A separable Banach space of nontrivial Baire order

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Introduction

We consider a separable Banach space X as a subspace of its bidual X^{**}.

Intrinsic Baire classes of X

We set $X_0^{**} = X$ and for a countable ordinal α we set

$$X_\alpha^{**} = \{x^{**} \in X^{**} : \text{there is a sequence } (x_n^{**}) \subseteq \bigcup_{\beta<\alpha} X_\beta^{**} \text{ that weak}^* \text{ converges to } x^{**}\}.$$

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- (Odell, Rosenthal '75)
  $X_1^{**} = X^{**}$ if and only if X does not contain ℓ_1.

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In general we need to distinguish between elements of X_α^{**} and elements of X^{**} that happen to be Baire-α.

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Answer

There is a separable Banach space of Baire order 2.

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The Azimi-Hagler space is like a "Baire-1" version of the James space.

$$\|x\|_{AH} = \sup \left\{ \sum_{j=1}^{n} \frac{1}{j} |\langle I_j, x \rangle| : \ (I_1, \ldots, I_n) \text{ sequence of successive intervals} \right\}$$

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Thanks for your attention.