Horofunction extension of metric spaces and Banach spaces

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Structures in Banach spaces.

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Compactifications

Let (X, τ) be a Hausdorff topological space.

Do there exist a compact space (Y, σ) and a homeomorphism $h : X \to h(X) \subset Y$ such that $h(X)$ is dense in Y.

- (X, τ) is locally compact $\Rightarrow$ Alexandrov compactification $X \cup \{\infty\}$.
- (X, τ) is completely regular $\Rightarrow$ Stone-Čech compactification βX.
- In $\mathbb{R}$ we also have $\mathbb{R} \cup \{-\infty, +\infty\}$.

Stone-Čech compactification

- $C_b(X)$: the space of bounded continuous functions from X to $\mathbb{R}$.

$$x \in X \hookrightarrow \Phi(x) := (f(x))_{f \in C_b(X)} \in \prod_{f \in C_b(X)} [\inf f, \sup f].$$

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- $\beta X = \overline{\Phi(X)}$.
- Each point $x \in X$ is seen as an evaluation map, $\Phi : X \to \mathbb{R}^{C_b(X)}$.
- Each function $f \in C_b(X)$ is a coordinate of the space βX.
- The space $C_b(X)$ is used as a pivot to define βX.

Horofunction extension of (X, d)

Equip $C(X)$ with the compact-open topology.
Gromov[1] proposed the following construction:

$$x \in X \hookrightarrow \iota_x(\cdot) := d(\cdot, x) \in C(X)$$

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Definition

The horofunction extension of (X, d) corresponds to

$$\overline{X}^h := \overline{\widehat{\iota}(X)} \subset C(X)/\{const\}.$$

Remark: $\widehat{\iota}$ is continuous and injective.

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A Lipschitz point of view.[2]

Fix $b \in X$ (base point).
$\mathrm{Lip}_b(X)$: Space of real-valued 1-Lipschitz functions that vanish at b.
Equip $\mathrm{Lip}_b(X)$ with the pointwise topology.

$$\mathrm{Lip}_b(X) \subset \prod_{x \in X} [-d(x,b), d(x,b)].$$

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Define

$$x \in X \hookrightarrow h_x(\cdot) := d(\cdot, x) - d(b, x) \in Lip_b^1(X).$$

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- h is continuous and injective.
- The horofunction extension $\overline{X}^h$ is homeomorphic to $\overline{h(X)}$.
- The horofunction extension of X is homeomorphic to the one of its completion.
- $\overline{X}^h$ is compact.

[2] A. Gutiérrez. Metric Compactification of Banach Spaces. Doctoral Dissertation, Aalto University (Finland) 2019.

Applications and related works

- Description of the horofunction boundary $\partial X := \overline{X}^h \setminus h(X)$:
  Finite dimensional spaces, Hilbert geometry (C. Walsh 2007, 2008), ℓ_p and L^p spaces (A. Gutierrez 2019, 2020).
- Dynamics of nonexpansive operators (S. Gaubert, G. Vigeral 2011)
- Fix point theory for nonexpansive operators (A. Karlsson 2024).
- And more...

Some examples

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Known examples:

- Finite dimensional spaces. For instance, $\overline{(\mathbb{R}^n, \|\cdot\|_2)}^h = \overline{B}(0,1)$. Indeed, for $x \in \mathbb{R}^d \setminus \{0\}$ and $y \in \mathbb{R}^d$, $h_{tx}(y) \underset{t\to\infty}{\longrightarrow} \langle -\frac{x}{\|x\|}, y\rangle$.

  So, $\partial\mathbb{R}^n := \overline{\mathbb{R}^n}^h \setminus h(\mathbb{R}^n) = \{\langle -x, \cdot\rangle : \ x \in S_{\mathbb{R}^n}\}$.
- Hilbert spaces.
- The sphere of an infinite dimensional Hilbert space. Moreover, $\overline{S_{\mathcal{H}}}^h = (\overline{B}(0,1), \omega)$.

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Proposition (N. Fisher, S. Nicolussi Golo 2021)

Let X be a proper metric space such that every ball is path connected. Then, h is an homeomorphism.

h is not always bicontinuous

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$$\begin{aligned} h_{e_n}(e_m) &= d(e_m, e_n) - d(0, e_n) \\ &= \|e_n - e_m\| - \|e_n\| \underset{n\to\infty}{\longrightarrow} 1, \quad \text{for all } m \in \mathbb{N}. \end{aligned}$$

Therefore, $h_{e_n} \to \|\cdot\| = h_0$ pointwise on X.
So, h is not a bicontinuous and $\partial X = \emptyset$.
A similar proof shows that h is not bicontinuous for $X = \ell_1$.

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Question

Characterize those metric spaces such that h is bicontinuous, or equivalently, when $\overline{X}^h$ is a compactification of X.

Main characterization: Metric spaces

Theorem (Daniilidis, Garrido, Jaramillo, T. 2024)

Let (X, d) be a metric space. TFSAE:

- *The horofunction extension $\overline{X}^h$ is a compactification of X.*
- *For every point $x \in X$ and every $r > 0$, there exist $\eta_r > 0$ and a compact set $K_r \subset X$ such that, for each $z \in X \setminus \overline{B}(x, r)$ there exists $w \in K_r$ satisfying*

$$d(w, z) \leq d(w, x) + d(x, z) - \eta_r.$$

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Application: unit sphere of any normed space $(S_X, d_{\|\cdot\|})$.

- Fix $x \in S_x$ and $r > 0$. Set $K = \{-x\}$ and $\eta_r = r$. Then

$$d(-x, z) \leq 2 = d(-x, x) + r - r \leq d(-x, x) + d(x, z) - \eta_r.$$

Main characterization: Normed spaces

Theorem (Daniilidis, Garrido, Jaramillo, T. 2024)

Let $(X, \|\cdot\|)$ be a normed space. TFSAE:

(a) *The horofunction extension $\overline{X}^h$ is a compactification of X.*

(b) *There is a finite dimensional subspace $F \subset X$ such that*

$$d_H(S_F, S_X) < 2,$$

where S_F and S_X denote the unit spheres of F and X respectively.

[3] G. Godefroy, Metric characterization of first Baire class linear forms and octahedral norms, Studia Math. **95** (1989), no. 1, 1–15.

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(c) *$(X, \|\cdot\|)$ is not an octahedral space.*[3]

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Some examples.

Theorem (G. Godefroy (1989))

Let $(X, \|\cdot\|)$ *be a Banach space. TFSAE:* (i). *X contains an isomorphic copy of* ℓ^1. (ii). *X admits an equivalent octahedral norm.*

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Theorem (G. Godefroy (1989))

Let $(X, \|\cdot\|)$ be a Banach space. TFSAE: (i). *X contains an isomorphic copy of ℓ^1.* (ii). *X admits an equivalent octahedral norm.*

Consequences: h is bicontinuous if

- X does not contain ℓ^1 (Reflexive spaces, Asplund, ...).
- $X = Y \oplus_p Z$, with $p \in (1, +\infty)$.[4]
- $X = Y \oplus_\infty Z$, if Y is finite dimensional.

[4] R. Haller, J. Langemets and M. Põldvere, On duality of diameter 2 properties, J. Convex Anal. **22** (2015), 465–483.

Further consequences

Corollary

Every normed space $(X, \|\cdot\|)$ admits an equivalent norm $|||\cdot|||$ such that $\overline{(X, |||\cdot|||)}^h$ is a compactification of $(X, |||\cdot|||)$.

Proof: Consider $X = Y \oplus \mathbb{R}\overline{x}$. Consider $(X, |||\cdot|||) = Y \oplus_2 \mathbb{R}\overline{x}$.

[5] A. Procházka and A. Rueda Zoca, A characterisation of octahedrality in Lipschitz-free spaces, Ann. Inst. Fourier (Grenoble) **68** (2018), no. 2, 569–588.

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Proof: Consider $X = Y \oplus \mathbb{R}\overline{x}$. Consider $(X, |||\cdot|||) = Y \oplus_2 \mathbb{R}\overline{x}$.
Denote by $\mathcal{F}(X)$ the Lipschitz-free space[5] and by $\mathcal{P}^1(X)$ the 1-Wasserstein space of (X, d).

Proposition

Let (X, d) be a metric space. Then

(i). $\overline{X}^h$ *is a compactification of X if $\overline{\mathcal{F}(X)}^h$ is a compactification of $\mathcal{F}(X)$.*

(ii). $\overline{X}^h$ *is a compactification of X if $\overline{\mathcal{P}^1(X)}^h$ is a compactification of $\mathcal{P}^1(X)$.*

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Thank you for your attention.