Compact sets of Baire class one functions in Banach space geometry

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Vienna, March 17, 2025

Rosenthal's ℓ_1 theorem

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Theorem (Rosenthal 1977)

Every sequence (f_n) of pointwise bounded continuous real functions on a Polish (or compact) space X contains a subsequence which is either pointwise convergent on X or has closure in $\mathbb{R}^X$ homeomorphic to $\beta\mathbb{N}$.

Remark

Compact subsets of the class $\mathcal{B}_1(X)$ of Baire-class-1 functions on X are called **Rosenthal compacta**.

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Theorem (Bourgain-Fremlin-Talagrand 1978)

Rosenthal compacta over Polish (or compact perfect) spaces are Fréchet.

Double dual characterizations of $\ell_1 \hookrightarrow X$

Theorem (Odell-Rosenthal 1975; Saab-Saab 1983)

A Banach space X contains a subspace isomorphic to ℓ_1 if and only if there is x^{**} in X^{**} and a weak compact subset K of X^* such that $x^{**} \restriction K$ has no point of continuity.*

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Corollary (Odell-Rosenthal 1975)

A separable Banach space X contains no ℓ_1 if and only if the double dual unit ball is a Rosenthal compactum over the dual unit ball with the weak topology.*

Metric characterizations

Let X be a Banach space and let y^{**} in X^{**}. The **set of smoothness** $\Omega(y^{**}, \|\cdot\|)$ is the set of all $x \in X \setminus \{0\}$ such that

$$\lim_{\lambda \to 0} (\|x + \lambda y^{**}\| + \|x - \lambda y^{**}\| - 2\|x\|) = 0.$$

Thus, $\Omega(y^{**}, \|\cdot\|)$ is the collection of nonzero vectors of X at which the bidual norm is differentiable in the direction y^{**}.

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1. *y^{**} has a point of continuity on every weak* compact subset of X^*.*
2. *For every equivalent norm $\|\cdot\|$ on X the set $\Omega(y^{**}, \|\cdot\|)$ is a dense G_δ subset of X.*

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$$\|x + y^{**}\|' = \|x\|' + \|y^{**}\|'.$$

Locally uniformly convex norms

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If K is a separable Rosenthal compact then $\mathcal{C}(K)$ admits a locally uniformly convex renorming.

Theorem (Haydon-Molto-Orihuela 2007)

If K is a separable Rosenthal compactum of functions on a Polish space with only countably many points of discontinuities then $\mathcal{C}(K)$ admits a pointwise lower semicontinuous locally uniformly convex renorming.

Namioka property

We say that a compactum K has the **Namiaka property** if for every Baire space B and a separately continuous function

$$f : X \times B \to \mathbb{R}$$

there is a comeager set G of B such that f is continuous on $X \times G$.

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This property is closed under products and continuous images, so in particular every metric compactum has the Namioka property. Its relevance to Haydon-Molto-Orihuela problem is supported by the following.

Theorem (Deville-Godefroy 1993)

If $\mathcal{C}(K)$ admits a locally uniformly convex equivalent norm that is pointwise lower semicontinuous, then K has the Namioka property

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Theorem (T., 2005)

There is a scattered Rosenthal compactum K without the Namioka property and therefore $\mathcal{C}(K)$ does not have an equivalent pointwise lower semicontinuous locally uniformly convex renorming.

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Let $1_{[t]} : 2^{\mathbb{Q}} \to 2$ be the characteristic function of $[t]$ on $2^{\mathbb{Q}}$, i.e.

$$1_{[t]}(x) = 1 \text{ iff } t \sqsubseteq x.$$

Note that $1_{[t]}$ is a Baire-class-1 function on $2^{\mathbb{Q}}$ and that $1_{[t]} = \delta_t$ iff $\sup(t) = \infty$.

Let $w\mathbb{Q}$ be the collection of all subsets of $\mathbb{Q}$ that are well-ordered under the induced ordering from $\mathbb{Q}$.
We consider $w\mathbb{Q}$ as a **tree** under the ordering $\sqsubseteq$ as well as **tree-space** equipped with the **locally compact topology** τ_{in} generated by subbasic clopen sets of the form

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Lemma

The set $K_{w\mathbb{Q}}$ is a relatively compact subset of $\mathcal{B}_1(2^{\mathbb{Q}})$ with only the constant mapping $\bar{0}$ as its proper accumulation point.

Lemma

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Proposition

Suppose T *is a Hausdorff tree of cardinality at most continuum which admits a strictly increasing mapping into* $\mathbb{R}$. *Then* (T, τ_{in}) *is homeomorphic to a subspace of* $(w\mathbb{Q}, \tau_{\mathrm{in}})$.

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Corollary

Every Hausdorff tree-space (T, τ_{in}) where T is of cardinality at most continuum admitting a strictly increasing map to $\mathbb{R}$ has a scattered compactification representable as a compact subset of $\mathcal{B}_1(2^{\mathbb{N}})$.

Separate versus joint continuity

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$$\{x \in \sigma\mathbb{Q} : t \sqsubseteq x\} \text{ and } \{x \in \sigma\mathbb{Q} : \sup(x) < q\},$$

where $t \in \sigma\mathbb{Q}$ and $q \in \mathbb{Q}$.

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Lemma

The mapping f is separately continuous but not continuous on any set of the form $G \times (w\mathbb{Q} \cup \{\infty\})$ for G is a comeager subset of $\sigma\mathbb{Q}$.

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Question

Does $(w\mathbb{Q}, \tau_{\rm in})$ (or $(\sigma\mathbb{Q}, \tau_{\rm in})$) admit a separable Rosenthal compactification?

Universally Baire subtrees of $w\mathbb{Q}$

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A subset A of a topological space X is a **universally Baire** subset of X if for every topological space Y (or equivalently, for every Baire space Y) and every continuous mapping $f : Y \to X$ the preimage $f^{-1}[A]$ has the **property of Baire** as a subset of Y.

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Lemma

If there is an uncountable co-analytic set of reals of cardinality smaller than $\mathfrak{p}$ then any subtree of $w\mathbb{Q}$ of cardinality $\aleph_1$ is universally Baire.

Spacial trees and σ-fragmentability

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Recall that a topological space (X, τ) is σ-**fragmented by a metric** ρ on X if for every $\epsilon > 0$ there is a decomposition $X = \bigcup_{n=0}^{\infty} X_n^{\epsilon}$ such that for every n and $A \subseteq X_n^{\epsilon}$ there is $U \in \tau$ such that $U \cap A \neq \emptyset$ and ρ-diam$(U \cap A) < \epsilon$.

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Corollary

If a subtree T of $w\mathbb{Q}$ is not spacial then $\mathcal{C}_0(T)$ admits no locally uniformly convex renorming.

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Fix a C-sequence C_α $(\alpha < \omega_1)$ such that $C_{\alpha+1} = \{\alpha\}$ and $C_\alpha \subseteq \alpha$ such that $\mathrm{otp}(C_\alpha) = \omega$ and $\sup(C_\alpha) = \alpha$ for limit α.

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Which choice of C_α $(\alpha < \omega_1)$ guarantees that $T(\rho_1)$ is universally Baire?

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Cohen generic choice of the C-sequence

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Let $C_\alpha(0) = 0$ and for $0 < n < \omega$, land et $C_\alpha(n)$ denote the n'th element of C_α according to its increasing enumeration with the convention that $C_{\alpha+1}(n) = \alpha$ for all $n > 0$.

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Fix also sequence $e_\alpha : \alpha \to \omega$ $(\alpha < \omega_1)$ of one-to-one mappings such that

$$\{\xi < \min\{\alpha, \beta\} : e_\alpha(\xi) \neq e_\beta(\xi)\}$$

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is finite for all α and β.
For each $r \in ([\omega]^{<\omega})^\omega$, we associate another sequence C^r_α $(\alpha < \omega_1)$ by letting for limit α, $C^r_\alpha = \bigcup_{n\in\omega} D^r_\alpha(n)$, where

$$D^r_\alpha(n) = \{\xi \in [C_\alpha(n), C_\alpha(n+1)) : e_\alpha(\xi) \in r(n)\}.$$

For $r \in ([\omega]^{<\omega})^\omega$, using C_α^r ($\alpha < \omega_1$), as before, we recursively define $\rho_1^r : [\omega_1]^2 \to \omega$ as follows:

$$\rho_1^r(\alpha, \beta) = \max\{\rho_1^r(\alpha, \min(C_\beta^r \setminus \alpha)), |C_\beta^r \cap \alpha|\}.$$

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Lemma

If r is a Cohen real then $T(\rho^r_1)$ is not special.

Theorem (T., 2005)

If there is an uncountable co-analytic set of cardinality $< \mathfrak{p}$ and if r is a Cohen real, then $T(\rho_1^r)$ admits a Rosenthal compactification K which has the Namioka property but the corresponding function space $\mathcal{C}(K)$ is not σ-fragmented by the norm.

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Theorem (Haydon 1990)

The topology $\tau_p$of pointwise convergence of $\mathcal{C}(K)$ for K a scattered compactum is σ-fragmented by the norm if and only if the restriction of τ_p to the function subspace $\mathcal{C}(K,2)$ of $\{0,1\}$-valued continuous maps on K is σ-scattered.

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Theorem (T., 2007)

If there is a compact cardinal and if T *is a subtree of* $w\mathbb{Q}$ *whose closure* K *in* $w\mathbb{Q}$ *satisfies the Namioka principle then the topology of pointwise convergence of function space* $\mathcal{C}(K)$ *is* σ*-fragmented by the norm.*

Problem (T., 2007)

Show that in the presence of sufficiently many compact cardinals the Namioka generic continuity principle captures norm-σ-fragmentability of the topology of pointwise convergence in function spaces of the form $\mathcal{C}(K)$.

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Question

Given sufficiently many compact cardinals, can the Namioka property of a compactum K guarantee locally uniformly convex renorming of $\mathcal{C}(K)$?

G_δ points in Rosenthal compacta

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Theorem (Bourgain 1978)

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Theorem (T., 1999)

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Non G_δ points and the Separable Quotient Problem

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The Cantor tree is the tree

$$P = (2^{\leq\omega}, \sqsubseteq)$$

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Theorem (T., 1999)

If x is a non G_δ-point in a Rosenthal compactum K then there is a homeomorphic embedding

$$f : P \cup \{\infty\} \to K$$

such that $f(\infty) = x$.

Corollary

Suppose that a separable Banach space X does not contain ℓ_1 but it has a non separable dual X^*. Then there is a homeomorphic embedding

$$f : P \cup \{\infty\} \to (X^{**}, w^{**})$$

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Lemma (Argyros-Dodos-Kannelopoulos 2008)

Given X and f as above, for every positive integer n, the set UNC_n,

$$\{(x_1, ..., x_n) \in (\{0,1\}^\omega)^{[n]} : \{f(x_1), ..., f(x_n)\} \textit{ is } 1\textit{-unconditional in } X^{**}\}$$

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Remark

By Mycielski's theorem there is a perfect set $P \subseteq \{0,1\}^\omega$ such that $[P]^n \subseteq UNC_n$ for all $n < \omega$. Hence $\{f(x) : x \in P\}$ is a 1-unconditional sequence in X^{**}.

The Separable Quotient Problem

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Rosenthal compacta close to metrizable

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1. K *is hereditarily separable.*
2. K *is hereditarily Lindelöf.*
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Theorem (T., 1999)

If a Rosenthal compacum has no uncountable discrete subspace then there is a compact metric space M and a continuous map $f : K \to M$ such that $|f^{-1}(x)| \leq 2$ for all $x \in M$.

Corollary

Every Rosenthal compactum K with no uncountable discrete subspace has the Namioka property.

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Proof.

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Then g is separatelly continuous and since M has the Namioka property there is a dense G_δ subset G of B such that g is continuous on $M \times G$. Then h is continuous on $K \times G$. $\square$

Problem

Let K be a Rosenthal compactum with no uncountable discrete subspace. Show that $\mathcal{C}(K)$ has admits a pointwise lower semicontinuous locally uniformly convex renorming.

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Question

Is there a fine structure theory of compact sets of Baire class one functions on K-analytic spaces analogous to that of the class of Baire class one functions on analytic (Polish) spaces?

Rosenthal and Bourgain-Fremlin-Talagrand dichotomies on non Polish spaces

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Example

Let $X \subseteq [\omega]^\omega$ be a fixed splitting family. For $n < \omega$, let $p_n : X \to \{0, 1\}$ be defined by,

$$p_n(x) = 1 \text{ if and only if } n \in x.$$

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Example

Let $X \subseteq [\omega]^\omega$ be a fixed infinite maximal almost disjoint family. For $n < \omega$, let $p_n : X \to \{0, 1\}$ be defined by

$$p_n(x) = 1 \text{ if and only if } n \in x.$$

Let $\bar{0}$ be the constantly equal 0 function.
Then $\bar{0}$ is the pointwise closure of $\{p_n : n < \omega\}$ on X but no subsequence of (p_n) pointwise converges to $\bar{0}$.

The function space $\mathbb{R}^X \cap L(\mathbb{R})$

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Let $L(\mathbb{R})$ be the constructible closure of the reals.

The existence of large cardinal is assumed that make every selective ultrafilter on ω generic over $L(\mathbb{R})$ and in particular that $L(\mathbb{R})$ is a **Solovay model** in which all sets of reals are Lebesgue measurable and have the property of Baire.

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We shall see that Rosenthal and Bourgain-Fremlin-Talagrand dichotomies **do hold** even in the larger function space $\mathbb{R}^X \cap L(\mathbb{R})$ rather than $\mathcal{B}_1(X)$ and with X not necessarily Polish but rather than just a separable metric space belonging to $L(\mathbb{R})$.

Theorem (Horowicz-T., 2019)

Let X be a separable metric space in $L(\mathbb{R})$.Let (f_n) be a sequence of pointwise bounded continuous functions on X. Then either

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Corollary

For every $\mathcal{S} \subseteq [\omega]^\omega$ be a splitting family in $L(\mathbb{R})$ there is a $B \in [\omega]^\omega$ such that

$$\mathcal{S} \restriction B = \mathcal{P}(B),$$

where $\mathcal{S} \restriction B = \{A \cap B : A \in \mathcal{S}\}$.

Theorem (Horowicz-T., 2019)

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Question

Let $\mathcal{U}$ be a selective ultrafilter on ω. How much of this can be transferred to the larger model $L(\mathbb{R})[\mathcal{U}]$?

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Remark

It can be shown that there are no infinite maximal almost disjoint families in $L(\mathbb{R})[\mathcal{U}]$ so in this model there are no counterexamples to the Bourgain-Fremlin-Talagrand dichotomy as in Example 2.

THANK YOU!