-Cervantes, and Abraham Rueda Zoca. *L-Orthogonal Elements and L-Orthogonal Sequences*. International Mathematics Research Notices, 2023(11):9128–9154, 05 2022.

[3] Gilles Godefroy. *Metric characterization of first baire class linear forms and octahedral norms*. Studia Mathematica, 95(1):1–15, 1989.

[4] Michael Hrušák, David Meza-Alcántara, and Hiroaki Minami. *Pair-splitting, pair-reaping and cardinal invariants of f Σ-ideals*. Journal of Symbolic Logic, 75(2):661–677, 2010.

[5] Michael Hrušák and Luis Sáenz. *Some applications of q-points and lebesgue filters to banach spaces*. Extracta Mathematicae, 2024.

[6] Vladimir Kadets, Varvara Shepelska, and Dirk Werner. *Thickness of the unit sphere, 1-types, and the almost daugavet property*. Houston journal of mathematics, 37:867–878, 01 2011.