The weak Extension Principle and its coarse version

Alessandro Vignati
IMJ-PRG - Université Paris Cité
Institut Universitaire de France

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1. The weak Extension Principle wEP
2. The coarse wEP
3. How does one prove these statements?

1. The weak Extension Principle wEP

The goal of this talk is to try to understand continuous functions between Čech–Stone remainders of locally compact topological spaces and boundary spaces arising from coarse geometry. (I live in France: compact means compact Hausdorff.)

If X is a locally compact noncompact topological space, the Čech–Stone **compactification** of X, βX, is the 'smallest compact space in which X sits densely'. βX is a compact space in which X embeds densely that has the following universal property: any continuous map $f\colon X \to K$ where K is a compact space extends uniquely to a continuous map $\beta f\colon \beta X \to K$ such that $\beta f \restriction X = f$.

The Čech–Stone **remainder** of X is the space $X^* = \beta X \setminus X$. Points in X^* are nonprincipal ultrafilters on the space of all closed subsets of X.

Dually, βX is the spectrum of the C^*-algebra of all continuous bounded functions on X, i.e., $C_b(X) = C(\beta X)$, and X^* is the spectrum of $C_b(X)/C_0(X)$.

One does one construct maps between Čech–Stone remainders?

Let us start with an *easy* space, ω. Continuous maps $\omega^* \to \omega^*$ correspond to

- unital C^*-algebra homomorphisms $\ell_\infty/c_0 \to \ell_\infty/c_0$, and to
- Boolean algebra homomorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$.

Consider a finite-to-one map $f\colon \omega \to \omega \subseteq \beta\omega$. Since f is proper, $\beta f \restriction \omega^* \subseteq \omega^*$. If f is an almost permutation, $\beta f \restriction \omega^*$ is an autohomeomorphism of ω^*. Maps arising this way are called **trivial**.

Question

Are all continuous functions $\omega^* \to \omega^*$ trivial?

This problem was originally studied (for homeomorphisms) as a by-product of the question of whether ω^* is homogeneous (Erdős).

Note, there are at most $2^{\aleph_0}$ trivial continuous maps $\omega^* \to \omega^*$.

A bit of history:

- Rudin ('50s) shows that assuming CH there are $2^{2^{\aleph_0}}$ nontrivial autohomeomorphisms of ω^* (plenty of nontrivial ones);
- Shelah ('80s) showed that it is consistent that all autohomeomorphisms of ω^* are trivial;
- Shelah and Steprans ('80s) showed that the statement "all autohomeomorphisms of ω^* are trivial" follows from PFA;
- Velikovic ('92) showed it follows from OCA and $\mathrm{MA}_{\aleph_1}$;
- De Bondt, Farah and V. ('23) showed it follows from just OCA.

PFA is the Proper Forcing Axiom of Shelah. Todorcevic's OCA (after Abraham–Rubin–Shelah) is a Ramseyan colouring axiom, and $\mathrm{MA}_{\aleph_1}$ is Martin's Axiom at level $\aleph_1$. Both OCA and $\mathrm{MA}_{\aleph_1}$ are consequences of PFA and contradict CH. Their conjugation does not have any large cardinal consistency strength.

What about continuous maps which are not necessarily homeomorphisms?

Definition (Farah, approx 2000)

We say that the **weak Extension Principle** holds for ω if for every continuous map $F\colon \omega^* \to \omega^*$ there exists a partition into clopen sets $\omega^* = U_0 \cup U_1$ such that $F[U_0]$ is nowhere dense and there is a proper $G\colon \omega \to \omega$ such that $\beta G \restriction U_1 = F \restriction U_1$.

Dually, if $F\colon \omega^* \to \omega^*$ has dual $\Phi\colon C(\omega^*) \to C(\omega^*)$, we get:

$$\begin{tikzcd} & C(U_0) \arrow[dr] & \\ C(\omega^*) \arrow[ur, "\Phi_0"] \arrow[rr, "\Phi = \Phi_0 \oplus \Phi_1"] \arrow[dr, "\Phi_1"] & & C(\omega^*) \\ & C(U_1) \arrow[ur] & \end{tikzcd}$$

Here $\Phi_i = \chi_{U_i}\Phi\chi_{U_i}$. Φ_0 has large kernel (i.e., its dual $F \restriction U_0$ has nowhere dense image), while Φ_1 has trivial dual, as witnessed by G.

Theorem (Farah, approx '00)

Assume OCA *and* $\mathrm{MA}_{\aleph_1}$. *Then the weak Extension Principle holds for* ω.

Farah's weak Extension Principle was indeed formulated and proved (under OCA and $\mathrm{MA}_{\aleph_1}$) for remainders of zero-dimensional spaces.

Can we get rid of the 'nowhere dense' part?

- Dow constructed ('14) in ZFC a nontrivial (nowhere dense) copy of ω^* inside ω^*.

Therefore a stronger version of the weak Extension Principle cannot hold. Also, clearly, the weak Extension Principle for ω cannot hold under CH (because of nontrivial homeomorphisms).

For general spaces, we need to fix the definition: consider for example $Y = (-\infty, 0] \cup [1, \infty)$ and $X = \mathbb{R}$.

X^* and Y^* are homeomorphic, as $[0, 1]$ is compact (and so all ultrafilters concentrating on $[0, 1]$ are principal).

On the other hand, there is no continuous map $X \to Y$ which induces an homeomorphism between Y^* and X^*, as Y is disconnected.

Definition

Let X and Y be locally compact noncompact second countable spaces. X and Y satisfy the weak Extension Principle (we write $\mathrm{wEP}(Y, X)$), if:

For every continuous map $F\colon Y^* \to X^*$ there exists a partition into clopen sets $Y^* = U_0 \cup U_1$, an open set with compact closure $V_Y \subseteq Y$ such that $F[U_0]$ is nowhere dense in X^*, and a proper continuous $G\colon Y \setminus V_Y \to X$ such that βG restricts to F on U_1.

By wEP we denote the statement "$\mathrm{wEP}(Y, X)$ holds whenever X and Y are locally compact, noncompact second countable spaces".

$$\begin{tikzcd}
 & C(U_0) \arrow[dr] & \\
C(X^*) \arrow[ur, "\Phi_0"] \arrow[rr, "\Phi"] \arrow[dr, "\Phi_1"] & & C(Y^*) \\
 & C(U_1) \arrow[ur] &
\end{tikzcd}$$

Note, if F is surjective, Φ is injective, meaning U_0 is empty.

Theorem (V.-Yilmaz, '24)

Assume OCA *and* $\mathrm{MA}_{\aleph_1}$. wEP *holds.*

We actually prove more: we show the weak Extension Principle for maps between powers of Čech–Stone remainders, meaning that

if X and Y are locally compact noncompact second countable spaces, and $d, \ell \geq 1$ then:

For every continuous map $F\colon (Y^*)^d \to (X^*)^\ell$ there exists a partition into clopen sets $(Y^*)^d = U_0 \cup U_1$, an open set with compact closure $V_Y \subseteq Y$ such that $F[U_0]$ is nowhere dense in $(X^*)^\ell$, and a continuous function $G\colon (\beta(Y \setminus V_Y))^d \to (\beta X)^\ell$ which restricts to F on U_1.

Before moving on, some consequences of (versions of) the wEP.

Theorem (V., '18)

Assume OCA and $\mathrm{MA}_{\aleph_1}$. Suppose that X and Y are locally compact noncompact second countable spaces such that X^* and Y^* are homeomorphic. Then there are open sets with compact closure $V_X \subseteq X$ and $V_Y \subseteq Y$ such that

$$X \setminus V_X \text{ and } Y \setminus V_Y \text{ are homeomorphic.}$$

Moreover all homeomorphisms between X^* and Y^* are trivial.

Question

Assume CH.

1. *Let X be locally compact and noncompact. Does X^* have nontrivial autohomeomorphisms? (Known for many spaces, including all manifolds , see Farah-McKenney, McKenney-V., V.).*
2. *Can you concretely describe a space X such that X^* and $\mathbb{R}^*$ are homeomorphic but X and $\mathbb{R}$ do not look alike (i.e., there is no trivial homeomorphism between X^* and $\mathbb{R}^*$)?. (Same for $\mathbb{R}^n$).*

Theorem (V., V.-Yilmaz)

Assume OCA and $\mathrm{MA}_{\aleph_1}$. Suppose that X and Y are locally compact noncompact second countable spaces such that X^* continuously surjects onto into Y^*. Then there are open sets with compact closure $V_X \subseteq X$ and $V_Y \subseteq Y$ such that $Y \setminus V_Y$ continuously surjects onto $X \setminus V_X$.

The conclusion of the above theorem is not true under CH. For example, if CH is assumed, $[0,1)^*$ continuously surjects onto every connected compact space of weight $\leq \mathfrak{c}$.

Theorem (V.-Yilmaz)

Assume OCA and $\mathrm{MA}_{\aleph_1}$. Suppose that X and Y are locally compact noncompact second countable spaces, and let $\kappa < \lambda$ be cardinals (finite or infinite). Then there is no continuous surjection $(X^*)^\kappa \to (Y^*)^\lambda$.

2. The coarse wEP

We want to bring the study of maps between remainder to the context of **coarse geometry**, the study of metric spaces from very very far.

Fix metric spaces (X, d_X) and (Y, d_Y). A function $f\colon X \to Y$ is **coarse** if it maps close points to close points:

$$\forall r > 0 \exists s > 0 \; d_X(x, x') < r \Rightarrow d_Y(f(x), f(x')) < s.$$

If $f\colon X \to Y$ and $g\colon Y \to X$ are coarse and

$$\sup_{x \in X} d_X(x, g(f(x))) < \infty \text{ and } \sup_{y \in Y} d_Y(y, f(g(y))) < \infty,$$

then f and g are **coarse equivalences** between X and Y, and the spaces are called **coarsely equivalent**.

Coarse equivalences do not have to be injective nor surjective. They only show the shapes of the spaces are the same, the local structure might change. This notion is really not ‘isomorphism’, and behaves more like ‘isomorphism up to multiplicity’ or Morita equivalence.

This approach underlies Gromov's tremendous success in the development of geometric group. Nowadays, coarse geometry is used for index theoretic purposes, applications to transportation theory, mathematical physics, etc etc.

We focus on very discrete spaces: If (X, d) is a metric space, we say that X is **uniformly locally finite** (u.l.f. for friends) if for every $R > 0$

$$\sup_{x \in X} |B_R(x)| < \infty.$$

Balls are not only finite, but the size of balls of a certain fixed radius is uniformly bounded.

Typical examples are Cayley graphs of finitely generated groups and discretisations of smooth Riemannian manifolds.

There is an abundance of operator algebras associated to u.l.f. metric space which are designed to remember the coarse structure, falling under the name of Roe-like algebras.

The program of understanding how much geometry is remembered by these Roe-like algebras (usually in ZFC) has attracted plenty of recent attention, and tremendous progress has been made (with coathours in recent years).

When these algebras arise from quotients, it looks like ZFC is not enough...

Definition

Let (X,d) be a u.l.f. metric space. A bounded function $f\colon X \to \mathbb{C}$ is **slowly oscillating** if for all $\epsilon > 0$ and $R > 0$ there is a finite $F \subseteq X$ such that for all $x, y \notin F$ if $d(x,y) < R$ then $|f(x) - f(y)| < \epsilon$.

As they go to infinity, the variation of these functions is less and less.

Definition

Let (X,d) be a u.l.f. metric space. The algebra of slowly oscillating functions $C_h(X)$ is a unital subalgebra of $\ell_\infty(X)$. Its spectrum is the **Higson compactification** of X, denoted hX.
The **Higson corona** of X, $\nu X = hX \setminus X$, is the spectrum of the algebra $C_\nu(X) = C_h(X)/c_0(X)$.

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Proposition (Roe)

Let X and Y be u.l.f. spaces. A proper coarse map $\varphi\colon Y \to X$ gives a continuous map $\nu\varphi\colon \nu Y \to \nu X$. If φ is a coarse equivalence, $\nu\phi$ is a homeomorphism.

So, if X and Y are the *same* space, their Higson coronas are homeomorphic. What about the converse?

Consider $X = \{n^2, n \in \omega\}$ and $Y = X^2$ with the metrics inherited from $\mathbb{R}$. These are not coarsely equivalent. The spectra νX and νY are zero-dimensional, and their associated Boolean algebras have size $\mathfrak{c}$, are atomless and countably saturated. So νX and νY are homeomorphic under CH.

Theorem (Banakh-Protasov)

Assume CH. If X is a u.l.f. space of asymptotic dimension zero then $C_\nu(X) \cong \ell_\infty / c_0$.

Asymptotic dimension (of Gromov) is the right notion of dimension here.

When the asymptotic dimension raises it is not true that (even under CH) all Higson coronas of spaces of a given dimension are homeomorphic.. yet:

Theorem (Brian-Farah, V.)

Let $n \in \omega$. If CH holds, then there exist $\mathfrak{c}$ pairwise not coarsely equivalent spaces of asymptotic dimension n whose Higson coronas are homeomorphic.

Let us focus on the other side, i.e., getting notions of triviality. Recall that coarse geometric preserving maps between the underlying spaces give continuous maps at the level of Higson coronas.

Definition

A continuous map $F\colon \nu Y \to \nu X$ is trivial if there is a proper coarse map $\varphi\colon Y \to X$ such that $F = \nu\varphi$.

Let X and Y be u.l.f. metric spaces. Let $F\colon \nu Y \to \nu X$ be continuous. The **coarse weak Extension Principle** for F asserts the existence of a clopen partition $\nu Y = U_0 \cup U_1$ such that $F \restriction U_1$ is trivial and $F[U_0]$ is nowhere dense.

Theorem (V., '25)

Assume OCA *and* $\mathrm{MA}_{\aleph_1}$. *Then the* cwEP *holds for all injective continuous maps between Higson coronas.*

Therefore, if X and Y are u.l.f. spaces with homeomorphic Higson coronas, then X and Y are coarsely equivalent. (More than that, all homeomorphisms are trivial). Further rigidity statements can be made about embeddings.

Question

1. *Does the* cwEP *hold for surjective continuous maps?*
2. *Is there a version of the* cwEP *allowing us to handle powers between Higson coronas? Is it true that for all cardinals $\kappa < \lambda$ we never have surjections $(\nu X)^\kappa \to (\nu Y)^\lambda$ under* OCA *and* $\mathrm{MA}_{\aleph_1}$*?*

3. How does one prove these statements?

Let us try to give some hints of how one proceeds when attacking such problems.

OCA, the Open Colouring Axiom is a consequence of PFA formulated in its current form by Todorcevic (present in some form in the work of Abraham, Rubin, and Shelah). OCA asserts that every open graph on a separable metric space is either countably chromatic or else has an uncountable complete subgraph. It implies that $\mathfrak{c} \geq \aleph_2$. (Big open problem: Does OCA imply that $\mathfrak{c} = \aleph_2$?)

$\mathrm{MA}_{\aleph_1}$ is Martin's Axiom at level $\aleph_1$. It is arguably the weakest possible Forcing Axiom (i.e., generalisations of Baire category) around, asserting that generic meet $\aleph_1$-many dense sets in c.c.c. posets, and contradicts CH.

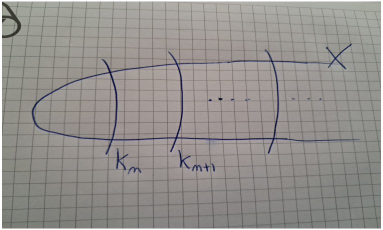
Let us focus on Čech–Stone remainders. Fix locally compact noncompact second countable X and Y, and a continuous $F\colon Y^* \to X^*$ with dual $\Phi\colon C_b(X)/C_0(X) \to C_b(Y)/C_0(Y)$.

We want a *nice* lifting $\tilde{\Phi}$ (where niceness is topological or algebraic) making the following diagram commute:

$$\begin{tikzcd}
C_b(X) \arrow[r, "\tilde{\Phi}"] \arrow[d, "\pi_X"'] & C_b(Y) \arrow[d, "\pi_Y"] \\
C_b(X)/C_0(X) \arrow[r, "\Phi"'] & C_b(Y)/C_0(Y)
\end{tikzcd}$$

and then extract topological information from $\tilde{\Phi}$. (π_X and π_Y are the canonical quotient maps.)

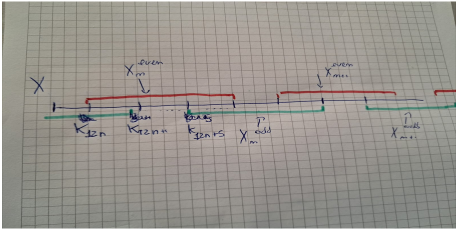
First, we filter $C_b(X)$. Since X is second countable and locally compact, we can find compact sets K_n with $X = \bigcup X_n$ and $K_n \subseteq \mathrm{int}(K_{n+1})$ for all n. We assume that $\mathrm{K_{n+1}} \setminus K_n \neq \emptyset$.



Consider the 'odd and even partition':

$$X_n^{\text{even}} = \mathrm{int}(K_{12n+7}) \setminus K_{12n}$$

$$X_n^{\text{odd}} = \mathrm{int}(K_{12n+13}) \setminus K_{12n+5}.$$

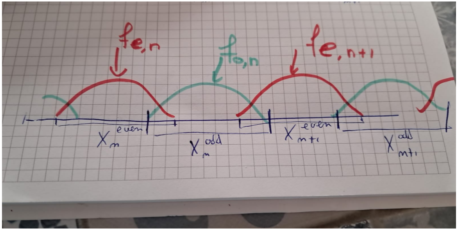


Modulo C_0, every $f \in C_n(X)$ function in $C_b(X)$ can be written as $f_e + f_o$ where

$$f_e = \sum f_{e,n}, \quad f_o = \sum f_{e,n}$$

are such that

$$\mathrm{supp}(f_{e,n}) \subseteq X_n^{even} \text{ and } \mathrm{supp}(f_{o,n}) \subseteq X_n^{odd}.$$



This shows that

$$C_b(X)/C_0(X) = \prod C_0(X_n^{even})/\bigoplus C_0(X_n^{even}) + \prod C_0(X_n^{odd})/\bigoplus C_0(X_n^{odd}).$$

We will lift maps

$$\prod C_0(X_n^{even})/\bigoplus C_0(X_n^{even}) \to C(Y^*)$$

and then glue everything together.
The first (and main) problem is that the range of the reduced product $\prod C_0(X_n^{even})/\bigoplus C_0(X_n^{even})$ is not contained in a reduced product. But we can lift in case the domain is a simple reduced product...

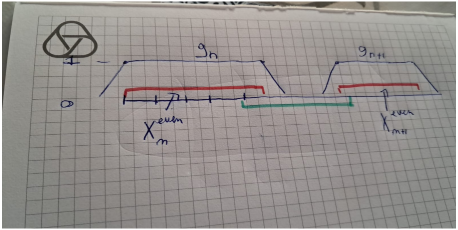
Theorem (McKenney-V., V., approx '18)

Assume OCA *and* $\mathrm{MA}_{\aleph_1}$. *Suppose that* $\rho\colon \ell_\infty/c_0 \to C(Y^*)$ *is a positive orthogonality preserving linear contraction. Then there is a linear positive orthogonality preserving map*

$$\tilde{\rho}\colon \ell_\infty \to C_b(Y)$$

and a nonmeager dense ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ *such that* $\tilde{\rho}$ *lifts* ρ *on functions supported on sets belonging to* $\mathcal{I}$.

Let $g = \sum g_n$ where the g_n's are pairwise orthogonal and $g_n[X_n^{even}] = 1$ and $g_n[X_m^{even}] = 0$ for all $n \neq m$.

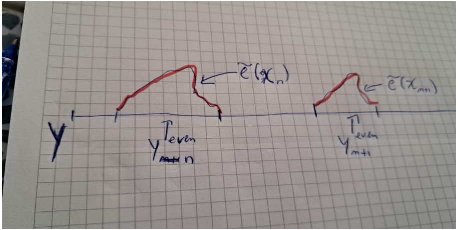


The map $S \mapsto \Phi(\pi_X(\sum_{n,\in S} g_n))$ induces a positive orthogonality preserving linear contraction

$$\rho\colon \ell_\infty/c_0 \to C(Y^*)$$

which we can lift to $\tilde{\rho}\colon \ell_\infty \to C_b(Y)$.
Let

$$Y_n^{even} = \mathrm{supp}(\tilde{\rho}(\chi_n)).$$



Do the same for the odd partition, and get the sets Y_n^{odd}.

Let

$$\tilde{U}_1 = \bigcup_n Y_n^{even} \cup Y_n^{odd}.$$

Modulo compact, $\tilde{U}_1$ is clopen and so this gives a clopen $U_1 \subseteq Y^*$. Moreover, with $U_0 = Y^* \setminus U_1$, $F[U_0]$ is nowhere dense.
We are not done yet. We want to show that

$$\Phi_1 := \chi_{U_1} \Phi \chi_{U_1} \colon C(X^*) \to C(U_1)$$

is trivial. First, if f is a contraction supported on $\bigcup X_n^{even}$, then $gf = f$. Hence for such f, we have that

$$\Phi_1(f) \in \prod_n C_0(Y_n^{even}) / \bigoplus C_0(Y_n^{even}).$$

meaning Φ_1 induces a function

$$\prod C_0(X_n^{even}) / \bigoplus C_0(X_n^{even}) \to \prod C_0(Y_n^{even}) / \bigoplus C_0(Y_n^{even}).$$

We showed Φ_1 induces a function

$$\prod C_0(X_n^{even})/\bigoplus C_0(X_n^{even}) \to \prod C_0(Y_n^{even})/\bigoplus C_0(Y_n^{even}).$$

Theorem (V. ,'18)

Assume OCA *and* $\mathrm{MA}_{\aleph_1}$. *Then you can lift this in a nice way.*

In particular there are functions

$$\alpha_n\colon C_0(X_n^{even}) \to C_0(Y_n^{even}).$$

such that $\prod \alpha_n$ lifts Φ_1 on functions supported on $\prod C_0(X_n^{even})$. These are not *-homomorphisms, but thanks to a result of Sěrml ('90s), we can modify them to *-homomorphisms $C_0(X_n^{even}) \to C_0(Y_n^{even})$.
Using duality again, we get continuous maps $Y_n^{even} \to X_n^{even}$, which we all glue to a get a continuous map $Y \to X$. (One would have to remove some V_Y to account for low values of n not behaving the right way).

The coarse version of this is fairly more complicated.

- It is not possible to factor $C_\nu(X)$ into a finite sum of meaningful reduced products of C^*-algebras (or at least I couldn't do it);
- But one can write $C_\nu(X)$ as a union of sums of reduced products of Banach spaces, and then apply a different set of lifting results (joint with De Bondt, '24);
- Also, obtaining informations about the spaces from well-behaved liftings proved itself to be ... painful, yet it can be done!

Thank you!