# On the extension of compact operators


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## Preliminaries

In this work we deal mostly with **categories**, **functors** and **diagrams**. As a summary, we have used homological techniques, applied to functional analysis, to obtain some results about the existence of extension of compact operators.

## Introduction

We establish $\mathbf{Ban}$ as the category of Banach spaces and (linear, continuous) operators, and $\boldsymbol{V}$ as the category of vector spaces and linear maps. Categories are related by **functors**. In this work, we focus on the functor $\mathcal{K}(-,E)$ associated to compact operators, which will be introduced in the next section. The basic diagrams we shall work with are **short exact sequences** in $\mathbf{Ban}$, that is, diagrams of the form

$$0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0 \qquad (\mathrm{x})$$

in which the kernel of every operator agrees with the image of the preceding. This amounts to saying that $Y$ is a subspace of $X$ and $Z$ is isomorphic to the quotient $X/j(Y)$.

- Equivalence of short exact sequences: if there exists $u : X_1 \to X_2$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow u & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

- Triviality: A short exact sequence is said to be **trivial** if it is equivalent to

$$0 \longrightarrow Y \xrightarrow{j_1} Y \times_\infty Z \xrightarrow{q_2} Z \longrightarrow 0$$

$$\mathrm{Ext}(Z,Y) = \frac{\{\ 0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0\ \}}{\equiv}$$

- $[\mathrm{x}]$ = class of all short exact sequences equivalent to $(\mathrm{x})$.
- $\mathrm{Ext}(Z,Y) = 0 \Rightarrow$ there are only trivial short exact sequences.

Proposition 1: **Every element of $\mathrm{Ext}(Z,Y)$ fits in a diagram of the form:**

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi & \xrightarrow{\rho} & \ell_1(I) & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

The upper row of the previous diagram is called *projective presentation of $Z$*, and the lower row is trivial precisely when $T$ admits an extension to $\ell_1(I)$. Hence the following identification holds:

$$\mathrm{Ext}(Z,Y) = \frac{\mathcal{L}(\ker\pi, Y)}{\equiv}, \qquad [T] = 0 \iff \text{there exists } \hat{T} : \ell_1(I) \to Y \text{ such that } T = \hat{T}\rho \qquad (1)$$

## The contravariant compact-operator functor

We consider the natural functor associated to the ideal $\mathcal{K}$ of compact operators:

$$\begin{aligned} \mathcal{K}(-,E) : \mathsf{Ban} &\rightsquigarrow \mathsf{V} \\ X &\longmapsto \mathcal{K}(X,E) \\ [T : X \to Y] &\longmapsto \mathcal{K}(T,E) = T^* : \mathcal{K}(Y,E) \to \mathcal{K}(X,E) \end{aligned}$$

How does $\mathcal{K}(-,E)$ behave on short exact sequences?

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ \mathcal{K}(-,E) \Downarrow & & & & & & & & \\ 0 & \longrightarrow & \mathcal{K}(Z,E) & \xrightarrow{q^*} & \mathcal{K}(X,E) & \xrightarrow{j^*} & \mathcal{K}(Y,E) & & \end{array}$$

✓ $q^*$ is injective, and $\mathrm{Im}\, q^* = \ker j^* \Rightarrow \mathcal{K}(-,E)$ is left-exact.

✗ In general, $j^*$ is not a surjective $\Rightarrow \mathcal{K}(-,E)$ is not right-exact.

Proposition 2: **The functor $\mathcal{K}(-,E)$ is exact if and only if $E$ is an $\mathcal{L}_\infty$-space.**

Proof: The map $j^* : \mathcal{K}(X,E) \to \mathcal{K}(Y,E)$ is surjective if, and only if, for each compact operator $K : Y \to E$ we can construct the following diagram in which $\hat{K} : X \to E$ has to be also compact:

$$\begin{array}{ccc} Y & \xhookrightarrow{j} & X \\ \downarrow K & \swarrow \hat{K} & \\ E & & \end{array}$$

By [6, Th. 4.1], this is equivalent to the fact that $E$ is an $\mathcal{L}_\infty$-space. □

## Compact short exact sequences

Since, in general, the functor $\mathcal{K}(-,E)$ is not right-exact, we can compute its right-derived functors using homological techniques. In particular, we are interested in the first right-derived functor:

$$\mathcal{R}^1\mathcal{K}(-,E)(X) = \mathrm{Ext}_{\mathcal{K}}(-,E)(X) = \mathrm{Ext}_{\mathcal{K}}(X,E)$$

But… who is $\mathrm{Ext}_{\mathcal{K}}(X,E)$?

- Short exact sequences fitting in a diagram of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi & \xrightarrow{\rho} & \ell_1(I) & \xrightarrow{\pi} & X & \longrightarrow & 0 \\ & & \downarrow K & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & \square & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

  where $K : \ker\pi \to E$ is a compact operator.
- Equivalence relation: $[\square]_{\mathcal{K}} = [\square']_{\mathcal{K}} \iff$ there exists $\widehat{K} \in \mathcal{K}(\ell_1(I), E)$ such that $\widehat{K}\rho = K_1 - K_2'$.
- $\mathcal{K}$-trivial element: $[\square]_{\mathcal{K}} = 0 \iff$ there exists $\widehat{K} \in \mathcal{K}(\ell_1(I), E)$ such that $\widehat{K}\rho = K$.

## Are the equivalence relations defined on $\mathrm{Ext}$ and $\mathrm{Ext}_{\mathcal{K}}$ compatible?

- Every $\mathcal{K}$-trivial element is trivial.
  - A non-trivial short exact sequence that is not compact. $c_0$ is not complemented in $\ell_\infty$, so the following short exact sequence is not trivial:

$$0 \longrightarrow c_0 \longrightarrow \ell_\infty \longrightarrow \ell_\infty/c_0 \longrightarrow 0$$

  Since $c_0$ is an $\mathcal{L}_\infty$-space, if it were an element of $\mathrm{Ext}_{\mathcal{K}}(\ell_\infty/c_0, c_0)$, it should be $\mathcal{K}$-trivial, hence trivial.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi & \longrightarrow & \ell_1(I) & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\ & & \downarrow K & \swarrow \hat{K} & \downarrow & & \| & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \end{array}$$

- Every trivial element is $\mathcal{K}$-trivial. Please, keep on reading…

## Extension of compact operators: main results

Theorem 3. **A compact operator that admits an extension to some superspace does not necessarily admit a compact extension.**

Proof. Let $0 \longrightarrow Y \xrightarrow{\rho} X \xrightarrow{\pi} Z \longrightarrow 0$ be a non-trivial short exact sequence and $K : Y \to E$ a compact operator without an extension:

$$\begin{array}{cccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\rho} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 & [\mathrm{x}] \\ & & \downarrow K & & \downarrow \overline{K} & & \| & & & \\ 0 & \longrightarrow & E & \xrightarrow{i} & PO & \xrightarrow{q} & Z & \longrightarrow & 0 & [K\mathrm{x}] \neq 0 \end{array}$$

Let us observe that $\overline{K}$ is a (non-compact) extension for the compact operator $iK : Y \to PO$:

$$\begin{array}{cccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\rho} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 & [\mathrm{x}] \\ & & \downarrow iK & \swarrow \overline{K} & \downarrow & & \| & & & \\ 0 & \longrightarrow & PO & \longrightarrow & \overline{PO} & \longrightarrow & Z & \longrightarrow & 0 & [iK\mathrm{x}] = 0 \end{array}$$

Is there a compact extesion $\beta : X \to PO$ for $iK$? Let us suppose so:

$$\begin{array}{cccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\rho} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 & [\mathrm{x}] \\ & & \downarrow K & & \downarrow \beta & & \downarrow \gamma & & & \\ 0 & \longrightarrow & E & \xrightarrow{i} & PO & \xrightarrow{q} & Z & \longrightarrow & 0 & \end{array}$$

where $\beta$, and therefore $\gamma$, are compact. The previous diagram can be decomposed as follows:

$$\begin{array}{cccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\rho} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 & [\mathrm{x}] \\ & & \downarrow K & & \downarrow \overline{K} & & \| & & & \\ 0 & \longrightarrow & E & \longrightarrow & PO & \longrightarrow & Z & \longrightarrow & 0 & [K\mathrm{x}] \\ & & \| & \beta & \downarrow u & & \| & & & \\ 0 & \longrightarrow & E & \longrightarrow & PB & \longrightarrow & Z & \longrightarrow & 0 & [K\mathrm{x}\gamma] \\ & & \| & & \downarrow \overline{\gamma} & & \downarrow \gamma & & & \\ 0 & \longrightarrow & E & \xrightarrow{j} & PO & \xrightarrow{q} & Z & \longrightarrow & 0 & [K\mathrm{x}] \end{array}$$

Therefore, $[K\mathrm{x}] = [K\mathrm{x}\gamma]$ which implies that $[K\mathrm{x}] = 0$ [1, Lem. 4.3.3], contradicting our hypothesis. □

We now show that **compact operators that cannot be extended occur quite often.** For this purpose, we need to represent short exact sequences by means of a certain type of *nonlinear map* defined between Banach spaces.

## Centralizers and quasi-linear maps

A homogeneous map $\Omega : X \to Y$ is ***quasi-linear*** if there is $C > 0$ such that, for every $x, y \in X$, the estimation $\|\Omega(x+y) - \Omega(x) - \Omega(y)\| \leq C(\|x\| + \|y\|)$ holds. Due to Kalton [4] we know that quasi-linear maps can be used to represent short exact sequences, and vice versa. Indeed, every quasi-linear map $\Omega : X \to Y$ gives rise to the exact sequence

$$0 \longrightarrow Y \longrightarrow Y \oplus_\Omega X \longrightarrow X \longrightarrow 0$$

where $Y \oplus_\Omega X$ is just the direct product $Y \times X$ endowed with the quasi-norm $\|(y,x)\|_\Omega = \|y - \Omega(x)\| + \|x\|$. Conversely, every short exact sequence arises in this way. It is easy to check that the short exact sequence induced by a quasi-linear map $\Omega$ is trivial precisely when $\Omega = B + L$ for a suitable linear map $L : X \to Y$ and a suitable bounded map $B : X \to Y$. Hence, we define the space

$$\mathcal{Q}(X,Y) = \frac{\mathrm{Q}(X,Y)}{\mathrm{L}(X,Y) + \mathrm{B}(X,Y)},$$

where $Q(X,Y)$ is the space of all quasi-linear maps from $X$ to $Y$. This way, it holds that $\mathcal{Q}(X,Y) \simeq \mathrm{Ext}(X,Y)$. Now, let us consider only spaces $X$ which are *Banach sequence spaces*, that is:

- the elements of $X$ are sequences,
- the unit vectors $(e_n)_{n=1}^\infty$ form a normalized basis of $X$,
- if $x \in X$ and $|y| \leq x$, then $y \in X$ and $\|y\| \leq \|x\|$.

The space $X$ is the completion of the subspace $X^0$ consisting of finitely supported sequences under a certain norm. Every sequence space $X$ is an $\ell_\infty$-*module*: the *pointwise product* $\ell_\infty \times X \to X$, $(a \cdot x)(n) = a(n) \cdot x(n)$ is norm-continuous. Now, a quasi-linear map $\Omega : X^0 \to Y$ is a ***centralizer*** if there is $C > 0$ such that $\|a \cdot \Omega(x) - \Omega(a \cdot x)\| \leq C\|a\|_\infty \|x\|$ for every $a \in \ell_\infty$ and for every $x \in X$.

Consider $X$ a Banach sequence space, and let us denote $X_n = [e_i : 1 \leq i \leq n]$. Let $\Omega : X^0 \to X$ be a centralizer and $n \in \mathbb{N}$, we define:

$$\lambda_n[\Omega] = \inf\{\|\pi\|, \pi : X_n \oplus_\Omega X_n \to X_n \text{ such that } \pi(x,0) = x\}$$

Lemma 4. **Assume $X$ is an ultrasummand. Then $\Omega$ is trivial if and only if $(\lambda_n[\Omega])_{n=1}^\infty$ is bounded.**

Proof. If $\Omega$ is trivial and $\pi : X \oplus_\Omega X \to X$ is a projection, then for every $n \in \mathbb{N}$ and every projection $\pi_n : X_n \oplus_\Omega X_n \to X_n$ we must have that $\|\pi_n\| \leq \|\pi\|$. Conversely, if $\sup_n \lambda_n[\Omega]$ is finite, consider $\mathbb{U}$ a free ultrafilter on $\mathbb{N}$ and the operator $\pi = [\pi_n] : X \oplus_\Omega X \to X_{\mathbb{U}}$. Since $X$ is complemented in $X_{\mathbb{U}}$ via some projection $P$, it turns out that $P\pi$ is a bounded projection, which implies $\Omega$ is trivial. □

Proposition 5. **Every non-trivial centralizer $\Omega : X^0 \to X$ admits a diagonal compact operator $d_\bullet : X \to X$ with no extension to $X \oplus_\Omega X$.**

Proof. It is a consequence from the above plus the fact that if $(d_n) \in c_0$ is decreasing, then $d_n \cdot \lambda_n[\Omega] \to \infty$ implies $\lambda_n[d_\bullet\Omega] \to \infty$. □

Theorem 6. **Let $E$ and $X$ be Banach spaces.**

(i) **If $E$ has the BAP, then $\mathrm{Ext}(X,E) = 0$ implies $\mathrm{Ext}_{\mathcal{K}}(X,E) = 0$.**

(ii) **If $E$ is an ultrasummand with the BAP, then $\mathrm{Ext}_{\mathcal{K}}(X,E) = 0$ implies $\mathrm{Ext}(X,E) = 0$.**

Proof. (i) .– Recall that $\mathrm{Ext}(X,E) = 0$ means –see [4, Prop. 3.3] or [2, Thm. 1]– that given a projective presentation of $X$

$$0 \longrightarrow \ker\pi \longrightarrow \ell_1(I) \longrightarrow X \longrightarrow 0 \qquad (\mathrm{x})$$

there exists $C > 0$ such that every operator $\tau : \ker\pi \to E$ admits an extension $\tau_C : \ell_1(I) \to E$ with $\|\tau_C\| \leq C\|\tau\|$. Using this fact and that $E$ has the $\lambda$-AP let us show that, if $\tau$ is compact, then $\tau_{2\lambda C}$ can be chosen compact. First: if $\tau$ has finite rank then $\tau_{2\lambda C}$ can be chosen to have finite rank. Indeed, pick $\varepsilon > 0$ and $T : E \to E$ a finite rank operator such that $T(\tau x) = \tau x$ for all $x$ and $\|T\| \leq \lambda + \varepsilon$ (see [1, Def. 0.0.1] and [3, Lem. 2.4]). Hence, $T\tau_C$ is a finite-rank operator extension of $\tau$ and $\|T\tau_C\| \leq (\lambda + \varepsilon)C$. Finally, compact operators can be extended to compact operators: if $\tau = \lim \tau_n$ is compact for a sequence of finite-rank operators with $\|\tau_n\| \leq \lambda$, and we assume $\|\tau_{n+1} - \tau_n\| \leq 2^{-n}$, then $T = \sum(\tau_{n+1} - \tau_n)_{2\lambda C}$ is a compact extension of $\tau$.

(ii) .– Let $\tau : \ker\pi \to E$ be an operator, and $(\tau_n)$ be a sequence of finite rank operators pointwisely convergent to the identity. Recall that, due to the $\lambda$-AP, if all compact operators admit $C$-extensions, all finite-rank operators admit finite-rank $2\lambda C$-extensions. Let $((\tau_n)_{2\lambda C})$ be the sequence of such extensions, consider $\mathbb{U}$ be a free ultrafilter on $\mathbb{N}$, and define $T = [(\tau_n)_{2\lambda C}] : X \to E_{\mathbb{U}}$. If $P : E_{\mathbb{U}} \to E$ is a projection onto the natural diagonal embedding $E \to E_{\mathbb{U}}$, then $PT$ is an extension of $\tau$. □

Finally, let us observe that equation (1) endows $\mathrm{Ext}(Z,Y)$ with a natural **(non-Hausdorff) vector topology**. In this line, we have:

Theorem 7. **Let $E$ be an ultrasummand with the BAP. Then:**

$$\text{(i) } \mathrm{Ext}(X,E) = 0 \iff \text{(ii) } \mathrm{Ext}(X,E) \text{ is Hausdorff} \iff \text{(iii) } \mathrm{Ext}_{\mathcal{K}}(X,E) = 0.$$

Proof. The implication (i) $\Rightarrow$ (ii) is obvious, and (ii) $\Rightarrow$ (iii) follows from [1, Lemma 4.5.10]. Finally, (iii) $\Rightarrow$ (i) is just Theorem 6. □

## References


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