

Q -measures, Q -points, and L -orthogonality.

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Workshop Structure in Banach Spaces
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Joint work with Antonio Avilés, Gonzalo Martínez-Cervantes, and Alejandro Poveda.

The plan

Banach Space questions

Special Ultrafilters and similar objects

Comments and Open questions

Some definitions

Definition

Let $(X, \|\cdot\|)$ be a Banach space

- ① $\|\cdot\|$ is **octahedral** if for any $x_0, \dots, x_{n-1} \in X$, $\epsilon > 0$, there is $y \in S_X$ such that $\|x_i + y\| > \|x_i\| + \|y\| - \epsilon$.

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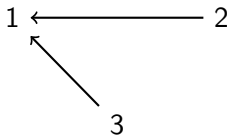
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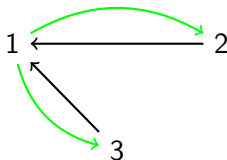


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Renorming results

Theorem (Godefroy (1989))

For a Banach space X , TFAE:

- *X contains a isomorphic copy of ℓ_1 .*
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Isometric results: L-orthogonal elements and sequences

Proposition (Avilés, Martínez-Cervantez, Rueda Zoca (2022))

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Spoiler

The answer to both questions is independent of the usual set theory axioms (ZFC).

Ultrafilters

Definition (Filters)

A family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is a **filter** over $\mathbb{N}$ if

- ① $\emptyset \notin \mathcal{F}$
 - ② If $F \in \mathcal{F}$, $F \subseteq E$ then $E \in \mathcal{F}$
 - ③ If $F, F' \in \mathcal{F}$, then $F \cap F' \in \mathcal{F}$
- A filter over $\mathbb{N}$ is **free** if it extends the Fréchet filter.
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Definition

A free ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ is a **Q-point** if for any partition of $\mathbb{N}$ into finite sets $(I_n)_{n \in \mathbb{N}}$, there is a selector $S \in \mathcal{U}$ such that for any $n \in \omega$, $|S \cap I_n| = 1$.

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The existence of Q-points is independent of ZFC.

Answering the second question.

Definition

Given a sequence $(x_n)_{n \in \mathbb{N}}$ in a topological space X , and filter $\mathcal{F}$ over $\mathbb{N}$, the $\mathcal{F}$ -limit with respect to the sequence is $x \in X$ ($x = \mathcal{F}\text{-}\lim x_n$) iff for every neighborhood V of x ,

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For any Banach Space X , $(x_n)_{n \in \mathbb{N}}$ an L -orthogonal sequence, and $\mathcal{U}$ a Q -point, $\mathcal{U}\text{-}\lim x_n \in X^{**}$ (taken in the w^* topology of X^{**}) is L -orthogonal.

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If there are no Q -points, then there is a Banach space X with an L -orthogonal sequence such that no $x^{**} \in \overline{\{x_n : n \in \mathbb{N}\}}^{w^*}$ is L -orthogonal.

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Theorem

Fix an ultrafilter $\mathcal{U}$. $\mathcal{U}$ is a Q -point iff for any Banach space X , and $(x_n)_{n \in \mathbb{N}}$ an L -orthogonal sequence, $\mathcal{U}\text{-}\lim x_n \in X^{**}$ is L -orthogonal.

Special Measures

Definition

Let $\mu : \mathcal{P}(\mathbb{N}) \rightarrow [0, +\infty)$ be a finitely additive measure defined on the subsets of $\mathbb{N}$ and vanishing on finite sets.

- ① μ is a **Q-measure** if every partition of $\mathbb{N}$ into finite sets has a selector of positive measure.
- ② μ is a **fit Q-measure** if for every partition of $\mathbb{N}$ into finite sets and every $\delta < \frac{1}{2}$ there is a finite union of selectors of measure greater than $\delta \cdot \mu(\mathbb{N})$.
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Definition

Given a sequence $(x_n)_{n \in \mathbb{N}}$ in a Banach space X , and a measure of bounded variation $\mu : \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}$. Let $T : \ell_1 \rightarrow X$ be defined as $T(e_n) := x_n$ for $n \in \mathbb{N}$. Define the μ -limit of $(x_n)_{n \in \mathbb{N}}$ as

$$\mu\text{-lim } x_n := T^{**}(\mu).$$

Here $T^{**} : \ell_1^{**} \rightarrow X^{**}$ and remember that $\ell_1^{**} = \ell_\infty^*$ is naturally identified with the set of finitely additive signed finite measures on $\mathcal{P}(\mathbb{N})$. In the case where μ stems from an ultrafilter $\mathcal{U}$ this coincides with the classical $\mathcal{U}$ -limit.

Answering the second question

Theorem

Let X be a Banach space, $(x_n)_{n \in \mathbb{N}}$ an L -orthogonal sequence, and $\mu: \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ a strong Q -measure with $\mu(\mathbb{N}) = 1$. Then $\mu\text{-}\lim x_n$ is an L -orthogonal element.

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Can you provide a characterization of when, for any measure μ , the $\mu\text{-}\lim x_n$ of an L -orthogonal sequence is L -orthogonal?

Comments on the proofs

Theorem (Mathias)

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, TFAE.

- ① $\mathcal{U}$ is selective.
- ② For any tall analytic ideal $\mathcal{I}$, $\mathcal{I} \cap \mathcal{U} \neq \emptyset$.

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Theorem (Hrušák, Meza, Minami (2010))

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, TFAE.

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Theorem

Let $\mu: \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ be a measure and $\varepsilon > 0$. TFAE.

- 1 μ is an ε -strong Q -measure.
- 2 For every analytic hereditary family $\mathcal{H}$ that is countably hitting, there is $A \in \mathcal{H}$, such that $\mu(A) \geq \varepsilon$.

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Theorem

TFAE

- ① $\mathfrak{d} = \text{cov}(\mathcal{M})$.
- ② Any filter $\mathcal{F}$ of character less than $\mathfrak{d}$ can be extended to a Q -point.
- ③ For any measure $\mu : \mathbb{B} \rightarrow [0, 1]$, such that $[\mathbb{N}]^{<\mathbb{N}} \subseteq \mathbb{B} \subseteq \mathcal{P}(\mathbb{N})$ is Boolean Algebra, μ vanishes on finite sets, and has density less than $\mathfrak{d}$, there exists an atomless strong Q -measure $\nu : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ extending μ .

Thanks!

References

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