True σ-Porisity for Alternating Projeciton Orders

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Problem (Convex Feasibility Problem)

$$C_1, C_2 \subseteq H \quad \textit{closed, convex subsets of Hilbert space}$$

Goal: *Find* $p \in C_1 \cap C_2 \neq \emptyset$

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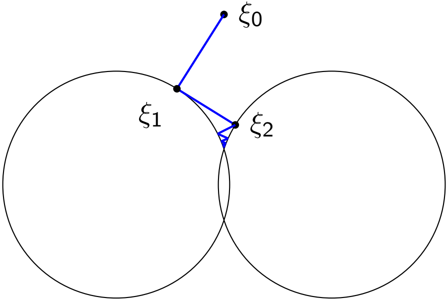
P_1 and P_2 nearest point projections onto C_1 and C_2.



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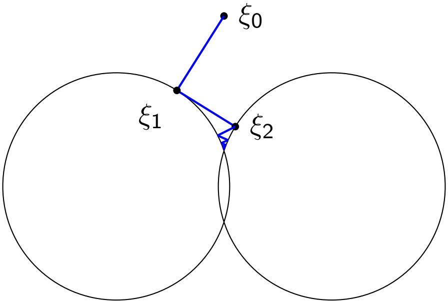
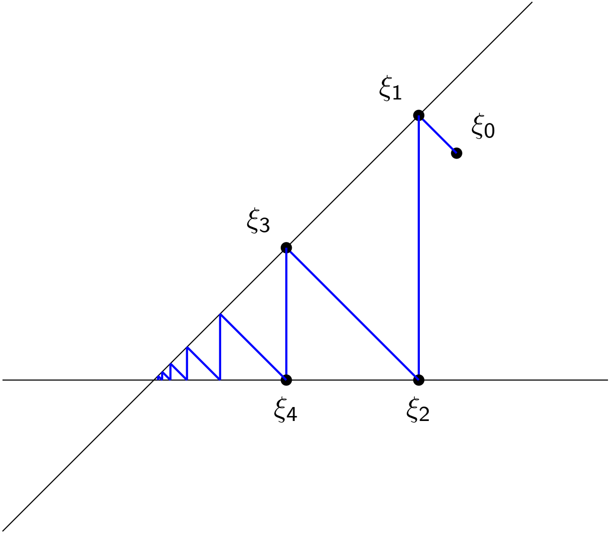
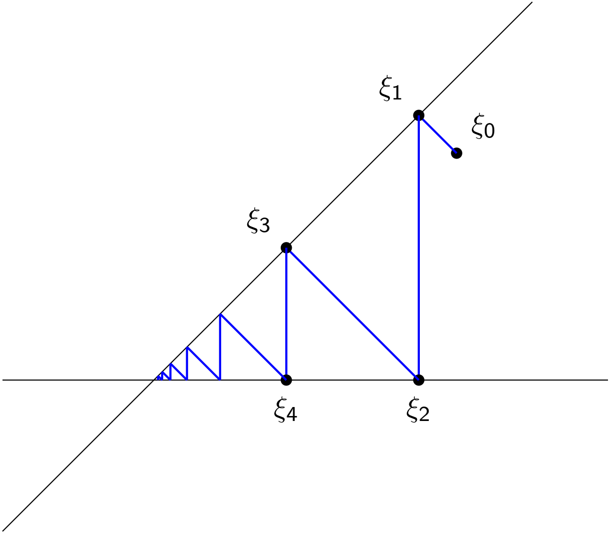


Hope that $(\xi_n)_{n\in\mathbb{N}}$ converges to some $p \in C_1 \cap C_2$.

Does it work?



ξ_1
ξ_0
ξ_3
ξ_4
ξ_2



Theorem (von Neumann, '49)

C_1, C_2 linear subspaces $\implies$ alternating projections work.
In fact: $\lim_{n\to\infty} \xi_n = P_{C_1 \cap C_2}(\xi_0)$

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x periodic $\Longrightarrow$ alternating projections work.
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Only periodic?

Definition (quasi-periodic)

A sequence $x \in \{1, \ldots, N\}^{\mathbb{N}}$ is **quasi-periodic** iff
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Idea: Don't let occurrences spread out too much.

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Theorem (Sakai, '95)

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More than periodic?

All reasonable projection orders?

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Theorem (Kopecká, Müller, Paszkiewicz, '14, '17)

H infinite-dimensional
$\exists$ special choice of C_1, C_2, C_3 linear subspaces such that
$\forall 0 \neq \xi_0 \in H$ $\exists$ projection order x that
leads to a non-convergent projection series ξ_n.

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$I := \{1, \ldots, N\}$
$K := I^{\mathbb{N}}$

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Definition (Measure on K)

Equip $I = \{1, \ldots, N\}$ with Bernoulli measure

$$\mathbb{P}_I(\{1\}) = \cdots = \mathbb{P}_I(\{N\}) = \frac{1}{N}$$

and $K = I^{\mathbb{N}}$ with the infinite product measure $\mathbb{P}$ of $\mathbb{P}_I$.

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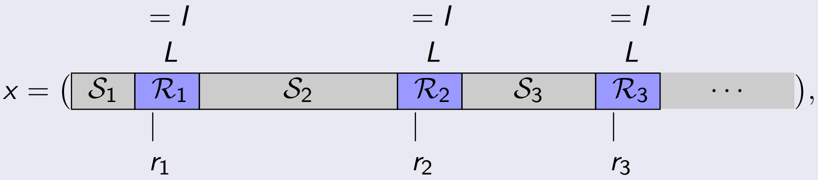
Theorem (Melo, da Cruz Neto, de Brito, '22)

$\mathbb{P}$-almost all orders $x \in K$ lead to $(\xi_n)_{n\in\mathbb{N}}$ being strongly convergent (under some constraints).

Definition (Greedy L-partition)



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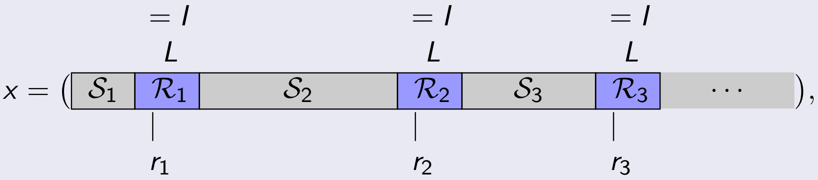
$(r_k)_{k\in\mathbb{N}}$ greedy L-partition: Choose blocks $\mathcal{R}_k$ as far left as possible.



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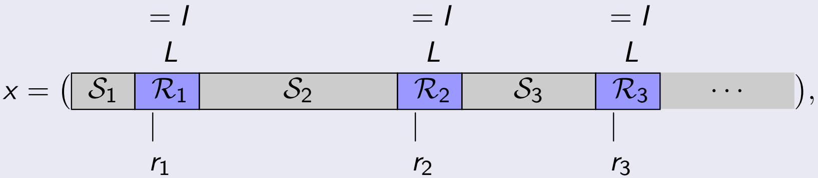

Definition (quasi-normal sequences)

$x \in K$ quasinormal $\iff$

$$\exists L \geq N : \bigwedge \begin{cases} \text{greedy } L\text{-partition } (r_k)_{k\in\mathbb{N}} \text{ exists} \\ \sum_{k\in\mathbb{N}} \frac{1}{r_k} = \infty \end{cases}$$

Theorem (Melo, da Cruz Neto, de Brito, '22)

$$\left.\begin{array}{ll}\text{(i)} & x \text{ quasi-normal} \\ \text{(ii)} & (\xi_n)_{n\in\mathbb{N}} \text{ has accumulation point}\end{array}\right\} \bigcirc\!\!\!\!\wedge \implies (\xi_n)_{n\in\mathbb{N}} \text{ converges}$$

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Guaranteed if:

- in Hilbert space
- one C_j compact, j in x infinitely often
- Hadamard manifold

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Definition (Metric on K)

On I choose discrete metric d_0.
On K choose

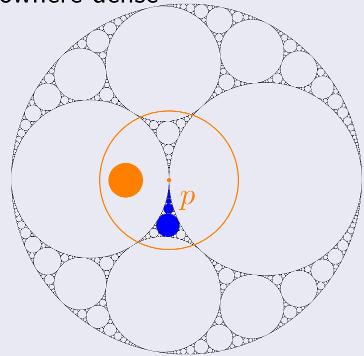
$$\begin{aligned} d(x,y) &:= \max\{2^{-j} d_0(x_j, y_j) : j \in \mathbb{N}\} \\ &= 2^{-(\text{first index where } x_j \neq y_j)}. \end{aligned}$$

Note that

$$B(x, 2^{-j}) = \{(x_1, \ldots, x_j, ?, ?, ?, ?, \ldots)\}.$$

Definition ((ϕ-)porosity)

Metric version of nowhere dense



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ϕ-**porous**	holes scale to given function
σ-(ϕ-)**porous**	countable union of ($\phi-$)porous sets.
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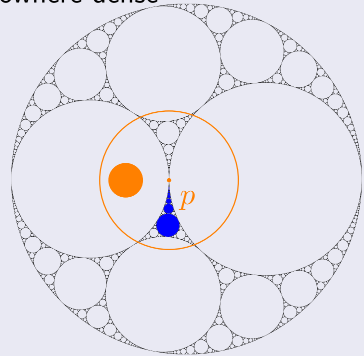

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Complement is large, **co-($\cdots$)-porous** $\implies$ dense G_δ.

How large is the set of sequences $x \in K$ for which $(\xi_n)_{n\in\mathbb{N}}$ is strongly convergent?

(in a metric sense)

Periodic sequences?

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Proposition (T., '23)

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$\mathcal{N} \subseteq (K, \mathcal{T})$ contains a co-σ-ϕ-porous set

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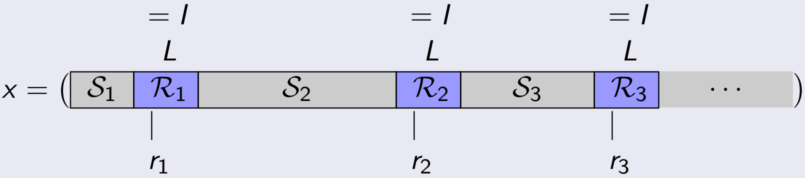
$$\mathcal{N}$$

Lemma

$$\sum_{k\in\mathbb{N}} \frac{1}{r_k} = \infty \iff \forall s \in \mathbb{N}\colon \sum_{k=s}^{\infty} \frac{1}{r_k - (r_s + 1)} = \infty$$



Definition

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$$\exists L \geq N\colon \bigcirc\!\!\!\!\wedge \begin{cases} \sum_{k\in\mathbb{N}} \frac{1}{r_k} = \infty & \text{(a)} \\ \text{greedy } L\text{-partition } (r_k)_{k\in\mathbb{N}} \text{ exists} & \text{(b)} \end{cases}$$

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Show that complement $K \setminus \mathcal{N}$ is small, σ-ϕ-porous.

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$$B_L = \{\nexists \text{ greedy } L\text{-part.}\}$$

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$$\exists L \geq N\colon \bigwedge \begin{cases} \sum_{k\in\mathbb{N}} \frac{1}{r_k} = \infty & \text{(a)} \\ \text{greedy } L\text{-partition } (r_k)_{k\in\mathbb{N}} \text{ exists} & \text{(b)} \end{cases}$$

Show that complement $K \setminus \mathcal{N}$ is small, σ-ϕ-porous.

$$x \notin \mathcal{N} \iff \forall L \geq N\colon \neg(a) \vee \neg(b)$$

$$x \in K \setminus \mathcal{N} \iff x \in \bigcap_{L\geq N} (\underbrace{A_L \cup B_L}_{\text{small}})$$

$$A_L = \{\sum \frac{1}{r_k} < \infty\} = \bigcup_{M\in\mathbb{N}} \{\sum \frac{1}{r_k} < M\}$$

$$(x_1, x_2, x_3, x_4, x_5, \underbrace{\mathcal{R}, \mathcal{R}, \mathcal{R}, \mathcal{R}, \mathcal{R}, \dots, \mathcal{R}}_{\text{enough to make } \sum > M}, \dots)$$

$$B_L = \{\nexists \text{ greedy } L\text{-part.}\} = \bigcup_{k\in\mathbb{N}} \{\text{g. L-p. only up to block } \mathcal{R}_k\}$$

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$$y = (x_1, x_2, x_3, x_4, x_5, x_6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots)$$

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Close: $= 2^{-6}$.

$$y = (x_1, x_2, x_3, x_4, x_5, x_6, \overbrace{1, 1, 1, 1, 1, 1, 1}^{m}, 1, 1, 1, 1, 1, 1, 1, 1, \ldots)$$

$$B(y, 2^{-s}) = (\underbrace{x_1, x_2, x_3, x_4, x_5, x_6, \overbrace{1, 1, 1, 1, 1, 1, 1}^{m}}_{s}, 1, 1, \ldots) \cap Q_m = \emptyset$$

□